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A COMPARISON THEOREM ON MAGNETIC JACOBI FIELDS

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A scalar multiple of the Kähler form of a Kähler manifold is called a Kähler magnetic field. We are focused on trajectories of charged particles under this action. As a variation of trajectories we define a magnetic Jacobi field. In this paper we discuss a comparison theorem on magnetic Jacobi fields, which corresponds to the Rauch's comparison theorem.

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Introduction

A static magnetic field in the Euclidean space $\mathbb{R}^3$ is a vector field $\mathbb{B} = (B_1, B_2, B_3)$ which satisfies the Gauss formula $\mathrm{div}(\mathbb{B}) = \partial B_1/\partial x_1 + \partial B_2/\partial x_2 + \partial B_3/\partial x_3 = 0$. On a charged particle with charge q and velocity vector $\mathbf{v} = (v_1, v_2, v_3)$, this yields the Lorentz force

$$\mathbb{F} = q \cdot \mathbf{v} \times \mathbb{B} = q(v_2B_3 - v_3B_2, v_3B_1 - v_1B_3, v_1B_2 - v_2B_1).$$

The Newton equation is hence given by $m(dv/dt) = q \cdot \mathbf{v} \times \mathbb{B}$, where m is the mass of this particle. In order to get rid of the choice of the orientation of $\mathbb{R}^3$ it is natural to identify $\mathbb{B}$ with the 2-form

$$\mathbb{B} = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2.$$

Under this identification the Gauss formula is equivalent to $d\mathbb{B} = 0$, and the Newton equation turns to $m(d\mathbf{v}/dt) = q \cdot \Omega_{\mathbf{B}}(\mathbf{v})$ with the skew symmetric matrix

$$\Omega_{\mathbf{B}} = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}.$$

With this observation we call a closed 2-form $\mathbb{B}$ on a Riemannian manifold $(M, \langle , \rangle)$ a *magnetic field*. Let $\Omega = \Omega_{\mathbf{B}} : TM \to TM$ denote the skew symmetric operator satisfying $\mathbb{B}(u, v) = \langle u, \Omega(v) \rangle$ for every $u, v \in TM$. A smooth curve γ on M is called a *trajectory* for $\mathbb{B}$ if it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \Omega(\dot\gamma)$. In terms of physics a $\mathbb{B}$-trajectory is a trajectory of a unit charged particle of unit mass under the action of

$\mathbb{B}$. For the trivial magnetic field $\mathbb{B}=0$, trajectories are geodesics. Needless to say, Rauch's comparison theorem plays a very important role in the study of geodesics. The aim of this paper is to prepare a corresponding comparison theorem.

An important class of magnetic fields is the class of uniform magnetic fields on a complete Riemannian manifold. A magnetic field $\mathbb{B}$ is called *uniform* if the associated skew symmetric operator is parallel; $\nabla\Omega=0$. Typical examples for uniform magnetic fields are scalar multiples $k\cdot\mathbb{B}_J$ of the Kähler form $\mathbb{B}_J$ on a Kähler manifold, and scalar multiples $k\cdot Vol$ of the volume form Vol on a Riemann surface. We shall call them Kähler magnetic fields. In the study of geodesics, real space forms, spheres, Euclidean spaces, and hyperbolic spaces, play as model spaces. For the study of (uniform) magnetic fields, since we have no nontrivial uniform magnetic fields on non-flat real space form of dimension greater than 2, the author thinks that complex space forms, complex projective spaces, complex Euclidean spaces, and complex hyperbolic spaces, play as model spaces. In the preceeding papers **[1]** and **[2]** we studied trajectories for Kähler magnetic fields on complex space forms, and pointed out that the feature of trajectories depends on the curvature condition of the base manifold and on the strength of a uniform magnetic field. In this context it is quite natural to study comparison theorems associated to uniform magnetic fields.

Let $\mathbb{B}$ be a uniform magnetic field on a complete Riemannian manifold M. A vector field V along a $\mathbb{B}$-trajectory γ is called a *magnetic Jacobi field* for $\mathbb{B}$ (or simply called $\mathbb{B}$-Jacobi field) if it satisfies the following magnetic Jacobi equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V-\Omega(\nabla_{\dot\gamma}V)+R(V,\dot\gamma)\dot\gamma=0, \tag{MJ}$$

where R denotes the curvature tensor on M. When $\mathbb{B}$ is the trivial magnetic field, magnetic Jacobi fields are usual Jacobi fields. Like Jacobi fields every magnetic Jacobi field is obtained by a variation of trajectories. An important thing in our study is that we should pay attention to the speed of trajectories. In general, since we have $d/dt(\|\dot\gamma(t)\|^2)=\langle\Omega(\dot\gamma(t)),\dot\gamma(t)\rangle+\langle\dot\gamma(t),\Omega(\dot\gamma(t))\rangle=0$, we see that every $\mathbb{B}$-trajectory γ has a constant speed. But once we change the speed of γ and consider the curve $\sigma(t)=\gamma(\lambda t)$ with a constant λ, then σ turns to a trajectory for $\lambda\mathbb{B}$. This is a different point from geodesics. We call a trajectory *normal* if it is parametrized by its arc length. In Section 1, we define a normal magnetic Jacobi field as a variation of normal trajectories and summarize some fundamental results. In Section 2, modifying the lines for the proof of Rauch's comparison theorem we define a magnetic index and show comparison theorems for Kähler magnetic Jacobi fields. As a consequence of this generalization we investigate in the final section some asymptotic behaviour of trajectories for uniform magnetic fields on a Hadamard surface.

1. Magnetic exponential maps and magnetic Jacobi fields

Let $\mathbb{B}$ be a uniform magnetic field on a complete Riemannian manifold M. Let γ be a trajectory for $\mathbb{B}$. Generally, on a complete Riemannian manifold every trajectory

$\gamma(t)$ is defined for $-\infty < t < \infty$. It is clear that the derivative $\dot{\gamma}$ of γ is a magnetic Jacobi field;

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\dot{\gamma} - \Omega(\nabla_{\dot{\gamma}}\dot{\gamma}) + R(\dot{\gamma}, \dot{\gamma})\dot{\gamma} = \nabla_{\dot{\gamma}}\Omega(\dot{\gamma}) - \Omega(\nabla_{\dot{\gamma}}\dot{\gamma}) = 0.$$

Since the magnetic Jacobi equation (MJ) is linear, magnetic Jacobi fields along γ form a $2\dim(M)$-dimensional vector space. We first point out that magnetic Jacobi fields are obtained by variations of trajectories. The proof is just similar as for usual Jacobi fields.

Lemma 1.1. *Let $\varphi : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a smooth map such that for each s the curve $\varphi(s, \cdot) : [a, b] \to M$ is a trajectory segment for a uniform magnetic field $\mathbb{B}$ on M. Then the vector field $V(t) = \partial\varphi/\partial s(0, t)$ is a magnetic Jacobi field for $\mathbb{B}$ along the $\mathbb{B}$-trajectory $\varphi(0, t)$.*

Unfortunately, for a magnetic Jacobi field V along a trajectory γ, the second derivative of the function $\langle V, \dot{\gamma}\rangle$ does not necessarily vanish along γ. But we have by a direct computation the following.

