

ANALYSIS OF A SPATIALLY INHOMOGENEOUS STOCHASTIC PARTIAL DIFFERENTIAL EQUATION EPIDEMIC MODEL

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Abstract

This work proposes and analyzes a family of spatially inhomogeneous epidemic models. This is our first effort to use stochastic partial differential equations (SPDEs) to model epidemic dynamics with spatial variations and environmental noise. After setting up the problem, the existence and uniqueness of solutions of the underlying SPDEs are examined. Then, definitions of permanence and extinction are given, and certain sufficient conditions are provided for permanence and extinction. Our hope is that this paper will open up windows for investigation of epidemic models from a new angle.

Keywords: SIR model; SPDE; mild solution; positivity; extinction; permanence

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1. Introduction

This work presents an attempt at studying stochastic epidemic models in which spatial inhomogeneity is allowed. The hope is that it will open up a new angle for investigating a large class of epidemic processes. In lieu of the usual formulation based on stochastic differential equations considered in the literature, we propose a new class of models by using stochastic partial differential equations. This greatly enriches the class of systems and offers great opportunities both mathematically and practically. Meanwhile, it poses greater challenges.

Epidemic models (compartment models) in which the density functions are spatially homogeneous were introduced in 1927 by Kermack and McKendrick [24, 25]. The main idea is to partition the population into susceptible, infected, and recovered (SIR) classes. The dynamics of these classes are given by a system of deterministic differential equations. One classical model takes the form

$$\begin{cases} dS(t) = \left[\Lambda - \mu_S S(t) - \frac{\alpha S(t)I(t)}{S(t) + I(t)} \right] dt, & t \geq 0, \\ dI(t) = \left[-(\mu_I + r)I(t) + \frac{\alpha S(t)I(t)}{S(t) + I(t)} \right] dt, & t \geq 0, \\ dR(t) = \left[-\mu_R R(t) + rI(t) \right] dt, & t \geq 0, \\ S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0) = R_0 \geq 0, \end{cases}$$

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where $S(t)$, $I(t)$, and $R(t)$ are the densities of the susceptible, infected, and recovered populations, respectively. In the above, Λ is the recruitment rate of the population; μ_S , μ_I , and μ_R are the death rates of susceptible, infected, and recovered individuals, respectively; α is the infection rate; and r is the recovery rate. To simplify the study, it has been noted that the dynamics of recovered individuals have no effect on the disease transmission dynamics. Thus, following the usual practice, the recovered individuals are removed from the formulation henceforth. SIR models are known to be useful and suited to such diseases as rubella, whooping cough, measles, smallpox, etc. It has also been recognized that random effects are not avoidable and a population is often subject to random disturbances. Thus, much effort has also been devoted to the investigation of stochastic epidemic models. One popular approach is adding stochastic noise perturbations to the above deterministic models. In recent years, resurgent attention has been devoted to analyzing and designing controls of infectious diseases for host populations; see [1, 4, 6, 14, 17, 20, 23, 26, 36, 37] and references therein.

For the deterministic models, studying the systems from a dynamic system point of view, certain threshold-type results have been found. In accordance with the threshold, the population tends to the disease-free equilibrium or approaches an endemic equilibrium under certain conditions. It has taken a long time to find the critical threshold value for the corresponding stochastic systems. A characterization of systems using critical thresholds was reported very recently in [15, 18, 21, 22], in which sufficient and almost necessary conditions were obtained using the idea of a Lyapunov exponent, so that the asymptotic behavior of the system has been completely classified. Such ideas can also be found in [16, 28] for related problems.

From another angle, it has been widely recognized that there should be spatial dependence in the model to better reflect the spatial variations. In the spatially inhomogeneous case, the epidemic reaction–diffusion system takes the form

$$\begin{cases} \frac{\partial}{\partial t} S(t, x) = k_1 \Delta S(t, x) + \Lambda(x) - \mu_1(x)S(t, x) - \frac{\alpha(x)S(t, x)I(t, x)}{S(t, x) + I(t, x)} & \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ \frac{\partial}{\partial t} I(t, x) = k_2 \Delta I(t, x) - \mu_2(x)I(t, x) + \frac{\alpha(x)S(t, x)I(t, x)}{S(t, x) + I(t, x)} & \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ \partial_\nu S(t, x) = \partial_\nu I(t, x) = 0 & \text{in } \mathbb{R}^+ \times \partial\mathcal{O}, \\ S(x, 0) = S_0(x), I(x, 0) = I_0(x) & \text{in } \mathcal{O}, \end{cases}$$

where Δ is the Laplacian with respect to the spatial variable, \mathcal{O} is a bounded domain with a C^2 boundary of \mathbb{R}^l ($l \geq 1$), $\partial_\nu S$ denotes the directional derivative with the ν being the outer normal direction on $\partial\mathcal{O}$, and k_1 and k_2 are positive constants representing the diffusion rates of the susceptible and infected population densities, respectively. In addition, $\Lambda(x)$, $\mu_1(x)$, $\mu_2(x)$, $\alpha(x) \in C^2(\mathcal{O})$ are non-negative functions. Recently, epidemic reaction–diffusion models have been studied in [2, 19, 32, 33, 40] and the references therein. In [36], some results were given for a general epidemic model with reaction–diffusion in terms of basic reproduction numbers. The above models are all noise free. However, random noise perturbations in the environment inevitably appear often. Therefore, a more suitable description requires considering stochastic epidemic diffusive models. Taking this into consideration, we propose a spatially non-homogeneous model using a system of stochastic partial differential equations given by

$$\begin{cases}
 dS(t, x) = \left[k_1 \Delta S(t, x) + \Lambda(x) - \mu_1(x)S(t, x) - \frac{\alpha(x)S(t, x)I(t, x)}{S(t, x) + I(t, x)} \right] dt \\
 \quad + S(t, x)dW_1(t, x) \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\
 dI(t, x) = \left[k_2 \Delta I(t, x) - \mu_2(x)I(t, x) + \frac{\alpha(x)S(t, x)I(t, x)}{S(t, x) + I(t, x)} \right] dt \\
 \quad + I(t, x)dW_2(t, x) \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\
 \partial_\nu S(t, x) = \partial_\nu I(t, x) = 0 \quad \text{in } \mathbb{R}^+ \times \partial\mathcal{O}, \\
 S(x, 0) = S_0(x), I(x, 0) = I_0(x) \quad \text{in } \mathcal{O},
 \end{cases} \tag{1.1}$$

where $W_1(t, x)$ and $W_2(t, x)$ are $L^2(\mathcal{O}, \mathbb{R})$ -value Wiener processes that represent noise in both time and space. We refer readers to [12] for more details on the $L^2(\mathcal{O}, \mathbb{R})$ -value Wiener process.

Because this is our first work in this direction, we have to settle a number of issues. First, we establish the existence and uniqueness of solutions in the sense of mild solution of the stochastic partial differential equations. Moreover, we examine some long-term behavior of the solutions. These are the main objectives of the current work.

The rest of the paper is arranged as follows. Section 2 gives some preliminary results and also formulates the problem that we wish to study. Section 3 establishes the existence and uniqueness of the solution of the stochastic partial differential equations. Section 4 provides sufficient conditions for extinction and permanence, while Section 5 provides an example. Finally, Section 6 concludes the paper with some further remarks.

2. Preliminary and formulation

Let \mathcal{O} be a bounded domain in \mathbb{R}^l (with $l \geq 1$) having a C^2 boundary, and let $H := L^2(\mathcal{O}; \mathbb{R})$ be the separable Hilbert space, endowed with the scalar product

$$\langle u, v \rangle_H := \int_{\mathcal{O}} u(x)v(x) \, dx$$

and the corresponding norm $|u|_H = \sqrt{\langle u, u \rangle_H}$. We will say $u \geq 0$ if $u(x) \geq 0$ almost everywhere in \mathcal{O} . Moreover, we denote by $L^2(\mathcal{O}, \mathbb{R}^2)$ the space of all functions $u(x) = (u_1(x), u_2(x))$ where $u_1, u_2 \in L^2(\mathcal{O}, \mathbb{R})$, on which the inner product is defined as

$$\begin{aligned}
 \langle u, v \rangle_{L^2(\mathcal{O}, \mathbb{R}^2)} &:= \int_{\mathcal{O}} \langle u(x), v(x) \rangle_{\mathbb{R}^2} \, dx = \int_{\mathcal{O}} (u_1(x)v_1(x) + u_2(x)v_2(x)) \, dx \\
 &= \langle u_1, v_1 \rangle_{L^2(\mathcal{O}, \mathbb{R})} + \langle u_2, v_2 \rangle_{L^2(\mathcal{O}, \mathbb{R})},
 \end{aligned}$$

for all $u, v \in L^2(\mathcal{O}, \mathbb{R}^2)$. Note that $L^2(\mathcal{O}, \mathbb{R}^2)$ is a separable Hilbert space. In what follows, we use u to denote a function that is either real valued or \mathbb{R}^2 -valued (as will be clear from the context). Denote by E the Banach space $C(\overline{\mathcal{O}}; \mathbb{R})$ endowed with the sup-norm

$$|u|_E := \sup_{x \in \overline{\mathcal{O}}} |u(x)|.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space, and $L^p(\Omega; C([0, t], C(\overline{\mathcal{O}}, \mathbb{R}^2)))$ be the space of all predictable $C(\overline{\mathcal{O}}, \mathbb{R}^2)$ -valued processes u in $C([0, t], C(\overline{\mathcal{O}}, \mathbb{R}^2))$, \mathbb{P} almost surely (a.s.), with the norm $L_{t,p}$ as follows:

$$|u|_{L_{t,p}}^p := \mathbb{E} \sup_{s \in [0, t]} |u(s)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p,$$

where

$$|u|_{C(\bar{\mathcal{O}}, \mathbb{R}^2)} = \left(\sum_{i=1}^2 \sup_{x \in \bar{\mathcal{O}}} |u_i(x)|^2 \right)^{\frac{1}{2}} \quad \text{if } u = (u_1, u_2) \in C(\bar{\mathcal{O}}, \mathbb{R}^2).$$

