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ON THE TIME SPENT ABOVE A LEVEL BY BROWNIAN MOTION WITH NEGATIVE DRIFT

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Abstract

Limit theorems of Berman involve the total time spent by Brownian motion with negative drift above a fixed or exponentially distributed negative level. We give explicitly the probability densities and distribution functions, obtained via an equivalence of laws.

LAST PASSAGE: DURATION OF POSITIVITY

1. Introduction

If B is standard Brownian motion, the total time ξ that $2^{\frac{1}{2}}B(t)-t+Y$ spends above 0, where Y is a negative exponential variable with parameter 1, independent of B, plays a substantial role in some limit theorems of Berman [1]. He has obtained the Laplace transform of ξ and shown that if $\Gamma(t)=\Pr(\xi>t)$, then $-\Gamma'(t)$ is a non-increasing function with $-\Gamma'(0)=1$. Using an equivalence of laws which explains the squared form of Berman's result we obtain Γ explicitly, identify $1+\Gamma'$ as a distribution function and give also the density and distribution function when y is constant.

2. Results

From now on, $\{X(t),t\geqq 0\}$ is the coordinate process on the space of continuous functions. We call $W_{-\delta}$ the law under which X is Brownian motion with $X(0)=0$ and constant drift $-\delta$; only values $\delta\geqq 0$ are considered. Under $W=W_0$, X is standard Brownian motion. Let $\tau(y)=\inf\{t>0:X(t)=y\}$, $M(t)=\max\{X(s):s\leqq t\}$, $\mu(t)=\inf\{s\leqq t:X(s)=M(t)\}$ and

$$\nu(y;t)=\int_0^t 1_{\{X(s)>y\}}\,ds.$$

When $\delta>0$, $\mu=\mu(\infty)$ and $\nu(y)=\nu(y,\infty)$ are almost surely finite, and we write $\nu=\nu(0)$. We use $\tau(-Y)$ and $\nu(-Y)$ for random positive Y also. Furthermore, $p(t;x)=(2\pi t)^{-\frac{1}{2}}\exp\{-x^2/2t\}$, $t>0$, $x\in\mathbb{R}$, and Φ is the standard normal distribution function.

Lemma 1. If $\delta>0$, the $W_{-\delta}$-laws of μ and ν are identical, with probability density

$$2\delta\psi(t;-\delta)=2\delta\{p(t;\delta t)-\delta\Phi(-\delta t^{\frac{1}{2}})\},\qquad t>0.$$

The Laplace transform is $L(\lambda;\delta)=2/\{1+(1+2\lambda/\delta^2)^{\frac{1}{2}}\}$, $\lambda>0$.

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Proof. First, let $\{X(t), 0 \leqq t \leqq s\}$ have the law of a Brownian bridge of duration s. Then $\mu(s)$ and $\nu(0, s)$ are well-known to be uniform over $[0, s]$. Considering the law of this Brownian bridge as weak limit, for $\varepsilon \downarrow 0$, of laws of Brownian motion conditioned to $X(s) \in [0, \varepsilon]$, one deduces the equality $W(\mu(s) \in dt, X(s) \in d0) = W(\nu(0; s) \in dt, X(s) \in d0)$, $0 < t < s$. Referring to Example 6 of [2] this in turn implies, when $\delta > 0$, $W_{-\delta}(\mu \in dt, \tau'(0) \in ds) = W_{-\delta}(\nu \in dt, \tau'(0) \in ds)$, where $\tau'(0) = \sup\{t > 0 : X(t) = 0\}$. Integration in s gives $W_{-\delta}(\nu \in dt) = W_{-\delta}(\mu \in dt) = 2\delta\psi(t; -\delta)$, the result for μ being known (see e.g. [2], Example 7). The Laplace transform is easily obtained.

Let Y be a negative exponential variable with parameter 2δ, independent of X which has law $W_{-\delta}$, $\delta > 0$. It is easy to see that when $\delta = 2^{-\frac{1}{2}}$, the total time $\nu(-Y)$ during which $X(t) > -Y$ holds has the same law as Berman's ξ.

Lemma 2. Let $\delta > 0$. The $W_{-\delta}$-law of $\tau(-Y)$ is the same as the law of μ, and ν, and the Laplace transform of $\nu(-Y)$ is $L^2(\lambda; \delta)$, $\lambda > 0$.

Proof. For $y > 0$, the $W_{-\delta}$-Laplace transform of $\tau(-y)$ is $\exp\{\delta y[1 - (1 + 2\lambda/\delta^2)^{\frac{1}{2}}]\}$ ([2], Lemma 3). That of $\tau(-Y)$ is therefore, at $\lambda > 0$,

$$2\delta \int_0^\infty \exp(-2\delta y) \exp\{\delta y[1 - (1 + 2\lambda/\delta^2)^{\frac{1}{2}}]\}\, dy = L(\lambda; \delta).$$

Let θ be the shift operator. The total time $\nu(-y) \circ \theta(\tau(-y))$ spent by X above $-y$, from $\tau(-y)$ on, is independent of $\tau(-y)$. As Y is independent of X, $\nu(-Y) \circ \theta(\tau(-Y))$ and $\tau(-Y)$ are also independent. But $\nu(-Y) = \tau(-Y) + \nu(-Y) \circ \theta(\tau(-Y))$ where the second summand has, conditionally on Y and therefore also unconditionally, the same law as ν. The conclusion follows.

Once it is known that the $W_{-\delta}$-density of $\nu(-Y)$ is the convolution of $2\delta\psi(t; -\delta)$ with itself, it is only a matter of lengthy computation to obtain this density, and the corresponding distribution function. The same holds true for $\nu(-y)$, the density of $\tau(-y)$ being known (e.g. [2], 5.2). We record the results as follows.

Theorem. When $\delta > 0$, $y > 0$, one has for $t > 0$:

(a) $W_{-\delta}(\mu > t) = 2\Phi(-\delta t^{\frac{1}{2}}) - 2\delta t\psi(t; -\delta)$.

(b) $W_{-\delta}(\nu(-Y) \in dt) = 2\delta^2 W_{-\delta}(\mu > t)\, dt$.

(c) $W_{-\delta}(\nu(-Y) > t) = 4\Phi(-\delta t^{\frac{1}{2}}) - (1 + \delta^2 t) W_{-\delta}(\mu > t)$.

(d) $W_{-\delta}(\nu(-y) \in dt) = 2\delta \exp(2\delta y)\{p(t; y + \delta t) - \delta\Phi(-t^{\frac{1}{2}}(y + \delta t))\}\, dt$.

(e) $W_{-\delta}(\nu(-y) > t) = \Phi(t^{-\frac{1}{2}}(y - \delta t)) + (1 + 2\delta y)\exp(2\delta y)\Phi(-t^{-\frac{1}{2}}(y + \delta t))$
$\qquad - tW_{-\delta}(\nu(-y) \in dt)/dt$.

As pointed out before, (c) gives for the particular value $\delta^* = 1/2^{\frac{1}{2}}$, Berman's function $\Gamma(t)$, and (b) then shows that $-\Gamma'(t) = W_{-\delta^*}(\mu > t)$.

References

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