Lemma 1.2. *Let V and W be $\mathbb{B}$-Jacobi fields along a $\mathbb{B}$-trajectory γ. Then $\langle\nabla_{\dot{\gamma}}V, W\rangle - \langle V, \nabla_{\dot{\gamma}}W\rangle + \langle V, \Omega(W)\rangle$ is constant along γ. In particular $\langle\nabla_{\dot{\gamma}}V, \dot{\gamma}\rangle$ is constant along γ.*

We shall call a magnetic Jacobi field V along a trajectory γ *normal* if it satisfies $\langle\nabla_{\dot{\gamma}}V, \dot{\gamma}\rangle \equiv 0$. Normal magnetic Jacobi fields are obtained naturally by variations of normal trajectories.

Lemma 1.3. *Let $\varphi : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a smooth map such that for each s the curve $\varphi(s, \cdot) : [a, b] \to M$ is a normal trajectory segment for a uniform magnetic field $\mathbb{B}$ on M. Then the vector field $V(t) = \partial\varphi/\partial s(0, t)$ is a normal magnetic Jacobi field for $\mathbb{B}$ along the $\mathbb{B}$-trajectory $\varphi(0, t)$.*

It is clear that normal magnetic Jacobi fields form a $(2\dim(M) - 1)$-vector space. Given a point $p \in M$ we define the $\mathbb{B}$-*exponential map* $\mathbb{B}\exp_p : T_pM \to M$ of the tangent space at p into M by

$$\mathbb{B}\exp_p(v) = \begin{cases} \gamma_{v_0}(\|v\|) & \text{if } v \neq o_p, \\ p & \text{if } v = o_p, \end{cases}$$

Here v_0 denotes the unit vector $v/\|v\| \in U_pM$, γ_{v_0} denotes the $\mathbb{B}$-trajectory with $\dot{\gamma}_{v_0}(0) = v_0$, and $o_p \in T_pM$ is the origin. The reader should note that $\gamma_v(1) \neq \gamma_{v_0}(\|v\|)$ in general. We first compute the differential map $D\mathbb{B}\exp_p$ of $\mathbb{B}\exp_p$ at the origin $o_p \in T_pM$. For $v \in T_pM\backslash\{o_p\}$ we have

$$(D\mathbb{B}\exp_p(o_p))v = \frac{d}{dt}\mathbb{B}\exp_p(tv)\,|_{t=0} = \|v\| \cdot \dot{\gamma}_{v_0}(0) = v,$$

hence $D\mathbb{B}\exp_p(o_p)$ is the identity map. Therefore, for some positive ε, the map $\mathbb{B}\exp_p|_{B_\varepsilon(o_p)}$ is a diffeomorphism onto some open neighbourhood of p. Similarly consider the map $F: TM \to M \times M$ defined by $F(v) = (\tau(v), \mathbb{B}\exp_{\tau(v)}(v))$, where $\tau: TM \to M$ denotes the projection. We find $DF(o_p)$ is not singular at each point $p \in M$. We have

Lemma 1.4. *For each $p_0 \in M$ we can find a neighbourhood U of p_0 and positive ε so that the following conditions hold.*

(1) *For each $p, q \in U$ $(p \neq q)$ there exists a unique normal $\mathbb{B}$-trajectory $\gamma_{p,q}$ and $\ell_{p,q}$ $(0 < \ell_{p,q} < \varepsilon)$ such that $\gamma_{p,q}(0) = p$ and $\gamma_{p,q}(\ell_{p,q}) = q$.*

(2) *$\dot{\gamma}_{p,q}(0) \in UM$ and $\ell_{p,q}$ depend smoothly on p, q.*

(3) *For each $p \in U$ the map $\mathbb{B}\exp_p|_{B_\varepsilon(o_p)}$ is a diffeomorphism onto some open set containing U.*

Let γ be a normal $\mathbb{B}$-trajectory. For a vector field W along γ we put $W^\natural(t) = \mathrm{Proj}_t(W(t))$, where

$$\mathrm{Proj}_t : T_{\gamma(t)}M \to \langle\!\langle \dot{\gamma}(t) \rangle\!\rangle^\perp = \{v \in T_{\gamma(t)}M \mid \langle v, \dot{\gamma}(t)\rangle = 0\}$$

denotes the projection. When we treat normal magnetic Jacobi fields their vertical component is important. We call a point $\gamma(t_0)$ a $\mathbb{B}$-*conjugate point* for $\gamma(0)$ along γ if there exists a nontrivial normal $\mathbb{B}$-Jacobi field V such that $V^\natural(0) = 0$ and $V^\natural(t_0) = 0$. When $\gamma(t_0)$ is a $\mathbb{B}$-conjugate point of $\gamma(0)$, the value t_0 is called a $\mathbb{B}$-*conjugate value* of $\gamma(0)$ along γ. We call the minimum positive $\mathbb{B}$-conjugate value $t_c(\gamma(0))$ the first $\mathbb{B}$-conjugate value of $\gamma(0)$ along γ. Similarly, we call $\gamma(t_1)$ a $\mathbb{B}$-*focal point* for $\gamma(0)$ along γ if there exists nontrivial normal $\mathbb{B}$-Jacobi field V such that $(\nabla_{\dot{\gamma}} V^\natural(0))^\natural = 0$ and $V^\natural(t_1) = 0$. The minimum positive value $t_f(\gamma(0))$ with this property is called the first $\mathbb{B}$-focal value.

Proposition 1. *Let γ be a normal $\mathbb{B}$-trajectory with $\dot{\gamma}(0) = u \in U_pM$. The map $\mathrm{Proj}_{t_0} \circ D(\mathbb{B}\exp_p(t_0u)) : T_{t_0u}(t_0 \cdot U_pM) \to \langle\!\langle \dot{\gamma}(t_0) \rangle\!\rangle^\perp$ is singular if and only if $\gamma(t_0)$ is a $\mathbb{B}$-conjugate point of p.*

Proof. Let $u(s) : (-\varepsilon, \varepsilon) \to U_pM$ be a smooth curve with $u(0) = u$, $\dot{u}(0) = \zeta \in T_u(U_pM)$. Set $\alpha(s, t) = \mathbb{B}\exp_p(tu(s))$. Then $\alpha(s, \cdot)$ is a normal $\mathbb{B}$-trajectory for each s, hence $V(t) = \partial\alpha/\partial s(0, t)$ is a normal $\mathbb{B}$-Jacobi field along γ. Since we have $V(t) = t(D\mathbb{B}\exp_p(tu))\zeta$. We get the conclusion.

We here notice that if γ is a normal $\mathbb{B}$-trajectory with $\dot{\gamma}(0) = u \in U_pM$ then $(D\mathbb{B}\exp_p(tu))u = \dot{\gamma}(t)$. Since a magnetic Jacobi field V is determined by the initial condition $V(0)$ and $\nabla V(0)$, the proof of Proposition 1 guarantees the following.