For $\varepsilon > 0, p \geq 1$, denote by $W^{\varepsilon, p}(\mathcal{O}, \mathbb{R}^2)$ the Sobolev–Slobodeckij space (the Sobolev space with non-integer exponent) endowed with the norm

$$|u|_{\varepsilon, p} := |u|_{L^p(\mathcal{O}, \mathbb{R}^2)} + \sum_{i=1}^2 \int_{\mathcal{O} \times \mathcal{O}} \frac{|u_i(x) - u_i(y)|^p}{|x - y|^{\varepsilon p + 1}} dx dy.$$

Assume that $B_{k,1}(t)$ and $B_{k,2}(t)$ with $k = 1, 2, \dots$ are independent $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted one-dimensional Wiener processes. Now, fix an orthonormal basis $\{e_k\}_{k=1}^\infty$ in H and assume that this sequence is uniformly bounded in $L^\infty(\mathcal{O}, \mathbb{R})$, i.e.

$$C_0 := \sup_{k \in \mathbb{N}} |e_k|_{L^\infty(\mathcal{O}, \mathbb{R})} = \sup_{k \in \mathbb{N}} \operatorname{ess\,sup}_{x \in \mathcal{O}} e_k(x) < \infty.$$

We define the infinite-dimensional Wiener processes $W_i(t)$, which are driving noise in (1.1), as follows:

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{a_{k,i}} B_{k,i}(t) e_k, \quad i = 1, 2,$$

where $\{a_{k,i}\}_{k=1}^\infty$ are sequences of non-negative real numbers satisfying

$$a_i := \sum_{k=1}^{\infty} a_{k,i} < \infty, \quad i = 1, 2. \quad (2.1)$$

Let A_1 and A_2 be Neumann realizations of $k_1 \Delta$ and $k_2 \Delta$ in H , respectively, i.e.

$$\begin{aligned} D(A_i) &= \{u \in H \mid \Delta u \in H \text{ and } \partial_\nu u = 0 \text{ on } \partial \mathcal{O}\}, \\ A_i u &= k_i \Delta u, \quad u \in D(A_i), \end{aligned}$$

where the Laplace operator in the above definition is understood in the distribution sense. Then, A_1 and A_2 are infinitesimal generators of analytic semigroups e^{tA_1} and e^{tA_2} with corresponding Neumann heat kernels denoted by $p_{\mathcal{O}}^{N,1}(t, x, y), p_{\mathcal{O}}^{N,2}(t, x, y)$, i.e.

$$(e^{tA_i} u)(x) = \int_{\mathcal{O}} p_{\mathcal{O}}^{N,i}(t, x, y) u(y) dy, \quad i = 1, 2,$$

respectively. In addition, if we denote by $A := (A_1, A_2)$ the operator defined in $L^2(\mathcal{O}, \mathbb{R}^2)$ by $Au := (A_1 u_1, A_2 u_2)$ for $u = (u_1, u_2) \in L^2(\mathcal{O}, \mathbb{R}^2)$, then it generates an analytic semigroup e^{tA} with $e^{tA} u = (e^{tA_1} u_1, e^{tA_2} u_2)$. In [13, Theorem 1.4.1], it is proved that the space $L^1(\mathcal{O}, \mathbb{R}^2) \cap L^\infty(\mathcal{O}, \mathbb{R}^2)$ is invariant under e^{tA} , so that e^{tA} may be extended to a non-negative one-parameter semigroup $e^{tA(p)}$ on $L^p(\mathcal{O}; \mathbb{R}^2)$, for all $1 \leq p \leq \infty$. All these semigroups are strongly continuous and consistent in the sense that $e^{tA(p)} u = e^{tA(q)} u$ for any $u \in L^p(\mathcal{O}, \mathbb{R}^2) \cap L^q(\mathcal{O}, \mathbb{R}^2)$ (see [8]). So, we will suppress the superscript p and denote them by e^{tA} whenever there is no confusion. Moreover, if we consider the part A_i^E of A_i in the space of continuous functions E , it generates an analytic semigroup (see [3, Chapter 2]), which has no dense domain in general. However, since we have assumed that \mathcal{O} has a C^2 boundary in our boundary condition, A_i^E has

dense domain in E (see [12, Appendix A.5.2]), and hence this analytic semigroup is strongly continuous. Finally, we recall some well-known properties of the operators A_i and analytic semigroups e^{tA_i} for $i = 1, 2$ as follows (for further details, we refer the reader to [3, 13, 31] and the references therein):

- For all $u \in H$, $\int_0^t e^{sA_i} u \, ds \in D(A_i)$ and $A_i(\int_0^t e^{sA_i} u \, ds) = e^{tA_i} u - u$.
- By Green's identity, it can be proved that A_i is symmetric, that A_i is self-adjoint in H , and that, for all $u \in D(A_i)$, $\int_{\mathcal{O}} (A_i u)(x) \, dx = 0$.
- For any $t > 0, x, y \in \mathcal{O}$,

$$0 \leq p_{\mathcal{O}}^{N,i}(t, x, y) \leq c_1(t \wedge 1)^{-\frac{l}{2}} \exp \left\{ -c_2 \frac{|x - y|^2}{t} \right\}$$

for some constants c_1 and c_2 which depend on \mathcal{O} but are independent of u and t .

- The semigroup e^{tA} satisfies

$$\left| e^{tA} u \right|_{L^\infty(\mathcal{O}, \mathbb{R}^2)} \leq c \|u\|_{L^\infty(\mathcal{O}, \mathbb{R}^2)} \quad \text{and} \quad \left| e^{tA} u \right|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)} \leq c \|u\|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)} \tag{2.2}$$

for some constant c which depends on \mathcal{O} but is independent of u and t .

- For any $t, \varepsilon > 0, p \geq 1$, the semigroup e^{tA} maps $L^p(\mathcal{O}, \mathbb{R}^2)$ into $W^{\varepsilon,p}(\mathcal{O}, \mathbb{R}^2)$ and, for all $u \in L^p(\mathcal{O}, \mathbb{R}^2)$,

$$\left| e^{tA} u \right|_{\varepsilon,p} \leq c(t \wedge 1)^{-\varepsilon/2} \|u\|_{L^p(\mathcal{O}, \mathbb{R}^2)} \tag{2.3}$$

for some constant c independent of u and t .

Now, we rewrite (1.1) as a stochastic differential equation in an infinite-dimension space:

$$\begin{cases} dS(t) = \left[A_1 S(t) + \Lambda - \mu_1 S(t) - \frac{\alpha S(t)I(t)}{S(t) + I(t)} \right] dt + S(t) dW_1(t), \\ dI(t) = \left[A_2 I(t) - \mu_2 I(t) + \frac{\alpha S(t)I(t)}{S(t) + I(t)} \right] dt + I(t) dW_2(t), \\ S(0) = S_0, I(0) = I_0. \end{cases} \tag{2.4}$$

As usual, we say that $(S(t), I(t))$ is a mild solution to (2.4) if

$$\begin{cases} S(t) = e^{tA_1} S_0 + \int_0^t e^{(t-s)A_1} \left(\Lambda - \mu_1 S(s) - \frac{\alpha S(s)I(s)}{S(s) + I(s)} \right) ds + W_S(t), \\ I(t) = e^{tA_2} I_0 + \int_0^t e^{(t-s)A_2} \left(-\mu_2 I(s) + \frac{\alpha S(s)I(s)}{S(s) + I(s)} \right) ds + W_I(t), \end{cases} \tag{2.5}$$

where

$$W_S(t) = \int_0^t e^{(t-s)A_1} S(s) dW_1(s) \quad \text{and} \quad W_I(t) = \int_0^t e^{(t-s)A_2} I(s) dW_2(s),$$

or, in vector form,

$$Z(t) = e^{tA} Z_0 + \int_0^t e^{(t-s)A} F(Z(s)) \, ds + \int_0^t e^{(t-s)A} Z(s) \, dW(s), \tag{2.6}$$

where $Z = (S, I)$, $F(Z) = (F_1(Z), F_2(Z)) := \left(\Lambda - \mu_1 S - \frac{\alpha SI}{S+I}, -\mu_2 I + \frac{\alpha SI}{S+I} \right)$, and

$$e^{(t-s)A} Z(s) dW(s) := (e^{(t-s)A_1} S(s) dW_1(s), e^{(t-s)A_2} I(s) dW_2(s)).$$

Because we are modeling SIR epidemic systems, we are only interested in the positive (≥ 0) solutions. Therefore, we define a *positive mild solution* of (2.4) as a mild solution $S(t, x), I(t, x)$ such that $S(t, x), I(t, x) \geq 0$, almost everywhere $x \in \mathcal{O}$, for all $t \geq 0$. Moreover, to have the term $\frac{si}{s+i}$ well defined, we assume that it is equal 0 whenever either $s = 0$ or $i = 0$.

Remark 2.1. The integrals on the right-hand side of (2.5) are understood as Bochner integrals (in the Banach space H), while $W_S(t)$ and $W_I(t)$ are stochastic integrals (stochastic convolutions). The $S(s)$ (resp. $I(s)$) in the stochastic integrals is understood as a multiplication operator, i.e.

$$S(s)(u) = S(s)u \quad \text{for all } u \in H.$$

The stochastic integral $\int_0^t e^{(t-s)A_i} U(s) dW_i(s)$ (see [12, Chapter 4] for more details on stochastic integrals) is well defined if the process $U(s)$ satisfies

$$\int_0^t \sum_{k=1}^{\infty} a_{k,i} |e^{(t-s)A_i} U(s) e_k|_H^2 ds < \infty.$$

Finally, in the vector form, to simplify the notation we do not write the vectors in column form. However, the calculations involving vectors are understood as in the usual sense.

To investigate epidemic models, an important question is whether the infected individuals will die out in the long term. That is, the consideration of extinction or permanence. Since the mild solution is used, let us introduce the definitions in the weak sense as follows.

Definition 2.1. A population with density $u(t, x)$ is said to be extinct in the mean if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E} \int_{\mathcal{O}} u(s, x) dx ds = 0,$$

and is said to be permanent in the mean if there exists a positive number R_I , independent of the initial conditions of the population, such that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\mathbb{E} \int_{\mathcal{O}} (u^2(s, x) \wedge 1) dx \right)^{\frac{1}{2}} ds \geq R_I.$$

Remark 2.2. It is well known that it is fairly difficult to confirm the existence of strong solutions for stochastic partial differential equations (or even weak solutions); see [12, Section 6.1]. As an alternative, we shall use the notion of mild solutions. Hence, the convergence in our situation is in the weak sense. Note, however, in the deterministic case, [2, 19, 32, 33, 40] obtained strong solutions of deterministic reaction–diffusion epidemic models and the convergence was taken in L^∞ , E , or Sobolev spaces. In what follows, for convenience, we often suppress the phrase ‘in the mean’ when we refer to extinction and permanence, because we are mainly working with mild solutions.

