Morphing Lord Brouncker's continued fraction for \( \pi \) into the product of Wallis

THOMAS J. OSLER

Introduction

Three of the oldest and most celebrated formulae for \( \pi \) are:

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} \ldots},
\]

(1)

\[
\frac{2}{\pi} = \frac{1 \times 3 \times 5 \times 7 \times 9}{2 \times 2 \times 4 \times 6 \times 8}, \quad \text{and}
\]

(2)

\[
\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \ldots}}},
\]

(3)

The first is Vieta's product of nested radicals from 1592 [1]. The second is Wallis's product of rational numbers [2] from 1656 and the third is Lord Brouncker's continued fraction [3, 2], also from 1656. (In the remainder of the paper, for continued fractions we will use the more convenient notation \( \frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \ldots}}}, \)

In a previous paper [4] the author showed that (1) and (2) are actually special cases of a more general formula

\[
\frac{2}{\pi} = \prod_{k=1}^{n} \sqrt{\frac{1}{2} + \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} + \ldots + \frac{1}{2} \frac{1}{2} \prod_{k=1}^{n} \frac{2^{n+1}k - 1}{2^{n+1}k - 1}.}
\]

(4)

By examining this formula for the sequence of special values \( n = 0, 1, 2, \ldots \), we observe that the product of Wallis (case \( n = 0 \)) appears to gradually morph into Vieta's product as \( n \) approaches infinity. We illustrate this below:

\( n = 0: \)

\[
\frac{2}{\pi} = \frac{1 \times 3 \times 5 \times 7 \times 9}{2 \times 2 \times 4 \times 6 \times 8 \times 8} \ldots \quad \text{(original Wallis's product)}
\]

\( n = 1: \)

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \ldots} \quad \text{13 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17 \times 19 \times 21}
\]

\( n = 2: \)

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \ldots} \quad \text{24 \times 8 \times 8 \times 16 \times 16 \times 16 \times 20 \times 20 \ldots}
\]
$n = 3: \frac{2}{\pi} = \sqrt{\frac{1}{2} + \frac{1}{2 \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2}} + \frac{15 \times 17 \times 31 \times 33 \times 47 \times 49}{16 \times 16 \times 32 \times 32 \times 48 \times 48 \times \cdots}$

$n \to \infty: \frac{2}{\pi} = \sqrt{\frac{1}{2} + \frac{1}{2 \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2}} + \frac{1}{2 \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2}} + \cdots}$ (original Vieta product).

Observe that as we progress through each step of the sequence, one additional factor of Vieta's product is added, while every other fraction, starting with the first, in the Wallis type product is removed.

We will show that Brouncker's continued fraction (3) and the product of Wallis (2) are both special cases of the general formula

$$\frac{4}{\pi} = W(n) \frac{1}{2n + 1} \left( \frac{(4n + 1)}{2(4n + 1)} + \frac{1^2}{2(4n + 1)} + \frac{3^2}{2(4n + 1)} + \frac{5^2}{2(4n + 1)} + \cdots \right),$$

in which

$$W(n) = \frac{1 \times 3 \times 5 \times 7 \times \cdots}{2 \times 2 \times 4 \times 6 \times 8 \times \cdots} \frac{(2n - 1)(2n + 1)}{2n \times 2n}$$

is the partial Wallis product. Just as above, this general formula allows us to start with Lord Brouncker's continued fraction (case $n = 0$) and gradually morph it into the Wallis product as $n$ approaches infinity.

$n = 0: \frac{4}{\pi} = 1 + \frac{1^2}{2} + \frac{3^2}{2} + \cdots$ (Lord Brouncker's continued fraction)

$n = 1: \frac{4}{\pi} = \frac{1 \times 3}{2 \times 2} \frac{1}{3} \left[ 5 + \frac{1^2}{10} + \frac{3^2}{10} + \frac{5^2}{10} + \cdots \right]

n = 2: \frac{4}{\pi} = \frac{2 \times 1 \times 3 \times 5 \times 1}{2 \times 2 \times 4 \times 4 \times 5} \left[ 9 + \frac{1^2}{18} + \frac{3^2}{18} + \frac{5^2}{18} + \cdots \right]

\ldots

n \to \infty: \frac{4}{\pi} = \frac{1 \times 3 \times 5 \times 7 \times 9}{2 \times 2 \times 4 \times 6 \times 8 \times \cdots}$ (original product of Wallis).

(Notice that

$$\frac{1^2}{2(4n + 1)} + \frac{3^2}{2(4n + 1)} + \frac{5^2}{2(4n + 1)} + \cdots < \frac{1}{2(4n + 1)}$$

and therefore as $n$ approaches infinity

$$\frac{1}{2n + 1} \left( \frac{(4n + 1)}{2(4n + 1)} + \frac{1^2}{2(4n + 1)} + \frac{3^2}{2(4n + 1)} + \frac{5^2}{2(4n + 1)} + \cdots \right)$$

approaches the number 2. This justifies the last limit in the above sequence.)