Lemma 1.5. *Every normal magnetic Jacobi field is obtained by a variation of normal $\mathbb{B}$-trajectories.*

We can also conclude the following.

Lemma 1.6. *Let γ be a normal $\mathbb{B}$-trajectory. Suppose $\gamma(t_0)$ is not a $\mathbb{B}$-conjugate point for $\gamma(0)$ along γ. Then for any $v \in T_{\gamma(0)}M$ and $w \in \langle\!\langle \dot\gamma(t_0) \rangle\!\rangle^{\perp}$ we can find a unique normal $\mathbb{B}$-Jacobi field V along with $V(0) = v$ and $\mathrm{Proj}_{t_0}(V(t_0)) = w$.*

2. A comparison theorem for magnetic Jacobi fields

We now discuss a comparison theorem for magnetic Jacobi fields along normal trajectories for Kähler magnetic fields on Kähler manifolds. For a Kähler manifold (M, J) with a complex structure J we denote by $\mathbb{B}_J$ the Kähler form. One of our main results is the following.

Theorem 1. *Let γ and $\hat\gamma$ be a normal trajectory for Kähler magnetic fields $k \cdot \mathbb{B}_J$ and $k \cdot \mathbb{B}_{\hat J}$ on Kähler manifolds (M, J) and $(\hat M, \hat J)$ respectively. Assume*

(a) $\min\{\langle R(v, \dot\gamma(t))\dot\gamma(t), v\rangle \mid \|v\| = 1, \langle v, \dot\gamma(t)\rangle = 0\}$
$\geq \max\{\langle R(\hat v, \dot{\hat\gamma}(t))\dot{\hat\gamma}(t), \hat v\rangle \mid \|\hat v\| = 1, \langle \hat v, \dot{\hat\gamma}(t)\rangle = 0\}$ *for* $0 \leq t \leq t_c(\gamma(0))$,

(b) *the dimension $\dim(M)$ of M is not smaller than $\dim(\hat M)$.*

We then have the following.

(1) $t_c(\gamma(0)) \leq t_c(\hat\gamma(0))$.

(2) *If normal magnetic Jacobi fields $V, \hat V$ for $k \cdot \mathbb{B}_J, k \cdot \mathbb{B}_{\hat J}$ along $\gamma, \hat\gamma$ satisfy $V^{\natural}(0) = 0$, $\hat V^{\natural}(0) = 0$, $\|\nabla V^{\natural}(0)\| = \|\nabla \hat V^{\natural}(0)\|$, then*

$$\|V^{\natural}(t)\| \leq \|\hat V^{\natural}(t)\| \quad \textit{for} \quad 0 \leq t \leq t_c(\gamma(0)).$$

Let γ be a normal trajectory for a Kähler magnetic field $\mathbb{B} = k \cdot \mathbb{B}_J$ on a Kähler manifold M of complex dimension n. For a vector field $X = hJ\dot\gamma + X^{\perp}$, where h is a function and $X^{\perp}$ is $\mathbb{C}$-perpendicular to $\dot\gamma$, that is, $X^{\perp}(t) \in \langle\!\langle \dot\gamma(t) \rangle\!\rangle_{\mathbb{C}}^{\perp} = \{v \in T_{\gamma(t)}M \mid \langle v, \dot\gamma(t)\rangle = \langle v, J\dot\gamma(t)\rangle = 0\}$, we define the index $\mathscr{I}_T$ by

$$\mathscr{I}_T(X) = \int_0^T \{h'^2 - k^2h^2 + \langle \nabla_{\dot\gamma}X^{\perp} - kJX^{\perp}, \nabla_{\dot\gamma}X^{\perp}\rangle - \langle R(X, \dot\gamma)\dot\gamma, X\rangle\}dt.$$

Here we should note that $\langle \nabla_{\dot\gamma}^m X^{\perp}, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}^m X^{\perp}, J\dot\gamma\rangle \equiv 0$ for every positive integer m. When V is a normal magnetic Jacobi field along γ, the index of $V^{\natural}$ is given as

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follows;

$$\mathscr{I}_T(V^\sharp) = \langle \nabla_{\dot\gamma} V^\sharp(T), V^\sharp(T)\rangle. \tag{2.1}$$

To see this we denote $V = f\dot\gamma + gJ\dot\gamma + V^\perp$. We find that V is a normal magnetic Jacobi field for $k\cdot\mathbb{B}_J$ if and only if the following equations hold:

$$\begin{cases} f' = kg \\ (g'' + k^2 g)J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}V^\perp - kJ\nabla_{\dot\gamma}V^\perp + R(V^\sharp, \dot\gamma)\dot\gamma = 0. \end{cases} \tag{2.2}$$

We therefore have

$$\begin{aligned} \mathscr{I}_T(V^\sharp) &= \int_0^T \bigg\{(gg')' - g(g'' + k^2 g) + \frac{d}{dt}\langle \nabla_{\dot\gamma}V^\perp - kJV^\perp, V^\perp\rangle \\ &\qquad - \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}V^\perp - kJ\nabla_{\dot\gamma}V^\perp, V^\perp\rangle - \langle R(V^\sharp, \dot\gamma)\dot\gamma, V^\sharp\rangle\bigg\}dt \\ &= g(T)g'(T) + \langle \nabla_{\dot\gamma}V^\perp(T), V^\perp(T)\rangle \\ &\qquad - \int_0^T \langle (g'' + k^2 g)J\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}V^\perp - kJ\nabla_{\dot\gamma}V^\perp + R(V^\sharp, \dot\gamma)\dot\gamma, V\rangle dt \\ &= \langle g'J\dot\gamma + \nabla_{\dot\gamma}V^\perp, gJ\dot\gamma + V^\perp\rangle_{(T)} \\ &= \langle \nabla_{\dot\gamma}V^\sharp(T), V^\sharp(T)\rangle, \end{aligned}$$

and get (2.1). Using this we obtain the following modified index lemma.

Lemma 2.1. *Let V be a normal magnetic Jacobi field with $V(0) = 0$ along a normal trajectory γ for a Kähler magnetic field. If $T < t_c(\gamma(0))$ and a vector field X along γ satisfies $X(0) = 0$, $X(T) = V^\sharp(T)$ and $X^\sharp = X$, then it satisfies $\mathscr{I}_T(V^\sharp) \le \mathscr{I}_T(X)$. The equality holds if and only if $X = V^\sharp$.*