3. Existence and uniqueness of a positive mild solution

In this section we shall prove the existence and uniqueness of a positive mild solution of the system as well as its continuous dependence on the initial conditions. In what follows, without loss of the generality we can assume $|\mathcal{O}| = 1$, where $|\mathcal{O}|$ is the volume of the bounded domain \mathcal{O} in \mathbb{R}^l , and the initial values are non-random for simplicity.

Theorem 3.1. *For any initial data $0 \leq S_0, I_0 \in E$, there exists a unique positive mild solution $(S(t), I(t))$ of (2.4) that belongs to $L^p(\Omega; C([0, T], C(\overline{\mathcal{O}}, \mathbb{R}^2)))$ for any $T > 0$ and $p \geq 1$. Moreover, this solution depends continuously on the initial data.*

3.1. Proof of Theorem 3.1

In this proof, the letter c denotes a positive constant whose value may change in different occurrences. We will write the dependence of the constant on parameters explicitly if it is essential. First, we rewrite the coefficients by defining f and f^* as follows:

$$f(x, s, i) = \left(\Lambda(x) - \mu_1(x)s - \frac{\alpha(x)si}{s+i}, -\mu_2(x)i + \frac{\alpha(x)si}{s+i} \right), \quad x \in \mathcal{O}, (s, i) \in \mathbb{R}^2,$$

and

$$f^*(x, s, i) = f(x, s \vee 0, i \vee 0).$$

Writing $z = (s, i)$, by noting that we are assuming the term $\frac{si}{s+i}$ will be equal to 0 whenever either $s = 0$ or $i = 0$, it is easy to see that $f^*(x, \cdot, \cdot): \mathbb{R}^2 \mapsto \mathbb{R}^2$ is Lipschitz continuous, uniformly in $x \in \mathcal{O}$, so that the composition operator $F^*(z)$ associated with f^* , i.e.

$$F^*(z)(x) = (F_1^*(z)(x), F_2^*(z)(x)) := f^*(x, z(x)), \quad x \in \mathcal{O},$$

is Lipschitz continuous in both $L^2(\mathcal{O}, \mathbb{R}^2)$ and $C(\overline{\mathcal{O}}, \mathbb{R}^2)$. Now, we consider the following problem:

$$dZ^*(t) = [AZ^*(t) + F^*(Z^*(t))] dt + (Z^*(t) \vee 0) dW(t), \quad Z^*(0) = Z_0 = (S_0, I_0), \quad (3.1)$$

where $Z^*(t) = (S^*(t), I^*(t))$ and $Z^*(t) \vee 0$ is defined by

$$(Z^*(t) \vee 0)(x) = (S^*(t, x) \vee 0, I^*(t, x) \vee 0).$$

For any

$$u(t, x) = (u_1(t, x), u_2(t, x)) \in L^p(\Omega; C([0, T], C(\overline{\mathcal{O}}, \mathbb{R}^2))),$$

consider the mapping

$$\gamma(u)(t) := e^{tA}Z_0 + \int_0^t e^{(t-s)A}F^*(u(s)) ds + \varphi(u)(t),$$

where

$$\begin{aligned} \varphi(u)(t) &:= \int_0^t e^{(t-s)A}(u(s) \vee 0) dW(s) \\ &:= \left(\int_0^t e^{(t-s)A_1}(u_1(s) \vee 0) dW_1(s), \int_0^t e^{(t-s)A_2}(u_2(s) \vee 0) dW_2(s) \right). \end{aligned}$$

We will prove that γ is a contraction mapping in $L^p(\Omega; C([0, T_0], C(\overline{\mathcal{O}}, \mathbb{R}^2)))$ for some $T_0 > 0$ and any $p \geq p_0$ for some p_0 .

Lemma 3.1. *There exists p_0 such that, for any $p \geq p_0$,*

- *the mapping φ maps $L^p(\Omega; C([0, t], C(\overline{\mathcal{O}}, \mathbb{R}^2)))$ into itself,*
- *for any $u = (u_1, u_2), v = (v_1, v_2) \in L^p(\Omega; C([0, t], C(\overline{\mathcal{O}}, \mathbb{R}^2)))$,*

$$|\varphi(u) - \varphi(v)|_{L^p} \leq c_p(t) |u - v|_{L^p}, \quad (3.2)$$

where $c_p(t)$ is some constant satisfying $c_p(t) \downarrow 0$ as $t \downarrow 0$.

Proof. Let p_0 be sufficiently large to ensure that, for any $p \geq p_0$, we can choose simultaneously $\beta, \varepsilon > 0$ such that

$$\frac{1}{p} < \beta < \frac{1}{2} \quad \text{and} \quad \frac{1}{p} < \varepsilon < 2\left(\beta - \frac{1}{p}\right).$$

Now, for any fixed $p \geq p_0$, let β, ε be chosen as above. By using a factorization argument (see, e.g., [12, Theorem 8.3]), we have

$$\varphi(u)(t) - \varphi(v)(t) = \frac{\sin \pi \beta}{\pi} \int_0^t (t-s)^{\beta-1} e^{(t-s)A} Y_\beta(u, v)(s) ds,$$

where

$$Y_\beta(u, v)(s) = \int_0^s (s-r)^{-\beta} e^{(s-r)A} (u(r) \vee 0 - v(r) \vee 0) dW(r).$$

If

$$\int_0^t |Y_\beta(u, v)(s)|_{L^p(\mathcal{O}, \mathbb{R}^2)}^p ds < \infty \text{ a.s.},$$

then it is easily seen from the properties in (2.3) of the semigroup e^{tA} and Hölder's inequality that

$$\begin{aligned} & |\varphi(u)(t) - \varphi(v)(t)|_{\varepsilon, p} \\ & \leq c_\beta \int_0^t (t-s)^{\beta-1} ((t-s) \wedge 1)^{-\varepsilon/2} |Y_\beta(u, v)|_{L^p(\mathcal{O}, \mathbb{R}^2)} ds \\ & \leq c_{\beta, p}(t) \left(\int_0^t ((t-s) \wedge 1)^{\frac{p}{p-1}(\beta-\varepsilon/2-1)} ds \right)^{\frac{p-1}{p}} \left(\int_0^t |Y_\beta(u, v)(s)|_{L^p(\mathcal{O}, \mathbb{R}^2)}^p ds \right)^{\frac{1}{p}} \quad (3.3) \\ & \leq c_{\beta, p}(t) \left(\int_0^t |Y_\beta(u, v)(s)|_{L^p(\mathcal{O}, \mathbb{R}^2)}^p ds \right)^{\frac{1}{p}} \text{ a.s.}, \end{aligned}$$

where $c_{\beta, p}(t)$ is some positive constant satisfying $c_{\beta, p}(t) \downarrow 0$ as $t \downarrow 0$. Rewriting $Y_\beta(u, v)(s) = (Y_{1\beta}(u, v)(s), Y_{2\beta}(u, v)(s))$, where

$$Y_{i\beta}(u, v)(s) := \int_0^s (s-r)^{-\beta} e^{(s-r)A_i} (u_i(r) \vee 0 - v_i(r) \vee 0) dW_i(r), \quad i = 1, 2.$$

Therefore, applying the Burkholder inequality, we obtain that for all $s \in [0, t]$ and almost every $x \in \mathcal{O}$,

$$\mathbb{E} |Y_{i\beta}(u, v)(s, x)|^p \leq c_p \mathbb{E} \left[\int_0^s (s-r)^{-2\beta} \sum_{k=1}^\infty a_{k,i} |M_i(s, r, k, x)|^2 dr \right]^{\frac{p}{2}},$$

where

$$M_i(s, r, k) = e^{(s-r)A_i}(u_i(r) \vee 0 - v_i(r) \vee 0)e_k.$$

In the above, we used the notations

$$Y_{i\beta}(u, v)(s, x) := Y_{i\beta}(u, v)(s)(x), \quad M_i(s, r, k, x) := M_i(s, r, k)(x), \quad i = 1, 2.$$

As a consequence,

$$\begin{aligned} & \mathbb{E} \int_0^t |Y_\beta(u, v)(s)|_{L^p(\mathcal{O}, \mathbb{R}^2)}^p ds \\ & \leq c_p(t) \mathbb{E} \int_0^t \int_{\mathcal{O}} (|Y_{1\beta}(u, v)(s, x)|^p + |Y_{2\beta}(u, v)(s, x)|^p) dx ds \\ & \leq c_p(t) \int_0^t \mathbb{E} \left(\int_0^s (s-r)^{-2\beta} (a_1 + a_2) \sup_{k \in \mathbb{N}} |M(s, r, k)|_{L^\infty(\mathcal{O}, \mathbb{R}^2)}^2 dr \right)^{\frac{p}{2}} ds, \end{aligned} \tag{3.4}$$

where $M(s, r, k) := (M_1(s, r, k), M_2(s, r, k))$ and a_1, a_2 are defined in (2.1). Moreover, from the uniformly boundedness property of $\{e_k\}_{k=1}^\infty$ and (2.2), we have

$$\sup_{k \in \mathbb{N}} |M(s, r, k)|_{L^\infty(\mathcal{O}, \mathbb{R}^2)} \leq c |u(r) - v(r)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)} \tag{3.5}$$

for some constant c independent of s, r, u , and v . Combining (3.4) and (3.5) implies that

$$\begin{aligned} & \mathbb{E} \int_0^t |Y_\beta(u, v)(s)|_{L^p(\mathcal{O}, \mathbb{R}^2)}^p ds \\ & \leq c_p(t) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |u(r) - v(r)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p \left(\int_0^s (s-r)^{-2\beta} dr \right)^{\frac{p}{2}} ds \\ & \leq c_{\beta,p}(t) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |u(r) - v(r)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p ds \leq c_{\beta,p}(t) |u - v|_{L^{t,p}}^p < \infty, \end{aligned} \tag{3.6}$$

where $c_{\beta,p}(t)$ is some positive constant satisfying $c_{\beta,p}(t) \downarrow 0$ as $t \downarrow 0$. Therefore, the inequality (3.3) holds and, as a consequence, $\varphi(u)(t) - \varphi(v)(t) \in W^{\varepsilon,p}(\mathcal{O}, \mathbb{R}^2)$. Since $\varepsilon > l/p$, the Sobolev embedding theorem implies that $\varphi(u)(t) - \varphi(v)(t) \in C(\overline{\mathcal{O}}, \mathbb{R}^2)$. Finally, (3.3) and (3.6) imply that

$$|\varphi(u) - \varphi(v)|_{L^{t,p}} \leq c_p(t) |u - v|_{L^{t,p}}$$

for some constant $c_p(t)$ satisfying $c_p(t) \downarrow 0$ as $t \downarrow 0$. The lemma is proved. □