Observe that as we progress through each step of the sequence, one additional factor of Wallis's product is added, while the Brouncker type continued fraction $\left[ x + \frac{1^2}{2x + \frac{3^2}{2x + \frac{5^2}{2x + \cdots}} \right]$ has the value of $x$ incremented by 4.
In a recent paper [5] Lange called attention to a continued fraction for \( \pi \) resembling Brouncker's fraction (3)

\[ \pi = 3 + \frac{1^2}{6} + \frac{3^2}{6 + 6} + \frac{5^2}{6 + 6 + 6} + \ldots . \]  

(7)

We show that the general formula

\[ \pi = \frac{1}{W(n)(2n+1)} \left[ \frac{(4n+3)}{2(4n+3)} + \frac{1^2}{2(4n+3)} + \frac{3^2}{2(4n+3)} + \frac{5^2}{2(4n+3)} + \ldots \right], \]  

(8)

like (5) contains Lange's continued fraction (7) and the product of Wallis (2) as special cases. Again by examining this formula as \( n = 0, 1, 2, \ldots \), we can morph (7) into (2) as shown below:

- \( n = 0: \) \( \pi = 3 + \frac{1^2}{6} + \frac{3^2}{6 + 6} + \frac{5^2}{6 + 6 + 6} + \ldots \) (Lange's continued fraction)

- \( n = 1: \) \( \pi = \frac{2 \times 2}{1 \times 3} \times \frac{1}{3} \left[ 7 + \frac{1^2}{14} + \frac{3^2}{14} + \frac{5^2}{14} + \ldots \right]. \)

- \( n = 2: \) \( \pi = \frac{2 \times 2 \times 4 \times 4}{1 \times 3 \times 3 \times 5} \times \frac{1}{5} \left[ 22 + \frac{1^2}{22} + \frac{3^2}{22} + \frac{5^2}{22} + \ldots \right]. \)

- \( n \to \infty: \) \( \pi = 2 \times \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6}{1 \times 3 \times 3 \times 5 \times 5 \times 7} \ldots \) (Original product of Wallis reciprocated.)

Extensions of the formulae (6) and (8) are also given.

**Derivation of the results and more morphing**

All the results of this paper are special cases of the known formula [6, p. 35]

\[ \frac{4 \Gamma \left( \frac{x+y+3}{4} \right) \Gamma \left( \frac{x-y+3}{4} \right)}{\Gamma \left( \frac{x+y+1}{4} \right) \Gamma \left( \frac{x-y+1}{4} \right)} = x + \frac{1^2 - y^2}{2x} + \frac{3^2 - y^2}{2x} + \frac{5^2 - y^2}{2x} + \ldots , \]  

(9)

valid for either \( y \) an odd integer and \( x \) any complex number or \( y \) any complex number and \( x \) real. The names of Euler, Stieltjes, and Ramanujan [7, p. 140] have been associated with this result. Using the very well known formulae \( \Gamma(n+1) = n! \), \( \Gamma(x) = \Gamma(x+1) \) and \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \) we have \( \Gamma \left( \frac{2k+1}{2} \right) = \frac{1 \times 3 \times 5 \times \ldots \times (2k-1)}{2^k} \sqrt{\pi} \), valid for \( k = 1, 2, 3, \ldots \). With this last result and appropriate values of \( x \) and \( y \), the left hand side of (9) can be expressed in terms of rational numbers and \( \pi \). For example, if we set \( y = 0 \) and \( x = 4n + 1 \) in (9) we get our general formula (5) and setting \( y = 0 \) and \( x = 4n + 3 \) we get our general formula (8). The manipulations are simple and the reader will have no difficulty verifying our formulae.

If in (9) we set \( x = 4n + 3 \) and \( y = 2(2j + 1) \) for \( j \) an integer in the range \( 0 \leq j \leq n \) we get an extension of (5)
\[
\frac{4}{\pi} = \frac{(2n-2j+1)(2n-2j+3) \ldots (2n+2j+1)}{(2n-2j+2)(2n-2j+4) \ldots (2n+2j+2)} \times W(n-j) \times \\
\frac{1}{2n-2j+1} \left[ 4n+3 + \frac{1^2 - 2^2(2j+1)^2}{2(4n+3)} + \frac{3^2 - 2^2(2j+1)^2}{2(4n+3)} + \frac{5^2 - 2^2(2j+1)^2}{2(4n+3)} + \ldots \right]. \tag{10}
\]

Let \(j = 0\) and (10) becomes
\[
\frac{4}{\pi} = \frac{2n+1}{2n+2} \times W(n) \times \\
\frac{1}{2n+1} \left[ 4n+3 + \frac{1^2 - 2^2}{2(4n+3)} + \frac{3^2 - 2^2}{2(4n+3)} + \frac{5^2 - 2^2}{2(4n+3)} + \ldots \right].
\]

We list the special cases of this formula for \(n = 0, 1, 2, \ldots\) below:

\(n = 0:\)
\[
\frac{4}{\pi} = \frac{1}{2} \left[ 3 + \frac{1^2 - 2^2}{6} + \frac{3^2 - 2^2}{6} + \frac{5^2 - 2^2}{6} + \ldots \right].
\]