Proof. Choose linearly independent normal magnetic Jacobi fields $V_1, \dots, V_{2n-1}$ along γ so that $V_j(0) = 0$. Since $T < t_c(\gamma(0))$, we have that $V_1^\sharp(t), \dots, V_{2n-1}^\sharp(t)$ are also linearly independent for $0 < t < T$. Hence we can represent $X = \sum_{j=1}^{2n-1}\zeta_j V_j^\sharp$ by using smooth functions $\zeta_1, \dots, \zeta_{2n-1}$. Denoting $V_j^\sharp = g_j J\dot\gamma + V_j^\perp$, we have

$$\begin{aligned} \mathscr{I}_T(X) = \int_0^T \bigg\{&\bigg(\sum_{j=1}^{2n-1}\zeta_j' g_j + \zeta_j g_j'\bigg)^2 - k^2\bigg(\sum_{j=1}^{2n-1}\zeta_j g_j\bigg)^2 \\ &+ \bigg\langle \sum_{j=1}^{2n-1}\zeta_j' V_j^\perp + \zeta_j\nabla_{\dot\gamma}V_j^\perp - k\zeta_j JV_j^\perp, \sum_{j=1}^{2n-1}\zeta_j' V_j^\perp + \zeta_j\nabla_{\dot\gamma}V_j^\perp\bigg\rangle \\ &+ \bigg\langle R\bigg(\sum_{j=1}^{2n-1}\zeta_j(g_j J\dot\gamma + V_j^\perp), \dot\gamma\bigg)\dot\gamma, \sum_{j=1}^{2n-1}\zeta_j(g_j J\dot\gamma + V_j^\perp)\bigg\rangle\bigg\}dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \Bigg[\Bigg(\sum_{j=1}^{2n-1} \zeta_j' g_j \Bigg)^2 + \Bigg\| \sum_{j=1}^{2n-1} \zeta_j' V_j^\perp \Bigg\|^2 \\
&\qquad + \frac{d}{dt} \Bigg\{ \Bigg(\sum_{j=1}^{2n-1} \zeta_j g_j' \Bigg) \Bigg(\sum_{\ell=1}^{2n-1} \zeta_\ell g_\ell \Bigg) + \Bigg\langle \sum_{j=1}^{2n-1} \zeta_j \nabla_{\dot\gamma} V_j^\perp, \sum_{\ell=1}^{2n-1} \zeta_\ell V_\ell^\perp \Bigg\rangle \Bigg\} \Bigg] dt \\
&+ \int_0^T \Bigg(\sum_{j,\ell=1}^{2n-1} \zeta_j' \zeta_\ell (g_j g_\ell' - g_j' g_\ell + \langle V_j^\perp, \nabla_{\dot\gamma} V_\ell^\perp \rangle - \langle \nabla_{\dot\gamma} V_j^\perp, V_\ell^\perp \rangle - k \langle V_j^\perp, J V_\ell^\perp \rangle) \Bigg) dt \\
&- \int_0^T \Bigg[\Bigg(\sum_{j=1}^{2n-1} \zeta_j (g_j'' + k^2 g_j) \Bigg) \Bigg(\sum_{j=1}^{2n-1} \zeta_\ell g_\ell \Bigg) \\
&\qquad + \Bigg\langle \sum_{j=1}^{2n-1} \zeta_j (\nabla_{\dot\gamma} \nabla_{\dot\gamma} V_j^\perp - kJ \nabla_{\dot\gamma} V_j^\perp), \sum_{\ell=1}^{2n-1} \zeta_\ell V_\ell^\perp \Bigg\rangle \\
&\qquad + \Bigg\langle R\Bigg(\sum_{j=1}^{2n-1} \zeta_j (g_j J\dot\gamma + V_j^\perp), \dot\gamma \Bigg) \dot\gamma, \sum_{\ell=1}^{2n-1} \zeta_\ell (g_\ell J\dot\gamma + V_\ell^\perp) \Bigg\rangle \Bigg] dt.
\end{aligned}
$$

By Lemma 1.2, we see that

$$
\begin{aligned}
g_j g_\ell' - g_j' g_\ell + \langle V_j^\perp, \nabla_{\dot\gamma} V_\ell^\perp \rangle - \langle \nabla_{\dot\gamma} V_j^\perp, V_\ell^\perp \rangle - k \langle V_j^\perp, J V_\ell^\perp \rangle \\
= \langle \nabla_{\dot\gamma} V_\ell^\sharp, V_j^\sharp \rangle - \langle V_\ell^\sharp, \nabla_{\dot\gamma} V_j^\sharp \rangle + \langle V_\ell^\sharp, kJ V_j^\sharp \rangle
\end{aligned}
$$

is constant along γ, hence equals to 0. Since $\sum_{j=1}^{2n-1} \zeta_j(T) V_j(0) = 0$ and $\sum_{j=1}^{2n-1} \zeta_j(T) V_j(T) = X(T) = V(T)$, we have $V = \sum_{j=1}^{2n-1} \zeta_j(T) V_j$. We get by using (2.1) and $X(0) = \sum_{\ell=1}^{2n-1} \zeta_\ell(0) V_\ell^\sharp(0) = 0$ that

$$
\begin{aligned}
\mathscr{I}_T(X) &= \Bigg\langle \sum_{j=1}^{2n-1} \zeta_j (g_j' J\dot\gamma + \nabla_{\dot\gamma} V_j^\perp), \sum_{\ell=1}^{2n-1} \zeta_\ell (g_\ell J\dot\gamma + V_\ell^\perp) \Bigg\rangle \Bigg|_{t=0}^{t=T} + \int_0^T \Bigg\| \sum_{j=1}^{2n-1} \zeta_j' V_j^\sharp \Bigg\|^2 dt \\
&- \int_0^T \Bigg\langle \sum_{j=1}^{2n-1} \zeta_j \Big((g_j'' + k^2 g_j) + \nabla_{\dot\gamma} \nabla_{\dot\gamma} V_j^\perp - kJ \nabla_{\dot\gamma} V_j^\perp + R(g_j J\dot\gamma + V_j^\perp, \dot\gamma)\dot\gamma \Big), \sum_{\ell=1}^{2n-1} \zeta_\ell (g_\ell J\dot\gamma + V_\ell^\perp) \Bigg\rangle dt \\
&= \Bigg\langle \sum_{j=1}^{2n-1} \zeta_j (\nabla_{\dot\gamma} V_j^\sharp)^\sharp, \sum_{\ell=1}^{2n-1} \zeta_\ell V_\ell^\sharp \Bigg\rangle \Bigg|_{t=0}^{t=T} + \int_0^T \Bigg\| \sum_{j=1}^{2n-1} \zeta_j' V_j^\sharp \Bigg\|^2 dt \\
&= \mathscr{I}_T(V^\sharp) + \int_0^T \Bigg\| \sum_{j=1}^{2n-1} \zeta_j' V_j^\sharp \Bigg\|^2 dt \geq \mathscr{I}_T(V^\sharp).
\end{aligned}
$$

Since the last inequality holds if and only if all ζ_j' vanish along γ, we obtain the conclusion.