Therefore, for $p \geq p_0$, with sufficiently large p_0 , γ maps $L^p(\Omega; C([0, t], C(\overline{\mathcal{O}}, \mathbb{R}^2)))$ into itself. Moreover, by using (2.2) and the Lipschitz continuity of F^* , we have

$$\begin{aligned} & \int_0^t \left| e^{(t-s)A} [F^*(u(s)) - F^*(v(s))] \right|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p ds \leq c \int_0^t |u(s) - v(s)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p ds \\ & \leq c \int_0^t \sup_{r \in [0, s]} |u(r) - v(r)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p ds \leq ct \sup_{s \in [0, t]} |u(s) - v(s)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p. \end{aligned} \tag{3.7}$$

Hence, (3.2) and (3.7) imply that

$$|\gamma(u) - \gamma(v)|_{L_t,p} \leq c_p(t) |u - v|_{L_t,p},$$

where $c_p(t)$ is some constant depending on p and t satisfying $c_p(t) \downarrow 0$ as $t \downarrow 0$. Therefore, for some T_0 sufficient small, γ is a contraction mapping in $L^p(\Omega; C([0, T_0], C(\bar{\mathcal{O}}, \mathbb{R}^2)))$. By a fixed point argument we can conclude that (3.1) admits a unique mild solution in $L^p(\Omega; C([0, T_0], C(\bar{\mathcal{O}}, \mathbb{R}^2)))$. Thus, by repeating the above argument in each finite time interval $[kT_0, (k + 1)T_0]$, for any $T > 0$ and $p \geq p_0$ (3.1) admits a unique mild solution $Z^*(t) = (S^*(t), I^*(t))$ in $L^p(\Omega; C([0, T], C(\bar{\mathcal{O}}, \mathbb{R}^2)))$. We proceed to prove the positivity of $S^*(t), I^*(t)$.

Lemma 3.2. *Let $(S^*(t), I^*(t))$ be the unique mild solution of (3.1). Then, for all $t \in [0, T]$, $S^*(t), I^*(t) \geq 0$ a.s.*

Proof. Equivalently, $(S^*(t), I^*(t))$ is the mild solution of the equation

$$\begin{cases} dS^*(t) = [A_1 S^*(t) + F_1(S^*(t) \vee 0, I^*(t) \vee 0)]dt + (S^*(t) \vee 0)dW_1(t), \\ dI^*(t) = [A_2 I^*(t) + F_2(S^*(t) \vee 0, I^*(t) \vee 0)]dt + (I^*(t) \vee 0)dW_2(t), \\ S^*(0) = S_0, I^*(0) = I_0. \end{cases} \tag{3.8}$$

For $i = 1, 2$, let $\lambda_i \in \rho(A_i)$ be the resolvent set of A_i and $R_i(\lambda_i) := \lambda_i R_i(\lambda_i, A_i)$, with $R_i(\lambda_i, A_i)$ being the resolvent of A_i . For each small $\varepsilon > 0$, $\lambda = (\lambda_1, \lambda_2) \in \rho(A_1) \times \rho(A_2)$, by [27, Proposition 1.3.6], there exists a unique strong solution $S_{\lambda,\varepsilon}(t, x), I_{\lambda,\varepsilon}(t, x)$ of the equation

$$\begin{cases} dS_{\lambda,\varepsilon}(t) = [A_1 S_{\lambda,\varepsilon}(t) + R_1(\lambda_1)F_1(\varepsilon\Phi(\varepsilon^{-1}S_{\lambda,\varepsilon}(t)), \varepsilon\Phi(\varepsilon^{-1}I_{\lambda,\varepsilon}(t)))]dt \\ \quad + R_1(\lambda_1)\varepsilon\Phi(\varepsilon^{-1}S_{\lambda,\varepsilon}(t))dW_1(t), \\ dI_{\lambda,\varepsilon}(t) = [A_2 I_{\lambda,\varepsilon}(t) + R_2(\lambda_2)F_2(\varepsilon\Phi(\varepsilon^{-1}S_{\lambda,\varepsilon}(t)), \varepsilon\Phi(\varepsilon^{-1}I_{\lambda,\varepsilon}(t)))]dt \\ \quad + R_2(\lambda_2)\varepsilon\Phi(\varepsilon^{-1}I_{\lambda,\varepsilon}(t))dW_2(t), \\ S_{\lambda,\varepsilon}(0) = R_1(\lambda_1)S_0, \quad I_{\lambda,\varepsilon}(0) = R_2(\lambda_2)I_0, \end{cases}$$

where

$$\Phi(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ 3\xi^5 - 8\xi^4 + 6\xi^3 & \text{if } 0 < \xi < 1, \\ \xi & \text{if } \xi \geq 1, \end{cases}$$

satisfying

$$\begin{cases} \Phi \in C^2(\mathbb{R}), \\ \varepsilon\Phi(\varepsilon^{-1}\xi) \rightarrow \xi \vee 0 \quad \text{as } \varepsilon \rightarrow 0. \end{cases}$$

Combined with the convergence property in [27, Proposition 1.3.6], we obtain that $(S_{\lambda(k),\varepsilon}(t), I_{\lambda(k),\varepsilon}(t)) \rightarrow (S^*(t), I^*(t))$ in $L^p(\Omega; C([0, T], L^2(\mathcal{O}, \mathbb{R}^2)))$ for some sequence $\{\lambda(k)\}_{k=1}^\infty \subset \rho(A_1) \times \rho(A_2)$ and $\varepsilon \rightarrow 0$.

Now, let

$$g(\xi) = \begin{cases} \xi^2 - \frac{1}{6} & \text{if } \xi \leq -1, \\ -\frac{\xi^4}{2} - \frac{4\xi^3}{3} & \text{if } -1 < \xi < 0, \\ 0 & \text{if } \xi \geq 0. \end{cases}$$

Then $g'(\xi) \leq 0$ for all ξ and $g''(\xi) \geq 0$ for all ξ . Hence, we are to compute

$$d_t \left(\int_{\mathcal{O}} g(I_{\lambda,\varepsilon}(t, x)) dx \right).$$

Since $g'(\xi)\Phi(\xi) = g''(\xi)\Phi(\xi) = 0$ for all ξ , by Itô's Lemma [10, Theorem 3.8], we get

$$\begin{aligned} \int_{\mathcal{O}} g(I_{\lambda,\varepsilon}(t, x)) dx &= k_2 \int_0^t \int_{\mathcal{O}} g'(I_{\lambda,\varepsilon}(s, x)) \Delta I_{\lambda,\varepsilon}(s, x) dx ds \\ &= -k_2 \int_0^t \int_{\mathcal{O}} g''(I_{\lambda,\varepsilon}(s, x)) |\nabla I_{\lambda,\varepsilon}(s, x)|^2 dx ds \\ &\leq 0. \end{aligned}$$

Since $g(\xi) > 0$ for all $\xi < 0$, we conclude that, for all $\lambda \in \rho(A_1) \times \rho(A_2)$, $\varepsilon > 0$, $I_{\lambda,\varepsilon}(t, x) \geq 0$ for all $t \in [0, T]$, almost everywhere in \mathcal{O} . Similarly, we have

$$\begin{aligned} \int_{\mathcal{O}} g(S_{\lambda,\varepsilon}(t, x)) dx &= \int_0^t \int_{\mathcal{O}} g'(S_{\lambda,\varepsilon}(s, x))(k_1 \Delta S_{\lambda,\varepsilon}(s, x) + (R_1(\lambda_1)\Lambda)(x)) dx ds \\ &= -k_1 \int_0^t \int_{\mathcal{O}} g''(S_{\lambda,\varepsilon}(s, x)) |\nabla S_{\lambda,\varepsilon}(s, x)|^2 dx ds \\ &\quad + \int_0^t \int_{\mathcal{O}} g'(S_{\lambda,\varepsilon}(s, x))(R_1(\lambda_1)\Lambda)(x) dx ds \\ &\leq 0, \end{aligned}$$

where the last inequality above follows from the fact that

$$R_1(\lambda_1, A_1) = \int_0^\infty e^{-\lambda_1 t} e^{tA_1} dt$$

preserves positivity. Again, since $g(\xi) > 0$ for all $\xi < 0$, we obtain the positivity of $S_{\lambda,\varepsilon}(t, x)$. Hence, $S^*(t, x), I^*(t, x) \geq 0$ almost everywhere in \mathcal{O} for all $t \in [0, T]$, a.s. \square

Returning to the proof of the theorem, since $(S^*(t), I^*(t))$ is a unique mild solution of (3.8) and is positive, it is a mild solution of (2.4). Therefore, (2.4) admits a unique positive mild solution $(S(t), I(t))$.