\(n = 1:\)
\[
\frac{4}{\pi} = \frac{3}{4} \times \frac{1}{2} \times \left[ 7 + \frac{1^2 - 2^2}{14} + \frac{3^2 - 2^2}{14} + \frac{5^2 - 2^2}{14} + \ldots \right].
\]

\(n = 2:\)
\[
\frac{4}{\pi} = \frac{5}{6} \times \frac{1}{2} \times \frac{3}{5} \times \frac{1}{2} \times \left[ 11 + \frac{1^2 - 2^2}{22} + \frac{3^2 - 2^2}{22} + \frac{5^2 - 2^2}{22} + \ldots \right].
\]

\(n \to \infty:\)
\[
\frac{4}{\pi} = \frac{2}{\pi} \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \ 	ext{(original Wallis product)}
\]

If in (9) we set \(x = 4n+1\) and \(y = 2(2j+1)\) for \(j\) an integer in the range \(1 \leq j < n\) we get an extension of (8)
\[
\frac{4}{\pi} = \frac{(2n-2j+2)(2n-2j+4) \ldots (2n+2j)}{(2n-2j+3)(n-j+5) \ldots (2n+2j+1)} \times \frac{1}{W(n-j)} \times \\
\frac{1}{2(n-j)} \left[ 4n+1 + \frac{1^2 - 2^2(2j+1)^2}{2(4n+1)} + \frac{3^2 - 2^2(2j+1)^2}{2(4n+1)} + \frac{5^2 - 2^2(2j+1)^2}{2(4n+1)} + \ldots \right]. \tag{11}
\]

Let \(j = 1\) and (11) becomes
\[
\pi = \frac{(2n)(2n+2)}{(2n+1)(2n+3)} \times \frac{1}{W(n-1)} \times \\
\frac{1}{2(n-1)} \left[ 4n + 1 + \frac{1^2 - 6^2}{2(4n+1)} + \frac{3^2 - 6^2}{2(4n+1)} + \frac{5^2 - 6^2}{2(4n+1)} + \ldots \right].
\]

Let us list this formula for various values on \(n:\)

\(n = 2:\)
\[
\pi = \frac{4 \times 6}{5} \times \frac{2 \times 2}{1 \times 3} \times \frac{1}{18} \times \left[ 9 + \frac{1^2 - 6^2}{18} + \frac{3^2 - 6^2}{18} + \frac{5^2 - 6^2}{18} + \ldots \right].
\]

\(n = 3:\)
\[
\pi = \frac{6 \times 8}{7} \times \frac{2 \times 4}{1 \times 3} \times \frac{1}{26} \times \left[ 13 + \frac{1^2 - 6^2}{26} + \frac{3^2 - 6^2}{26} + \frac{5^2 - 6^2}{26} + \ldots \right].
\]
MORPHING LORD BROWNCKER'S CONTINUED FRACTION FOR $\pi$

$n \to \infty$: $\pi = 2 \times \frac{2 \times 2.4 \times 4.6 \times 6.8 \times 8}{1 \times 3 \times 5 \times 7 \times 9} \ldots$ (original Wallis product reciprocated)

We see that the general formulae (10) and (11) start with a generalised Brouncker type continued fraction $\left[ x + \frac{1^2 - y^2}{2x + \frac{3^2 - y^2}{2x + \frac{5^2 - y^2}{2x + \cdots}}} \right]$ and gradually morph it into the Wallis product as $n$ approaches infinity.

**Final remarks**

Wallis described the ingenious way in which he obtained his product (2) in [2]. He states that he showed his product to Lord Brouncker who then obtained the continued fraction (3). It appears that Brouncker never published his method of finding this continued fraction and only partially explained his reasoning to Wallis. Wallis gives some hints in [2, pp. 167-178] as to how Brouncker proceeded but the explanation is incomplete. Euler took keen interest in Brouncker's continued fractions and gave his own derivations and generalisations in [8]. Stedall in [3, pp. 300-310] made her own conjecture as to how Brouncker might have reasoned. In his discussion of this question, Wallis published a table in [2, p. 172] which we reproduce here.

![Table Image]

In the third row of this table we see the continued fractions obtained from our general formulae (5) and (8). Stedall [3] has recently called attention to these fractions that appear to have been overlooked in recent years. She also points out [3, p. 307] that both Wallis and Brouncker could easily have written the value of these fractions in terms of rational numbers and $\pi$. See also [9] and [10] for recent discussions of these formulae. Thus we see that the continued fractions that we obtained from (5) and (8) are among the oldest continued fractions and their values were conjectured as early as 1656!

**Acknowledgement**

The author wishes to thank James Smoak for his generous assistance with the historical items in this paper.
References


6. O. Perron, *Die Lehre von den Kettenbrüchen*, Band II, Teubner, Stuttgart (1957). (There is a Chelsea edition of this book in which our equation (9) appears on p. 255 and not p. 35.)


THOMAS J. OSLER

Mathematics Department, Rowan University, Glassboro, NJ 08028 USA
e-mail: osler@rowan.edu