Proof of Theorem 1. In the case $\nabla V^\sharp(0) = 0$, we have $V^\sharp \equiv 0$ by the assumption $V(0) = 0$, hence we may suppose $\nabla V^\sharp(0) \neq 0$. Since

$$\lim_{t\downarrow 0}\frac{\|\hat{V}^\sharp(t)\|^2}{\|V^\sharp(t)\|^2}=\lim_{t\downarrow 0}\frac{\langle\nabla_{\dot{\hat\gamma}}\hat{V}^\sharp,\hat{V}^\sharp\rangle}{\langle\nabla_{\dot\gamma}V^\sharp,V^\sharp\rangle}=\lim_{t\downarrow 0}\frac{\langle\nabla_{\dot{\hat\gamma}}\nabla_{\dot{\hat\gamma}}\hat{V}^\sharp,\hat{V}^\sharp\rangle+\|\nabla_{\dot{\hat\gamma}}\hat{V}^\sharp\|^2}{\langle\nabla_{\dot\gamma}\nabla_{\dot\gamma}V^\sharp,V^\sharp\rangle+\|\nabla_{\dot\gamma}V\|^2}=1,$$

we shall show

$$\frac{d}{dt}\left(\frac{\|\hat{V}^\sharp(t)\|^2}{\|V^\sharp(t)\|^2}\right)=2\frac{\|\hat{V}^\sharp(t)\|^2}{\|V^\sharp(t)\|^2}\left(\frac{\langle\nabla_{\dot{\hat\gamma}}\hat{V}^\sharp,\hat{V}^\sharp\rangle}{\|\hat{V}^\sharp(t)\|^2}-\frac{\langle\nabla_{\dot\gamma}V^\sharp,V^\sharp\rangle}{\|V^\sharp(t)\|^2}\right)$$

is not negative. For a positive $T<\min\{t_c(\gamma(0)),\,t_c(\hat\gamma(0))\}$ we put

$$X_T(t)=\frac{V^\sharp(t)}{\|V^\sharp(T)\|},\ \hat{X}_T(t)=\frac{\hat{V}^\sharp(t)}{\|\hat{V}^\sharp(T)\|},\quad\text{and}\quad \tilde{X}_T(t)=P^t_\gamma\circ I\circ\bar{P}^t_{\hat\gamma}(\hat{X}_T(t)),$$

where

$$P^t_\gamma: T_{\gamma(0)}M\to T_{\gamma(t)}M,\ \bar{P}^t_{\hat\gamma}: T_{\hat\gamma(t)}\hat{M}\to T_{\hat\gamma(0)}\hat{M}$$

are parallel transformation along γ and $\hat\gamma$ respectively, and $I: T_{\hat\gamma(t)}\hat{M}\to T_{\gamma(0)}M$ is an injective inner product preserving holomorphic linear map which satisfies $X_T(0)=P^t_\gamma\cdot I\cdot\bar{P}^t_{\hat\gamma}(\hat{X}_T(0))$. Denoting $X_T=hJ\dot\gamma+X_T^\perp$, $\hat{X}_T=\hat{h}\hat{J}\dot{\hat\gamma}+\hat{X}_T^\perp$ we have by the assumption (a) that

$$\begin{aligned}\frac{\langle\nabla_{\dot{\hat\gamma}}\hat{V}^\sharp(T),\hat{V}^\sharp(T)\rangle}{\|\hat{V}^\sharp(T)\|^2}&=\langle\nabla_{\dot{\hat\gamma}}\hat{X}_T(T),\hat{X}_T(T)\rangle=\mathscr{I}_T(\hat{X}_T)\\&=\int_0^T\{\hat{h}'^2-k^2\hat{h}^2+\|\nabla_{\dot{\hat\gamma}}\hat{X}_T^\perp\|^2-k\langle\hat{J}\hat{X}_T^\perp,\nabla_{\dot{\hat\gamma}}\hat{X}_T^\perp\rangle-\langle R_{\hat M}(\hat{X}_T,\dot{\hat\gamma})\dot{\hat\gamma},\hat{X}_T\rangle\}dt\\&\geq\int_0^T\{\hat{h}'^2-k^2\hat{h}^2+\|\nabla_{\dot{\hat\gamma}}\hat{X}_T^\perp\|^2-k\langle\hat{J}\hat{X}_T^\perp,\nabla_{\dot{\hat\gamma}}\hat{X}_T^\perp\rangle-\langle R_M(\tilde{X}_T,\dot\gamma)\dot\gamma,\tilde{X}_T\rangle\}dt=\mathscr{I}_T(\tilde{X}_T)\end{aligned}$$

Since $\hat{V}/\|\hat{V}(T)\|$ is a normal magnetic Jacobi field,

$$\mathscr{I}_T(\tilde{X}_T)\geq\mathscr{I}_T(X_T)=\langle\nabla_{\dot\gamma}X_T(T),X_T(T)\rangle=\frac{\langle\nabla_{\dot\gamma}V^\sharp(T),V^\sharp(T)\rangle}{\|V^\sharp(T)\|^2}.$$

We therefore have $\|\hat{V}^\sharp(t)\|\geq\|V^\sharp(t)\|$ for $0\leq t<\min\{t_c(\gamma(0)),t_c(\hat\gamma(0))\}$. This yields $t_c(\hat\gamma(0))\geq t_c(\gamma(0))$ and leads us to the conclusion.

By the same argument we have

Lemma 2.2. *Let V be a normal magnetic Jacobi field with $(\nabla_{\dot\gamma}V^\sharp)^\sharp(0)=0$ along a normal trajectory γ for a Kähler magnetic field. If $T<t_f(\gamma(0))$ and a vector field X along γ satisfies $(\nabla_{\dot\gamma}X)^\sharp(0)=0$, $X(T)=V^\sharp(T)$ and $X^\sharp=X$, then it satisfies $\mathscr{I}_T(V^\sharp)\leq\mathscr{I}_T(X)$.*

The equality holds if and only if $X = V^\sharp$.

Theorem 2. *Let* γ *and* $\hat{\gamma}$ *be normal trajectories for Kähler magnetic fields* $k \cdot \mathbb{B}_J$ *and* $k \cdot \mathbb{B}_{\hat{J}}$ *on Kähler manifolds* (M, J) *and* $(\hat{M}, \hat{J})$ *respectively. Assume*

(a) $\min\{\langle R(v, \dot\gamma(t))\dot\gamma(t), v\rangle \mid \|v\| = 1, \langle v, \dot\gamma(t)\rangle = 0\}$
$\geq \max\{\langle R(\hat{v}, \dot{\hat\gamma}(t))\dot{\hat\gamma}(t), \hat{v}\rangle \mid \|\hat{v}\| = 1, \langle \hat{v}, \dot{\hat\gamma}(t)\rangle = 0\}$ *for* $0 \leq t \leq t_f(\gamma(0))$,

(b) $\dim(M) \geq \dim(\hat{M})$.

We then have the following.

(1) $t_f(\gamma(0)) \leq t_f(\hat\gamma(0))$.

(2) *If normal magnetic Jacobi fields* $V, \hat{V}$ *for* $k \cdot \mathbb{B}_J, k \cdot \mathbb{B}_{\hat{J}}$ *along* $\gamma, \hat\gamma$ *satisfy* $\|V^\sharp(0)\| = \|\hat{V}^\sharp(0)\|$, $(\nabla V^\sharp)^\sharp(0) = 0$, $(\nabla \hat{V}^\sharp)^\sharp(0) = 0$, *then*

$$\|V^\sharp(t)\| \leq \|\hat{V}^\sharp(t)\| \quad \textit{for} \quad 0 \leq t \leq t_c(\gamma(0)).$$

When we compare Kähler magnetic Jacobi fields on Riemann sufaces, we can relax the assumption a bit.