Now we prove the second part. For convenience, we use subscripts to indicate the dependence of the solution on the initial value. Let $Z_{z_0}(t), Z_{z'_0}(t)$ be the positive mild solutions of (2.6) with the initial conditions $Z(0) = z_0$ and $Z(0) = z'_0$, respectively. That means,

$$Z_{z_0}(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} F^*(Z_{z_0}(s)) ds + \int_0^t e^{(t-s)A} Z_{z_0}(s) dW(s)$$

and

$$Z_{z'_0}(t) = e^{tA} z'_0 + \int_0^t e^{(t-s)A} F^*(Z_{z'_0}(s)) ds + \int_0^t e^{(t-s)A} Z_{z'_0}(s) dW(s).$$

This implies that

$$\begin{aligned} Z_{z_0}(t) - Z_{z'_0}(t) &= e^{tA} z_0 - e^{tA} z'_0 + \int_0^t e^{(t-s)A} (F^*(Z_{z_0}(s)) - F^*(Z_{z'_0}(s))) ds \\ &\quad + \int_0^t e^{(t-s)A} (Z_{z_0}(s) - Z_{z'_0}(s)) dW(s). \end{aligned}$$

Since (3.3) and (3.6) hold, we can obtain that

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} \left| \int_0^s e^{(s-r)A} (Z_{z_0}(r) - Z_{z'_0}(r)) \, dW(r) \right|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p \\ & \leq c_p(t) \int_0^t \mathbb{E} \sup_{r \in [0, s]} |Z_{z_0}(r) - Z_{z'_0}(r)|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p \, ds \\ & \leq c_p(t) \int_0^t |Z_{z_0} - Z_{z'_0}|_{L^{s,p}}^p \, ds. \end{aligned} \quad (3.9)$$

Therefore, by virtue of (3.7) and (3.9), it is possible to get

$$|Z_{z_0} - Z_{z'_0}|_{L^{t,p}}^p \leq c_p |z_0 - z'_0|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p + c_p(t) \int_0^t |Z_{z_0} - Z_{z'_0}|_{L^{s,p}}^p \, ds.$$

Hence, it is easy to obtain, from Gronwall's inequality, that

$$|Z_{z_0} - Z_{z'_0}|_{L^{T,p}}^p \leq c_p(T) |z_0 - z'_0|_{C(\overline{\mathcal{O}}, \mathbb{R}^2)}^p.$$

Therefore, the continuous dependence of the solution on the initial values is proved.

4. Long-time behavior

This section investigates the properties of the positive mild solution $(S(t), I(t))$ of system (2.4) when $t \rightarrow \infty$. In particular, we provide the sufficient conditions for extinction and permanence. For each function $u \in E$, denote

$$u_* = \inf_{x \in \overline{\mathcal{O}}} u(x).$$

Define the number

$$\widehat{R} = \int_{\mathcal{O}} \alpha(x) \, dx - \int_{\mathcal{O}} \mu_2(x) \, dx - \frac{a_2}{2}.$$

Theorem 4.1. *If $\Lambda_* > 0$ and $\widehat{R} > 0$, then the infected class is permanent in the sense that, for any initial values $0 \leq S_0, I_0 \in E$ satisfying*

$$\int_{\mathcal{O}} -\ln I_0(x) \, dx < \infty,$$

we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\mathbb{E} \int_{\mathcal{O}} (I^2(s, x) \wedge 1) \, dx \right)^{\frac{1}{2}} \, ds \geq R_I$$

for some $R_I > 0$ independent of initial values.

Proof. To obtain the long-time properties of $(S(t), I(t))$, one of tools we use is Itô's formula. Unfortunately, in general Itô's formula is not valid for mild solutions. Hence, our idea is to approximate the solution by a sequence of strong solutions when the noise is finite dimensional. First, we assume that $S_0, I_0 \in D(A_i^E)$, where $D(A_i^E)$ is the domain of A_i^E , the part of A_i in E . For

each fixed $n \in \mathbb{N}$, let $\bar{S}_n(t, x), \bar{I}_n(t, x)$ be the strong solution (see [12] for more details about strong, weak, and mild solutions) of the following equations:

$$\begin{cases} d\bar{S}_n(t, x) = \left[A_1 \bar{S}_n(t, x) + \Lambda(x) - \mu_1(x) \bar{S}_n(t, x) - \frac{\alpha(x) \bar{S}_n(t, x) \bar{I}_n(t, x)}{\bar{S}_n(t, x) + \bar{I}_n(t, x)} \right] dt \\ \quad + \sum_{k=1}^n \sqrt{a_{k,1}} e_k(x) \bar{S}_n(t, x) dB_{k,1}(t), \\ d\bar{I}_n(t, x) = \left[A_2 \bar{I}_n(t, x) - \mu_2(x) \bar{I}_n(t, x) + \frac{\alpha(x) \bar{S}_n(t, x) \bar{I}_n(t, x)}{\bar{S}_n(t, x) + \bar{I}_n(t, x)} \right] dt \\ \quad + \sum_{k=1}^n \sqrt{a_{k,2}} e_k(x) \bar{I}_n(t, x) dB_{k,2}(t), \\ \bar{S}_n(x, 0) = S_0(x), \quad \bar{I}_n(x, 0) = I_0(x). \end{cases} \tag{4.1}$$

The existence and uniqueness of a strong solution of (4.1) follow the results in [11] or [12, Section 7.4]. To see that the conditions in these references are satisfied, we note that the semi-groups e^{tA_1} and e^{tA_2} (as well as their restrictions to E) are analytic (see [3, Chapter 2]) and strongly continuous (see [12, Appendix A.5.2]). Moreover, from the characterizations of fractional power of elliptic operators in [38, Chapter 16] or [12, Appendix A], it is easy to confirm that the coefficients in (4.1) satisfy condition (e) in Hypothesis 2 of [11]. Moreover, a detailed argument can be also found in [29, 30].

In addition, from the continuous dependence on parameter ξ of the fixed points of the family of uniform contraction mappings $T(\xi)$, by a similar ‘parameter-dependent contraction mapping’ argument, it is easy to obtain (see [12] or [30, Proposition 4.2]) that, for any fixed t ,

$$\lim_{n \rightarrow \infty} \mathbb{E} |S(t) - \bar{S}_n(t)|_H^2 \rightarrow 0$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E} |I(t) - \bar{I}_n(t)|_H^2 \rightarrow 0.$$

To proceed, we state and prove the following auxiliary lemmas.

Lemma 4.1. *Let $\mu_* := \inf_{x \in \mathcal{O}} \min\{\mu_1(x), \mu_2(x)\}$. If $\mu_* > 0$ then*

$$\mathbb{E} \int_{\mathcal{O}} (S(t, x) + I(t, x)) dx \leq e^{-\mu_* t} \int_{\mathcal{O}} (S_0(x) + I_0(x)) dx + \frac{|\Lambda|_E}{\mu_*}.$$

Proof. In view of Itô’s formula ([10, Theorem 3.8]), we can obtain

$$\begin{aligned} \mathbb{E} e^{\mu_* t} \int_{\mathcal{O}} (\bar{S}_n(t, x) + \bar{I}_n(t, x)) dx &\leq \int_{\mathcal{O}} (S_0(x) + I_0(x)) dx + \mathbb{E} \int_0^t e^{\mu_* s} \int_{\mathcal{O}} \Lambda(x) dx ds \\ &\leq \int_{\mathcal{O}} (S_0(x) + I_0(x)) dx + \frac{|\Lambda|_E}{\mu_*} e^{\mu_* t}. \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain the desired result. □

Now we are in a position to estimate $\mathbb{E}(\int_{\mathcal{O}} \frac{1}{\bar{S}_n^p(t, x)} dx)$ by the following lemma.

Lemma 4.2. *For any $p > 0$, if $\int_{\mathcal{O}} \frac{1}{S_0^p(x)} dx < \infty$, there exists $\tilde{K}_p > 0$, which is independent of n and the initial conditions, such that*

$$\mathbb{E} \int_{\mathcal{O}} \frac{1}{\bar{S}_n^p(t, x)} dx \leq e^{-t} \int_{\mathcal{O}} \frac{1}{S_0^p(x)} dx + \tilde{K}_p.$$

Proof. For any $0 < \varepsilon < \frac{p\Lambda_*}{2}$, using Itô's Lemma ([10, Theorem 3.8]) and by direct calculations, we have

$$\begin{aligned}
 & e^t \int_{\mathcal{O}} \frac{1}{(\bar{S}_n(t, x) + \varepsilon)^p} dx \\
 &= \int_{\mathcal{O}} \frac{1}{(S_0(x) + \varepsilon)^p} dx + \int_0^t e^s \int_{\mathcal{O}} \frac{1}{(\bar{S}_n(s, x) + \varepsilon)^p} dx ds + \int_0^t e^s \int_{\mathcal{O}} \frac{-p}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} \\
 &\quad \times \left(k_1 \Delta \bar{S}_n(s, x) + \Lambda(x) - \mu_1(x) \bar{S}_n(s, x) - \frac{\alpha(x) \bar{S}_n(s, x) \bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} \right) dx ds \\
 &\quad + \frac{1}{2} \int_0^t e^s \sum_{k=1}^n \int_{\mathcal{O}} \frac{p(p+1) a_{k,1} e_k^2(x) \bar{S}_n^2(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{p+2}} dx ds \\
 &\quad + \sum_{k=1}^n \int_0^t e^s \left[\sqrt{a_{k,1}} \int_{\mathcal{O}} \frac{-p e_k(x) \bar{S}_n(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} dx \right] dB_{k,1}(s) \tag{4.2} \\
 &\leq \int_{\mathcal{O}} \frac{1}{(S_0(x) + \varepsilon)^p} dx + \int_0^t e^s \int_{\mathcal{O}} \frac{-p k_1 \Delta \bar{S}_n(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} dx ds \\
 &\quad + \int_0^t e^s \int_{\mathcal{O}} \frac{p}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} \left(-\Lambda(x) + \frac{\varepsilon}{p} + \left(|\mu_1|_E + |\alpha|_E + \frac{1}{p} + \frac{p+1}{2} a_1 C_0^2 \right) \right. \\
 &\quad \left. \times \bar{S}_n(s, x) \right) dx ds + \sum_{k=1}^n \int_0^t e^s \left[\sqrt{a_{k,1}} \int_{\mathcal{O}} \frac{-p e_k(x) \bar{S}_n(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} dx \right] dB_{k,1}(s) \\
 &\leq \int_{\mathcal{O}} \frac{1}{(S_0(x) + \varepsilon)^p} dx + \int_0^t \frac{p K_p^{p+1} 2^p}{\Lambda_*^p} e^s ds \\
 &\quad + \sum_{k=1}^n \int_0^t e^s \left[\sqrt{a_{k,1}} \int_{\mathcal{O}} \frac{-p e_k(x) \bar{S}_n(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} dx \right] dB_{k,1}(s),
 \end{aligned}$$

where $K_p = |\mu_1|_E + |\alpha|_E + \frac{1}{p} + \frac{p+1}{2} a_1 C_0^2$. In the above, we used the facts that

$$\int_{\mathcal{O}} \frac{-p k_1 \Delta \bar{S}_n(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} dx = -p(p+1) k_1 \int_{\mathcal{O}} \frac{|\nabla \bar{S}_n(s, x)|^2}{(\bar{S}_n(s, x) + \varepsilon)^{p+2}} dx \leq 0 \text{ a.s.}$$

and

$$\begin{aligned}
 & \int_{\mathcal{O}} \frac{p}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} \left(-\Lambda(x) + \frac{\varepsilon}{p} + \left(|\mu_1|_E + |\alpha|_E + \frac{1}{p} + \frac{p+1}{2} a_1 C_0^2 \right) \bar{S}_n(s, x) \right) dx \\
 &\leq \int_{\mathcal{O}} \frac{p}{(\bar{S}_n(s, x) + \varepsilon)^{p+1}} \left(-\frac{\Lambda_*}{2} + K_p \bar{S}_n(s, x) \right) \mathbf{1}_{\{\bar{S}_n(s, x) \geq \frac{\Lambda_*}{2K_p}\}} dx \\
 &\leq \frac{p K_p^{p+1} 2^p}{\Lambda_*^p} \text{ a.s.}
 \end{aligned}$$