Proposition 2. *Let* γ *and* $\hat\gamma$ *be normal trajectories for the uniform magnetic fields* $k \cdot vol_M$ *and* $\hat{k} \cdot vol_{\hat{M}}$ *on Riemann surface* M *and* $\hat{M}$ *respectively. Assume for each* $t \in [0, \ell]$ *that* $\mathrm{Riem}_M(t) + k^2 \geq \mathrm{Riem}_{\hat{M}}(t) + \hat{k}^2$. *Here* $\mathrm{Riem}_M(t)$ *and* $\mathrm{Riem}_{\hat{M}}(t)$ *denote the sectional curvature at* $\gamma(t)$ *and* $\hat\gamma(t)$ *respectively.*

(I) *Let* $V, \hat{V}$ *be normal magnetic Jacobi fields for* $k \cdot vol_M$, $\hat{k} \cdot vol_{\hat{M}}$ *along* $\gamma, \hat\gamma$ *such that* $\langle V(0), J\dot\gamma(0)\rangle = \langle \hat{V}(0), \hat{J}\dot{\hat\gamma}(0)\rangle = 0$ *and* $d/dt\langle V(t), J\dot\gamma(t)\rangle\,|_{t=0} = d/dt\langle \hat{V}(t), \hat{J}\dot{\hat\gamma}(t)\rangle\,|_{t=0}$.

(1) *If* $\ell < t_c(\gamma(0))$, *we have* $|\langle V(t), J\dot\gamma(t)\rangle| \leq |\langle \hat{V}(t), \hat{J}\dot{\hat\gamma}(t)\rangle|$ *for all* $t \in [0, \ell]$, *and hence have* $t_c(\gamma(0)) \leq t_c(\hat\gamma(0))$.

(2) *If we further assume* $|k| \leq |\hat{k}|$ *and* $\langle V(0), \dot\gamma(0)\rangle = \langle \hat{V}(0), \dot{\hat\gamma}(0)\rangle = 0$, *then we have* $|\langle V(t), \dot\gamma(t)\rangle| \leq |\langle \hat{V}(t), \dot{\hat\gamma}(t)\rangle|$ *for all* $t \in [0, \ell]$.

(II) *Let* $V, \hat{V}$ *be normal magnetic Jacobi fields for* $k \cdot vol_M$, $\hat{k} \cdot vol_{\hat{M}}$ *along* $\gamma, \hat\gamma$ *such that* $\langle V(0), J\dot\gamma(0)\rangle = \langle \hat{V}(0), \hat{J}\dot{\hat\gamma}(0)\rangle \neq 0$, *and* $d/dt\langle V(t), J\dot\gamma(t)\rangle\,|_{t=0} = d/dt\langle \hat{V}(t), \hat{J}\dot{\hat\gamma}(t)\rangle\,|_{t=0} = 0$.

(1) *If* $\ell < t_f(\gamma(0))$, *we have* $|\langle V(t), J\dot\gamma(t)\rangle| \leq |\langle \hat{V}(t), \hat{J}\dot{\hat\gamma}(t)\rangle|$ *for all* $t \in [0, \ell]$ *and* $t_f(\gamma(0)) \leq t_f(\hat\gamma(0))$.

(2) *If we further assume* $|k| \leq |\hat{k}|$ *and* $\langle V(0), \dot\gamma(0)\rangle = \langle \hat{V}(0), \dot{\hat\gamma}(0)\rangle = 0$, *then we have* $|\langle V(t), \dot\gamma(t)\rangle| \leq |\langle \hat{V}(t), \dot{\hat\gamma}(t)\rangle|$ *for all* $t \in [0, \ell]$.

3. Kähler magnetic Jacobi fields on complex space forms

In our study of magnetic fields, complex space forms with Kähler magnetic fields seem to play the role of model spaces. Here we study magnetic Jacobi fields on these spaces.

Example 1. (Complex Euclidean space $\mathbb{C}^n$). On a complex Euclidean space $\mathbb{C}^n$, magnetic Jacobi fields for $k \cdot \mathbb{B}_J$ along a normal trajectory γ are expressed as

$$V(t) = (\gamma(t), A + Be^{kit})$$

for some $A, B \in \mathbb{C}^n$. In particular, a normal $k \cdot \mathbb{B}_J$-Jacobi field V with $V(0) = 0$ is given by

$$V(t) = a\{(1 - \cos kt)\dot{\gamma}(t) + \sin kt \cdot J\dot{\gamma}(t)\} + (\gamma(t), C(1 - e^{kit})),$$

with some constant $a, \in \mathbb{R}$ and $C \in \langle\!\langle \dot{\gamma}(0) \rangle\!\rangle_{\mathbb{C}}^{\perp}$. We therefore find out that $\pi j/k, j = \pm 1, \pm 2, \ldots$, are $k \cdot \mathbb{B}_J$-conjugate values of $\gamma(0)$ along γ.

Let γ be a $k \cdot \mathbb{B}_J$-trajectory on a complex space form. A vector field $V = f\dot{\gamma} + gJ\dot{\gamma} + V^{\perp}$ along γ is a $k \cdot \mathbb{B}_J$-Jacobi field if and only if

$$\begin{cases} f''(t) = kg'(t) \\ g''(t) + kf'(t) + \alpha g(t) = 0, \\ \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V^{\perp} - k \cdot J(\nabla_{\dot{\gamma}} V^{\perp}) + \frac{\alpha}{4} \cdot V^{\perp} = 0 \end{cases} \tag{3.1}$$

where α denotes the holomorphic sectional curvature of the base space. Solving these equations we get the following.

Example 2. (Complex projective space $\mathbb{C}P^n(\alpha)$). On a complex projective space $\mathbb{C}P^n(\alpha)$ of holomorphic sectional curvature α, $k \cdot \mathbb{B}_J$-Jacobi fields $V = f\dot{\gamma} + gJ\dot{\gamma} + V^{\perp}$ along a normal $k \cdot \mathbb{B}_J$-trajectory γ are expressed as

$$\begin{aligned} f(t) &= a + k(b \cdot \cos\sqrt{k^2 + \alpha}\, t + c \cdot \sin\sqrt{k^2 + \alpha}\, t) + \alpha dt \\ g(t) &= -kd + \sqrt{k^2 + \alpha}\,(-b \cdot \sin\sqrt{k^2 + \alpha}\, t + c \cdot \cos\sqrt{k^2 + \alpha}\, t) \\ V^{\perp}(t) &= D\pi\left(\left(\tilde{\gamma}(t), e^{kit/2}\left(A \cdot \sin\frac{1}{2}\sqrt{k^2 + \alpha}\, t + B \cdot \cos\frac{1}{2}\sqrt{k^2 + \alpha}\, t\right)\right)\right), \end{aligned}$$

with some constants $a, b, c, d \in \mathbb{R}$ and some $A, B \in \mathbb{C}^{n+1}$ satisfying $\langle\!\langle A, \tilde{\gamma}(0) \rangle\!\rangle = \langle\!\langle A, \dot{\tilde{\gamma}}(0) \rangle\!\rangle = \langle\!\langle B, \tilde{\gamma}(0) \rangle\!\rangle = \langle\!\langle B, \dot{\tilde{\gamma}}(0) \rangle\!\rangle = 0$. Here $\langle\!\langle\, , \rangle\!\rangle$ denotes the natural Hermitian inner product on $\mathbb{C}^{n+1}$, $\pi : S^{2n+1} \to \mathbb{C}P^n$ is the Hopf fibration and $\tilde{\gamma}$ is a horizontal lift of γ into $S^{2n+1} \subset \mathbb{C}^{n+1}$ (cf. [1]).