Hence, (4.2) implies that, for all $t \geq 0$ and $n \in \mathbb{N}$,

$$\mathbb{E} \int_{\mathcal{O}} \frac{1}{(\bar{S}_n(t, x) + \varepsilon)^p} dx \leq e^{-t} \int_{\mathcal{O}} \frac{1}{(S_0(x) + \varepsilon)^p} dx + e^{-t} \int_0^t \frac{pK_p^{p+1} 2^p}{\Lambda_*^p} e^s ds. \tag{4.3}$$

Letting $\varepsilon \rightarrow 0$, we have, from the monotone convergence theorem,

$$\mathbb{E} \int_{\mathcal{O}} \frac{1}{\bar{S}_n^p(t, x)} dx \leq e^{-t} \int_{\mathcal{O}} \frac{1}{S_0^p(x)} dx + e^{-t} \int_0^t \frac{pK_p^{p+1} 2^p}{\Lambda_*^p} e^s ds.$$

The proof of the lemma is completed. □

Noting that our initial conditions are not assumed to satisfy $\int_{\mathcal{O}} \frac{1}{S_0^2(x)} dx < \infty$, we will prove in the following lemma that, after some finite time, the solutions have inverse functions that belong to $L^2(\mathcal{O}, \mathbb{R})$.

Lemma 4.3. *For any $n \in \mathbb{N}$,*

$$\mathbb{E} \int_{\mathcal{O}} \frac{1}{\bar{S}_n^2(4, x)} dx \leq \ell_0,$$

where ℓ_0 depends only on the initial conditions (independent of n).

Proof. Using

$$\begin{aligned} \mathbb{E} \int_{\mathcal{O}} \bar{S}_n(t, x) dx &= \int_{\mathcal{O}} S_0(x) dx + \int_0^t \mathbb{E} \int_{\mathcal{O}} \left(k_1 \Delta \bar{S}_n(s, x) + \Lambda(x) \right. \\ &\quad \left. - \mu_1(x) \bar{S}_n(s, x) - \frac{\alpha(x) \bar{S}_n(s, x) \bar{I}_n(s, x)}{\bar{S}_n(s, x) \bar{I}_n(s, x)} \right) dx ds \\ &\leq \int_{\mathcal{O}} S_0(x) dx + t |\Lambda|_E \end{aligned}$$

and $s^q \leq s + 1$ for all $s \in \mathbb{R} > 0, q \in [0, 1]$, it is easy to show that there exists $\ell_1 > 0$ such that

$$\mathbb{E} \int_{\mathcal{O}} \bar{S}_n^q(t, x) dx \leq \ell_1 \quad \text{for any } t \in [0, 1], q \in [0, 1], \tag{4.4}$$

where ℓ_1 is independent of n . For any $\varepsilon > 0$, using Itô's Lemma ([10, Theorem 3.8]) again, we have

$$\begin{aligned} &\mathbb{E} \int_{\mathcal{O}} (\bar{S}_n(1, x) + \varepsilon)^{\frac{1}{2}} dx \\ &= \int_{\mathcal{O}} (S_0(x) + \varepsilon)^{\frac{1}{2}} dx + \int_0^1 \mathbb{E} \int_{\mathcal{O}} \frac{1}{2(\bar{S}_n(s, x) + \varepsilon)^{\frac{1}{2}}} \left(k_1 \Delta \bar{S}_n(s, x) \right. \\ &\quad \left. + \Lambda(x) - \mu_1(x) \bar{S}_n(s, x) - \frac{\alpha(x) \bar{S}_n(s, x) \bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} \right) dx ds \\ &\quad - \frac{1}{8} \int_0^1 \mathbb{E} \sum_{k=1}^n \int_{\mathcal{O}} \frac{a_{k,1} e_k^2(x) \bar{S}_n^2(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{\frac{3}{2}}} dx ds \\ &\geq \frac{1}{2} \int_0^1 \mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(s, x) + \varepsilon)^{\frac{1}{2}}} dx ds - N_1 \int_0^1 \left(\mathbb{E} \int_{\mathcal{O}} \bar{S}_n^{\frac{1}{2}}(s, x) dx \right) ds, \end{aligned} \tag{4.5}$$

where

$$N_1 = \frac{|\mu_1|_E + |\alpha|_E + \frac{a_1 C_0^2}{4}}{2}.$$

In view of (4.4) and (4.5), we have

$$\frac{1}{2} \int_0^1 \mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(s, x) + \varepsilon)^{\frac{1}{2}}} dx ds \leq (1 + N_1)\ell_1 + \sqrt{\varepsilon} \quad \text{for all } \varepsilon > 0,$$

which implies that

$$\int_0^1 \mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(s, x))^{\frac{1}{2}}} dx ds \leq 2(1 + N_1)\ell_1,$$

or there exists $t_1 = t_1(n) \in [0, 1]$ such that

$$\mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(t_1, x))^{\frac{1}{2}}} dx \leq \frac{2(1 + N_1)\ell_1}{\Lambda_*}.$$

Applying Lemma 4.2 and the Markov property of (S_n, I_n) , we have

$$\mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(t, x) + \varepsilon)^{\frac{1}{2}}} dx \leq \ell_2 \quad \text{for all } t \in [1, 2],$$

for some ℓ_2 independent of n . We again have

$$\begin{aligned} & \mathbb{E} \int_{\mathcal{O}} (\bar{S}_n(2, x) + \varepsilon)^{-\frac{1}{2}} dx \\ &= \mathbb{E} \int_{\mathcal{O}} (\bar{S}_n(1, x) + \varepsilon)^{-\frac{1}{2}} dx - \int_1^2 \mathbb{E} \int_{\mathcal{O}} \frac{1}{2(\bar{S}_n(s, x) + \varepsilon)^{\frac{3}{2}}} \left(k_1 \Delta \bar{S}_n(s, x) \right. \\ & \quad \left. + \Lambda(x) - \mu_1(x) \bar{S}_n(s, x) - \frac{\alpha(x) \bar{S}_n(s, x) \bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} \right) dx ds \\ & \quad + \frac{3}{8} \int_1^2 \mathbb{E} \sum_{k=1}^n \int_{\mathcal{O}} \frac{a_{k,1} e_k^2(x) \bar{S}_n^2(s, x)}{(\bar{S}_n(s, x) + \varepsilon)^{\frac{3}{2}}} dx ds \\ & \leq -\frac{1}{2} \int_1^2 \mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(s, x) + \varepsilon)^{\frac{3}{2}}} dx ds + \mathbb{E} \int_{\mathcal{O}} (\bar{S}_n(1, x) + \varepsilon)^{-\frac{1}{2}} dx \\ & \quad + N_2 \int_1^2 \left(\mathbb{E} \int_{\mathcal{O}} \bar{S}_n^{-\frac{1}{2}}(s, x) dx \right) ds, \end{aligned} \tag{4.6}$$

where

$$N_2 = \frac{|\mu_1|_E + |\alpha|_E + \frac{3a_1 C_0^2}{4}}{2}.$$

Thus,

$$\int_1^2 \mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(s, x) + \varepsilon)^{\frac{3}{2}}} dx ds \leq \ell_3$$

for some ℓ_3 depending only on the initial conditions. Letting $\varepsilon \rightarrow 0$, we obtain that, for some $t_2 = t_2(n) \in [1, 2]$,

$$\mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{\bar{S}_n^{\frac{3}{2}}(t_2, x)} dx \leq \ell_3,$$

which together with Lemma 4.2 implies that

$$\mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(t, x) + \varepsilon)^{\frac{3}{2}}} dx \leq \ell_4, \quad \text{for all } t \in [2, 3],$$

where ℓ_4 is some constant independent of n . Using the same process we can obtain that there exist $t_3 = t_3(n) \in [0, 4]$ and ℓ_5 such that

$$\mathbb{E} \int_{\mathcal{O}} \frac{\Lambda(x)}{(\bar{S}_n(t_3, x))^{\frac{5}{2}}} dx \leq \ell_5.$$

Therefore, it is possible to obtain the existence of two constants, $t_4 = t_4(n) \in [0, 4]$ and ℓ_6 , satisfying

$$\mathbb{E} \int_{\mathcal{O}} \frac{1}{(\bar{S}_n(t_4, x))^2} dx < \ell_6.$$

The lemma is proved by applying Lemma 4.2. □

In view of Lemmas 4.2 and 4.3, we have

$$\mathbb{E} \int_{\mathcal{O}} \frac{1}{\bar{S}_n^2(t, x)} dx \leq e^{-t} \ell_0 + \tilde{K}_2 \quad \text{for all } n \in \mathbb{N}, t \geq 4. \tag{4.7}$$