In view of (2.2), normal $k \cdot \mathbb{B}_J$-Jacobi fields V along γ with $V(0) = 0$ are hence given by

$$\begin{aligned} V(t) = b\{k(1 - \cos\sqrt{k^2 + \alpha}\, t)\dot{\gamma}(t) + \sqrt{k^2 + \alpha} \cdot \sin\sqrt{k^2 + \alpha}\, t \cdot J\dot{\gamma}(t)\} \\ + D\pi\left(\left(\tilde{\gamma}(t), A \cdot e^{kit/2} \cdot \sin\frac{1}{2}\sqrt{k^2 + \alpha}\, t\right)\right). \end{aligned}$$

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Therefore $\pi j/\sqrt{k^2+\alpha}, j=\pm 1, \pm 2, \ldots$, are $k\cdot\mathbb{B}_J$-conjugate values of $\gamma(0)$ along γ.

Example 3. (Complex hyperbolic space $\mathbb{C}H^n(-\alpha)$). On a complex hyperbolic space $\mathbb{C}H^n(-\alpha)$ of holomorphic sectional curvature $-\alpha$ the feature of $k\cdot\mathbb{B}_J$-Jacobi fields $V=f\dot\gamma+gJ\dot\gamma+V^{\perp}$ depends on $|k|$. Along a normal $k\cdot\mathbb{B}_J$-trajectory γ they are expressed as follows:

$$f(t)=\begin{cases} a+k(b\cdot\cosh\sqrt{\alpha-k^2}\,t+c\cdot\sinh\sqrt{\alpha-k^2}\,t)+\alpha dt, & \text{if } |k|<\sqrt{\alpha},\\ a+(\alpha d-2c)t+\alpha\left(\dfrac{b}{2}t^2+\dfrac{c}{3}t^3\right) & \text{if } k=\pm\sqrt{\alpha},\\ a+k(b\cdot\cos\sqrt{k^2-\alpha}\,t+c\cdot\sin\sqrt{k^2-\alpha}\,t)+\alpha dt & \text{if } |k|>\sqrt{\alpha},\end{cases}$$

$$g(t)=\begin{cases} kd+\sqrt{\alpha-k^2}\,(b\cdot\sinh\sqrt{\alpha-k^2}\,t+c\cdot\cosh\sqrt{\alpha-k^2}\,t) & \text{if } |k|<\sqrt{\alpha},\\ k(d+bt+ct^2) & \text{if } k=\pm\sqrt{\alpha},\\ kd+\sqrt{k^2-\alpha}\,(-b\cdot\sin\sqrt{k^2-\alpha}\,t+c\cdot\cos\sqrt{k^2-\alpha}\,t) & \text{if } |k|>\sqrt{\alpha},\end{cases}$$

$$V^{\perp}(t)=\begin{cases} D\pi\left(\left(\tilde\gamma(t), e^{kit/2}\left(A\cdot\sinh\dfrac{1}{2}\sqrt{\alpha-k^2}\,t+B\cdot\cosh\dfrac{1}{2}\sqrt{\alpha-k^2}\,t\right)\right)\right) & \text{if } |k|<\sqrt{\alpha},\\ D\pi((\tilde\gamma(t), e^{kit/2}(A+tB))) & \text{if } k=\pm\sqrt{\alpha},\\ D\pi\left(\left(\tilde\gamma(t), e^{kit/2}\left(A\cdot\sin\dfrac{1}{2}\sqrt{k^2-\alpha}\,t+B\cdot\cos\dfrac{1}{2}\sqrt{k^2-\alpha}\,t\right)\right)\right) & \text{if } |k|>\sqrt{\alpha}.\end{cases}$$

Here $a,b,c,d\in\mathbb{R}$, and $A,B\in\mathbb{C}^{n+1}$ satisfy $\langle\!\langle A,\tilde\gamma(0)\rangle\!\rangle=\langle\!\langle A,\dot{\tilde\gamma}(0)\rangle\!\rangle=\langle\!\langle B,\tilde\gamma(0)\rangle\!\rangle=\langle\!\langle B,\dot{\tilde\gamma}(0)\rangle\!\rangle=0$. Here $\langle\!\langle\,,\rangle\!\rangle$ denotes the Hermitian form on $\mathbb{C}^{n+1}$ defined by $\langle\!\langle z,w\rangle\!\rangle=\ -z_0\bar w_0+\sum_{j=1}^n z_j\bar w_j$, $\pi: H_1^{2n+1}\to\mathbb{C}H^n$ is the standard S^1-fibration with $H_1^{2n+1}=\{z\in\mathbb{C}^{n+1}\mid\langle\!\langle z,z\rangle\!\rangle=-1\}$, and $\tilde\gamma$ is a horizontal lift of γ into H_1^{2n+1} (see [2]).

Therefore normal $k\cdot\mathbb{B}_J$-Jacobi fields along γ with $V(0)=0$ are given by

$$V(t)=\begin{cases} b\{k(\cosh\sqrt{\alpha-k^2}\,t-1)\cdot\dot\gamma(t)+\sqrt{\alpha-k^2}\cdot\sinh\sqrt{\alpha-k^2}\,t\cdot J\dot\gamma(t)\} & \\ \qquad +D\pi\left(\left(\tilde\gamma(t), A\cdot e^{kit/2}\cdot\sinh\dfrac{1}{2}\sqrt{\alpha-k^2}\,t\right)\right) & \text{if } |k|<\sqrt{\alpha}\\ b\{(\alpha t^2/2\cdot\dot\gamma(t)+kt\cdot J\dot\gamma(t))\}+D\pi\left((\tilde\gamma(t), B\cdot te^{kit/2})\right) & \text{if } k=\pm\sqrt{\alpha}\\ b\{k(1-\cos\sqrt{k^2-\alpha}\,t)\cdot\dot\gamma(t)+\sqrt{k^2-\alpha}\cdot\sin\sqrt{k^2-\alpha}\,t\cdot J\dot\gamma(t)\} & \\ \qquad +D\pi\left(\left(\tilde\gamma(t), A\cdot e^{kit/2}\cdot\sin\dfrac{1}{2}\sqrt{k^2-\alpha}\,t\right)\right) & \text{if } |k|>\sqrt{\alpha}.\end{cases}$$

We hence find that

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(1) if $|k| \leq \sqrt{\alpha}$, there are no $k \cdot \mathbb{B}_J$-conjugate points of $\gamma(0)$ along γ,

(2) if $|k| > \sqrt{\alpha}$, then $\pi j/\sqrt{k^2 - \alpha}, j = \pm 1, \pm 2, \ldots,$ are $k \cdot \mathbb{B}_J$-conjugate values of $\gamma(0)$ along γ.