Note that both ℓ_0 and \tilde{K}_2 are independent of n , and ℓ_0 may depend on the initial point but \tilde{K}_2 is independent. By Itô's Lemma ([10, Theorem 3.8]) again, and with calculations similar to obtaining (4.3), we have

$$\begin{aligned} \mathbb{E} \int_{\mathcal{O}} \bar{I}_n(t, x) dx &\geq \mathbb{E} \int_{\mathcal{O}} \ln(\bar{I}_n(t, x) + \varepsilon) dx \\ &= \int_{\mathcal{O}} \ln(I_0(x) + \varepsilon) dx + \int_0^t \mathbb{E} \int_{\mathcal{O}} \frac{1}{\bar{I}_n(s, x) + \varepsilon} \left(k_2 \Delta \bar{I}_n(s, x) \right. \\ &\quad \left. - \mu_2(x) \bar{I}_n(s, x) + \frac{\alpha(x) \bar{S}_n(s, x) \bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} dx ds \right) \\ &\quad - \frac{1}{2} \int_0^t \mathbb{E} \sum_{k=1}^n \int_{\mathcal{O}} \frac{a_{k,2} \bar{I}_n^2(s, x) e_k^2(x)}{(\bar{I}_n(s, x) + \varepsilon)^2} dx ds \\ &\geq \int_{\mathcal{O}} \ln(I_0(x) + \varepsilon) dx - \left(\frac{a_2}{2} + |\mu_2|_E \right) t \text{ for all } n \in \mathbb{N}, t > 0, 0 < \varepsilon < 1. \end{aligned}$$

As a consequence,

$$\mathbb{E} \int_{\mathcal{O}} \bar{I}_n(t, x) dx \geq \mathbb{E} \int_{\mathcal{O}} \ln \bar{I}_n(t, x) dx \geq \int_{\mathcal{O}} \ln I_0(x) dx - \left(\frac{a_2}{2} + |\mu_2|_E \right) t > -\infty \text{ for all } t > 0.$$

That means,

$$\mathbb{P}\{\bar{I}_n(t, x) > 0 \text{ almost everywhere in } \mathcal{O}\} = 1 \quad \text{for all } n \in \mathbb{N}, t > 0. \quad (4.8)$$

On the other hand, combining Itô's Lemma and basic calculations implies that

$$\begin{aligned} 0 &\geq \mathbb{E} \int_{\mathcal{O}} \ln \frac{\bar{I}_n(t, x) + \varepsilon}{1 + \bar{I}_n(t, x)} dx \geq \int_{\mathcal{O}} \ln \frac{I_0(x) + \varepsilon}{1 + I_0(x)} dx + \widehat{R}t \\ &\quad - \int_0^t \mathbb{E} \int_{\mathcal{O}} \left(\frac{\alpha(x)\bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} + \frac{\alpha(x)\bar{S}_n(s, x)\bar{I}_n(s, x)}{(\bar{S}_n(s, x) + \bar{I}_n(s, x))(\bar{I}_n(s, x) + 1)} \right) dx ds \\ &\quad - \int_0^t \mathbb{E} \int_{\mathcal{O}} \frac{\alpha(x)\varepsilon}{\bar{I}_n(s, x) + \varepsilon} dx ds \quad \text{for all } t > 0, n \in \mathbb{N}, 0 < \varepsilon < 1. \end{aligned}$$

Thus, for all $t > 0, n \in \mathbb{N}, 0 < \varepsilon < 1$,

$$\begin{aligned} &\int_0^t \mathbb{E} \int_{\mathcal{O}} \left(\frac{\alpha(x)\bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} + \frac{\alpha(x)\bar{S}_n(s, x)\bar{I}_n(s, x)}{(\bar{S}_n(s, x) + \bar{I}_n(s, x))(\bar{I}_n(s, x) + 1)} \right) dx ds \\ &\quad \geq \mathbb{E} \int_{\mathcal{O}} \ln \frac{I_0(x) + \varepsilon}{1 + I_0(x)} dx + \widehat{R}t - |\alpha|_E \int_0^t \mathbb{E} \int_{\mathcal{O}} \frac{\varepsilon}{\bar{I}_n(s, x) + \varepsilon} dx ds. \end{aligned} \quad (4.9)$$

Let $\varepsilon \rightarrow 0$; using (4.8) and (4.9), we have

$$\begin{aligned} &\int_0^t \mathbb{E} \int_{\mathcal{O}} \left(\frac{\alpha(x)\bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} + \frac{\alpha(x)\bar{S}_n(s, x)\bar{I}_n(s, x)}{(\bar{S}_n(s, x) + \bar{I}_n(s, x))(\bar{I}_n(s, x) + 1)} \right) dx ds \\ &\quad \geq \int_{\mathcal{O}} \ln \frac{I_0(x)}{1 + I_0(x)} dx + \widehat{R}t \quad \text{for all } t > 0, n \in \mathbb{N}. \end{aligned} \quad (4.10)$$

We have the following estimates:

$$\begin{aligned} |\alpha|_E \left(\mathbb{E} \int_{\mathcal{O}} \frac{\bar{I}_n^2(s, x)}{(1 + \bar{I}_n(s, x))^2} dx \right)^{\frac{1}{2}} &\geq \mathbb{E} \int_{\mathcal{O}} \frac{\alpha(x)\bar{I}_n(s, x)}{1 + \bar{I}_n(s, x)} dx \\ &\geq \mathbb{E} \int_{\mathcal{O}} \frac{\alpha(x)\bar{S}_n(s, x)\bar{I}_n(s, x)}{(\bar{S}_n(s, x) + \bar{I}_n(s, x))(\bar{I}_n(s, x) + 1)} dx \end{aligned}$$

and

$$\begin{aligned} |\alpha|_E \left(\mathbb{E} \int_{\mathcal{O}} \frac{\bar{I}_n^2(s, x)}{(1 + \bar{I}_n(s, x))^2} dx \right)^{\frac{1}{2}} \left(\mathbb{E} \int_{\mathcal{O}} \left(\frac{1}{\bar{S}_n(s, x)} + 1 \right)^2 dx \right)^{\frac{1}{2}} \\ \geq \mathbb{E} \int_{\mathcal{O}} \frac{\alpha(x)\bar{I}_n(s, x)}{1 + \bar{I}_n(s, x)} \left(\frac{1}{\bar{S}_n(s, x)} + 1 \right) dx \geq \mathbb{E} \int_{\mathcal{O}} \frac{\alpha(x)\bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} dx, \end{aligned}$$

since

$$\frac{1 + I}{S + I} = \frac{1}{S + I} + \frac{I}{S + I} \leq \frac{1}{S} + 1.$$

Therefore, after some basic estimates, we can get from (4.10) that

$$\begin{aligned} & \int_4^t |\alpha|_E \left(\mathbb{E} \int_{\mathcal{O}} \frac{\bar{I}_n^2(s, x)}{(1 + \bar{I}_n(s, x))^2} dx \right)^{\frac{1}{2}} \left(1 + \left(\mathbb{E} \int_{\mathcal{O}} \left(\frac{1}{\bar{S}_n(s, x)} + 1 \right)^2 dx \right)^{\frac{1}{2}} \right) ds \\ & \geq \int_4^t \mathbb{E} \int_{\mathcal{O}} \left(\frac{\alpha(x) \bar{I}_n(s, x)}{\bar{S}_n(s, x) + \bar{I}_n(s, x)} + \frac{\alpha(x) \bar{S}_n(s, x) \bar{I}_n(s, x)}{(\bar{S}_n(s, x) + \bar{I}_n(s, x))(\bar{I}_n(s, x) + 1)} \right) dx ds \\ & \geq \int_{\mathcal{O}} \ln \frac{I_0(x)}{1 + I_0(x)} dx + \widehat{R}t - 8 |\alpha|_E, \end{aligned}$$

which together with (4.7) leads to

$$\begin{aligned} & \int_4^t |\alpha|_E \left(\mathbb{E} \int_{\mathcal{O}} \frac{\bar{I}_n^2(s, x)}{(1 + \bar{I}_n(s, x))^2} dx \right)^{\frac{1}{2}} \left(2\sqrt{e^{-s} \ell_0} + 2\widetilde{K}_2^{\frac{1}{2}} + 3 \right) ds \\ & \geq \int_{\mathcal{O}} \ln \frac{I_0(x)}{1 + I_0(x)} dx - 8 |\alpha|_E + \widehat{R}t. \end{aligned}$$

Letting $n \rightarrow \infty$ yields

$$\begin{aligned} & \int_4^t |\alpha|_E \left(\mathbb{E} \int_{\mathcal{O}} \frac{I^2(s, x)}{(1 + I(s, x))^2} dx \right)^{\frac{1}{2}} \left(2\sqrt{e^{-s} \ell_0} + 2\widetilde{K}_2^{\frac{1}{2}} + 3 \right) ds \\ & \geq \int_{\mathcal{O}} \ln \frac{I_0(x)}{1 + I_0(x)} dx - 8 |\alpha|_E + \widehat{R}t, \end{aligned} \tag{4.11}$$

which is easily followed by

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\mathbb{E} \int_{\mathcal{O}} \frac{I^2(s, x)}{(1 + I(s, x))^2} dx \right)^{\frac{1}{2}} ds \geq \frac{\widehat{R}}{|\alpha|_E (2\widetilde{K}_2^{\frac{1}{2}} + 3)}.$$

As a consequence,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\mathbb{E} \int_{\mathcal{O}} (I^2(s, x) \wedge 1) dx \right)^{\frac{1}{2}} ds \geq R_I > 0,$$

where R_I is independent of the initial points. The proof of the theorem is completed by using the dense property of $D(A_i^E)$ in E and the continuous dependence on initial data of the solution. In more detail, since the constants \widetilde{K}_2 and \widehat{R} are independent of the initial points, the estimates (4.7) and (4.11) still hold for the solution starting from arbitrary initial points $S_0, I_0 \in E$ with $\int_{\mathcal{O}} -\ln I_0(x) dx < \infty$. □

Theorem 4.2. *For any non-negative initial data $S_0, I_0 \in E$, if*

$$(\mu_2 - \alpha)_* = \inf_{x \in \mathcal{O}} (\mu_2(x) - \alpha(x)) > 0,$$

then the infected class will be extinct with exponential rate.

Proof. First, we define the linear operator $J : H \mapsto \mathbb{R}$ as, for all $u \in H$,

$$Ju := \int_{\mathcal{O}} u(x) \, dx.$$

By the properties of e^{tA_i} , for all $u \in H$, $J(e^{tA_i}u - u) = 0$ or $Ju = Je^{tA_i}u$, for all $i = 1, 2$.

Now, as in the definition of the mild solution, we have

$$I(t) = e^{tA_2}I_0 + \int_0^t e^{(t-s)A_2} \left(-\mu_2 I(s) + \frac{\alpha S(s)I(s)}{S(s) + I(s)} \right) ds + \int_0^t e^{(t-s)A_2} I(s) \, dW_2(s).$$

Hence, applying the operator J to both sides, using the properties of operator J and stochastic convolution (see [12, Proposition 4.15]), we obtain

$$\begin{aligned} \int_{\mathcal{O}} I(t, x) \, dx &= \int_{\mathcal{O}} I_0(x) \, dx + \int_0^t \int_{\mathcal{O}} \left(-\mu_2(x)I(s, x) + \frac{\alpha(x)S(s, x)I(s, x)}{S(s, x) + I(s, x)} \right) dx \, ds \\ &\quad + \int_0^t J(e^{(t-s)A_2}I(s)) \, dW_2(s), \end{aligned}$$

where $J(e^{(t-s)A_2}I(s))$ in the stochastic integral is understood as the process taking values in spaces of the linear operator from H to \mathbb{R} that is defined by

$$J(e^{(t-s)A_2}I(s))u := \int_{\mathcal{O}} (e^{(t-s)A_2}I(s)u)(x) \, dx \quad \text{for all } u \in H.$$

From (2.1), it is easy to see that these integrals are well defined. By taking the expectation on both sides and using the properties of stochastic integral [10, Proposition 2.9],

$$\mathbb{E} \int_{\mathcal{O}} I(t, x) \, dx = \int_{\mathcal{O}} I_0(x) \, dx + \mathbb{E} \int_0^t \int_{\mathcal{O}} \left(-\mu_2(x)I(s, x) + \frac{\alpha(x)S(s, x)I(s, x)}{S(s, x) + I(s, x)} \right) dx \, ds$$

As a consequence,

$$\begin{aligned} \mathbb{E} \int_{\mathcal{O}} I(t, x) \, dx - \mathbb{E} \int_{\mathcal{O}} I(s, x) \, dx &= \int_s^t \mathbb{E} \int_{\mathcal{O}} \left(-\mu_2(x)I(r, x) + \frac{\alpha(x)S(r, x)I(s, x)}{S(r, x) + I(r, x)} \right) dx \, dr \\ &\leq -(\mu_2 - \alpha)_* \int_s^t \mathbb{E} \int_{\mathcal{O}} I(r, x) \, dx \, dr. \end{aligned}$$

Hence, we can obtain the following estimate for the upper Dini derivative:

$$\frac{d}{dt^+} \mathbb{E} \int_{\mathcal{O}} I(t, x) \, dx \leq -(\mu_2 - \alpha)_* \mathbb{E} \int_{\mathcal{O}} I(t, x) \, dx \quad \text{for all } t \geq 0.$$

Since $(\mu_2 - \alpha)_* > 0$, we can get that $\mathbb{E} \int_{\mathcal{O}} I(t, x) \, dx$ converges to 0 with exponential rate as $t \rightarrow \infty$. Hence, it is easy to claim that the infected class goes extinct. \square

Theorem 4.3. *Suppose that $W_2(t)$ is a space-independent Brownian motion with covariance $a_2 t$. For any non-negative initial data $S_0, I_0 \in E$, if*

$$(\mu_2 - \alpha)_* + \frac{a_2}{2} := \inf_{x \in \mathcal{O}} (\mu_2(x) - \alpha(x)) + \frac{a_2}{2} > 0,$$

then when $p > 0$ is sufficiently small that

$$R_p := (\mu_2 - \alpha)_* + \frac{(1 - p)a_2}{2} > 0,$$

we have

$$\limsup_{t \rightarrow \infty} \frac{\ln \mathbb{E} \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p}{t} \leq -pR_p < 0.$$

Proof. Since $W_2(t)$ is a space-independent Brownian motion, as in the arguments in the proof of Theorem 4.1, the mild solution $I(t)$ is also the solution in the strong sense if $I_0 \in D(A_i^E)$. Hence, with an initial value in $D(A_i^E)$, we have

$$\int_{\mathcal{O}} I(t, x) \, dx = \int_0^t \int_{\mathcal{O}} \left(-\mu_2(x)I(s, x) + \frac{\alpha(x)S(s, x)I(s, x)}{S(s, x) + I(s, x)} \right) dx \, ds + \int_0^t \int_{\mathcal{O}} I(s, x) \, dW_2(s).$$

By Itô’s formula, we obtain that

$$\begin{aligned} & \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p \\ &= \int_s^t \left[p \left(\int_{\mathcal{O}} I(r, x) \, dx \right)^{p-1} \int_{\mathcal{O}} \left(-\mu_2(x)I(r, x) + \frac{\alpha(x)S(r, x)I(r, x)}{S(r, x) + I(r, x)} \right) dx \right] dr \\ & \quad + \int_s^t p(1 - p) \frac{a_2}{2} \left(\int_{\mathcal{O}} I(r, x) \, dx \right)^p dr + \int_s^t \left(\int_{\mathcal{O}} I(r, x) \, dx \right)^p dW_2(r) \\ & \leq -pR_p \int_s^t \left(\int_{\mathcal{O}} I(r, x) \, dx \right)^p dr + \int_s^t \left(\int_{\mathcal{O}} I(r, x) \, dx \right)^p dW_2(r). \end{aligned}$$

Since $\mathbb{E} \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p < \infty$, we have

$$\mathbb{E} \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p = \mathbb{E} \left(\int_{\mathcal{O}} I(s, x) \, dx \right)^p - pR_p \int_s^t \mathbb{E} \left(\int_{\mathcal{O}} I(r, x) \, dx \right)^p dr,$$

which easily leads to

$$\frac{d}{dt^+} \mathbb{E} \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p \leq -pR_p \mathbb{E} \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p.$$

An application of the differential inequality shows that

$$\mathbb{E} \left(\int_{\mathcal{O}} I(t, x) \, dx \right)^p \leq e^{-pR_p t} \left(\int_{\mathcal{O}} I(0, x) \, dx \right)^p \tag{4.12}$$

for any $t \geq 0$ and initial values in $D(A_i^E)$. Since $D(A_i^E)$ is dense in E , (4.12) holds for each fixed t and any initial values in E . Then the desired result can be obtained. \square

5. An example

In this section, to demonstrate our results we consider an example where the noise processes in (1.1) are standard Brownian motions and the recruitment, death, infection, and recovery rates

are independent of the space variable. Precisely, we consider

$$\begin{cases}
dS(t, x) = \left[k_1 \Delta S(t, x) + \Lambda - \mu_1 S(t, x) - \frac{\alpha S(t, x)I(t, x)}{S(t, x) + I(t, x)} \right] dt \\
\quad + \sigma_1 S(t, x) dB_1(t) \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\
dI(t, x) = \left[k_2 \Delta I(t, x) - \mu_2 I(t, x) + \frac{\alpha S(t, x)I(t, x)}{S(t, x) + I(t, x)} \right] dt \\
\quad + \sigma_2 I(t, x) dB_2(t) \quad \text{in } \mathbb{R}^+ \times \mathcal{O}, \\
\partial_\nu S(t, x) = \partial_\nu I(t, x) = 0 \quad \text{in } \mathbb{R}^+ \times \partial\mathcal{O}, \\
S(x, 0) = S_0(x), I(x, 0) = I_0(x) \quad \text{in } \overline{\mathcal{O}},
\end{cases} \tag{5.1}$$

where $\Lambda, \mu_1, \mu_2,$ and α are positive constants, and $B_1(t), B_2(t)$ are independent standard Brownian motions. As we obtained above, for any initial values $0 \leq S_0, I_0 \in E,$ (5.1) has a unique positive mild solution $S(t, x), I(t, x) \geq 0.$ Moreover, the long-time behavior of the system is shown in the following theorem.

Theorem 5.1. *Let $S(t, x), I(t, x)$ be the positive mild solution (in fact also in the strong sense) of (5.1).*

- (i) *For any non-negative initial values $S_0, I_0 \in E,$ if $\alpha < \mu_2 + \frac{\sigma_2^2}{2},$ then the infected individual is extinct.*
- (ii) *For initial values $0 \leq S_0, I_0 \in E$ satisfying*

$$\int_{\mathcal{O}} -\ln I_0(x) \, dx < \infty,$$

If $\alpha > \mu_2 + \frac{\sigma_2^2}{2},$ then the infected class is permanent.

Remark 5.1. As in Theorem 5.1, the sufficient condition for permanence is almost a necessary condition. It is similar to the result for the SIS reaction–diffusion epidemic model shown in [33, Theorem 1.2].

6. Concluding remarks

Being possibly one of the first papers on spatially inhomogeneous stochastic partial differential equation epidemic models, we hope that our effort will provide some insights for subsequent study and investigation. For possible future study, we mention the following topics.

- First, there is growing interest in using the so-called regime-switching stochastic models in various applications; see [39] for the treatment of switching diffusion models, in which both continuous dynamics and discrete events coexist. Such switching diffusion models have gained popularity with applications ranging from networked control systems to financial engineering. For instance, in a financial market model, one may use a random switching process to model the mode of the market (bull or bear). Such a random switching process can be built into the stochastic partial differential equation models considered here. The switching is used to reflect different random environments that are not covered by the stochastic partial differential equation part of the model.

- Second, instead of systems driven by Brownian motions, we may consider systems driven by Lévy process; some recent work can be seen in [5]. One could work with stochastic partial differential equation models driven by Lévy processes. Recent work on switching jump diffusions [9] may also be adopted in stochastic partial differential equation models.
- Finally, in terms of mathematical development, various estimates of the long-time properties were given in the average norm, although the solution is in the better space E . Future work will be to obtain stochastic regularity of the solution by using the methods in [7, 34, 35] so that it is possible to provide estimates in the sup-norm $(\cdot|_E)$. Nevertheless, some mathematical details need to be carefully worked out. The result, in turn, will be of interest to people working on real data. Some other properties such as the strict positivity of the solutions and sharper conditions for extinction and permanence are worthy of consideration.

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