As a direct consequence of our comparison theorem we obtain

Proposition 3. (1) *On a Kähler manifold M of curvature* $\mathrm{Riem}_M \leq -\alpha \leq 0$ *we have no $k \cdot \mathbb{B}_J$-conjugate points if $|k| \leq \sqrt{\alpha}/4$.*

(2) *On a Riemann surface M of curvature* $\mathrm{Riem}_M \leq -\alpha \leq 0$, *we have no $k \cdot Vol_M$-conjugate points if $|k| \leq \sqrt{\alpha}$.*

4. Magnetic exponential maps and the ideal boundary

The Hopf–Rinow theorem guarantees that on a complete Riemannian manifold there exists a minimal geodesic joining given distinct points. But for trajectories of magnetic fields such a type of theorem does not hold in general. In [2] the author showed that for any distinct points on $\mathbb{C}H^n(-\alpha)$ there are only two $k \cdot \mathbb{B}_J$-trajectories joining them if $|k| < \sqrt{\alpha}$. This suggests a Hope–Rinow type theorem holds for trajectories of weak strength magnetic fields on a negatively curved manifold. In this section we study some feature of trajectories for uniform magnetic fields on a Hadamard surface M.

In [3] we showed the following by applying Rauch's comparison theorem for Jacobi fields along geodesics which are perpendicular to a given trajectory: If the curvature of M satisfies $K_M \leq -\alpha \leq 0$, then any trajectory rays $\gamma|_{[0,\infty)}, \gamma|_{(-\infty,0]}$ for $k \cdot Vol_M$ ($|k| \leq \sqrt{\alpha}$) are unbounded, crosses once to every geodesic circle $S_r(\gamma(0)) = \{p \in M \mid d(p, \gamma(0)) = r\}$ of radius r centred with $\gamma(0)$, and is not tangent to these geodesic circles. We shall show the following by applying our comparison theorem for magnetic Jacobi fields.

Theorem 3. *Let M be a Riemann surface of curvature $K_M \leq -\alpha \leq 0$, and $\mathbb{B} = k \cdot Vol_M$ be a uniform magnetic field with $|k| \leq \sqrt{\alpha}$. Then the $\mathbb{B}$-exponential map $\mathbb{B}\exp_p : T_pM \to M$ is a covering map at any point $p \in M$.*

Proof. Let $\pi : \widetilde{M} \to M$ denotes the universal covering. If $\pi(\tilde{p}) = p$ then we have $\mathbb{B}\exp_p \circ D\pi = \pi \circ \mathbb{B}\exp_{\tilde{p}} : T_{\tilde{p}}\widetilde{M} \to M$. We hence suppose M is a Hadamard surface, and show that $\mathbb{B}\exp_p$ is a diffeomorphism at any point $p \in M$.

We first show that $\mathbb{B}\exp_p$ is injective. For given $u_0 \in U_pM$, we set

$$\mathscr{T} = \{t > 0 \mid \text{there exist } u \in U_pM\backslash\{u_0\} \text{ and } s > 0 \text{ with } \gamma_{u_0}(t) = \gamma_u(s)\}.$$

By our comparison theorem we find that $\mathscr{T}$ is open in $(0, \infty)$. Since $\mathbb{B}\exp_p$ is continuous, we get that $\mathscr{T}$ is closed in $(0, \infty)$, hence either $\mathscr{T} = (0, \infty)$ or $\mathscr{T}$ is empty. As $D\mathbb{B}\exp_p$ is non-degenerate at the origin, we get $\mathscr{T}$ is empty, and $\mathbb{B}\exp_p$ is injective.

We now show that $\mathbb{B}\exp_p$ is surjective. For each positive r, we consider the subset $S = \mathbb{B}\exp_p(T_pM) \cap S_r(p)$ of the geodesic circle $S_r(p)$ of radius r centred at p. By our comparison theorem on magnetic Jacobi fields, we find that S is an open subset of $S_r(p)$. On the other hand, since $\mathbb{B}\exp_p$ is continuous, S is closed in $S_r(p)$. As S is not empty we get $S = S_r(p)$ and have that $\mathbb{B}\exp_p$ is surjective. Since $D\mathbb{B}\exp_p$ does not degenerate we get our conclusion.

Remark. Let M be a Riemann surface of curvature $K_M \leq -\alpha \leq 0$. Consider the uniform magnetic field $k \cdot Vol_M$ with $|k| \leq \sqrt{\alpha}$. For any two points $p, q \in M$, we have at least two normal trajectories for $\mathbb{B}$ joining them. One is from p to q, and the other is from q to p. If we further assume M is simply connected, the number of joining normal $\mathbb{B}$-trajectories is two.

In the last stage we mention an asymptotical feature of trajectories for uniform magnetic fields on a Hadamard surface M. We denote its ideal boundary by $M(\infty)$. As we have seen in [3], if the curvature of M satisfies $K_M \leq -\alpha \leq 0$, then every trajectory γ for $k \cdot Vol_M$ ($|k| \leq \sqrt{\alpha}$) has points at infinity; $\gamma(\infty) = \lim_{t\to\infty} \gamma(t) \in M(\infty)$ and $\gamma(-\infty) = \lim_{t\to-\infty} \gamma(t) \in M(\infty)$. In the above Remark, by letting q tend to the ideal boundary we immediately get the following.

Corollary. *Let M be a Hadamard surface of curvature $K_M \leq -\alpha \leq 0$. Consider the uniform magnetic field $\mathbb{B} = k \cdot Vol_M$ with $|k| \leq \sqrt{\alpha}$. For any points $p \in M$ and $y \in M(\infty)$, we have two and only two normal trajectories γ, σ for $\mathbb{B}$ such that $\gamma(0) = \sigma(0) = p$ and $\gamma(\infty) = \sigma(-\infty) = y$.*

Proof. When $k = 0$ this is a classical result on a Hadamard manifold, so we suppose $\alpha > 0$. We shall show the uniqueness. Suppose there are two distinct normal trajectories γ_0, γ_1 with $\gamma_0(0) = \gamma_2(0) = p$ and $\gamma_0(\infty) = \gamma_1(\infty) = y$. By Theorem 3 and by the theorem for Jordan curves we can conclude the strip Δ between γ_0 and γ_1 consists of a family of normal trajectories $\{\gamma_s\}_{0\leq s\leq 1}$ for $\mathbb{B}$ with $\gamma_s(0) = p$ and $\gamma_s(\infty) = y$. We compare this strip with the geodesic strip $\{\lambda\dot{\gamma}_s(0) \mid \lambda \geq 0, 0 \leq s \leq 1\}$ in $\mathbb{R}^2 \simeq T_pM$. Regarding the magnetic exponential map $\mathbb{B}\exp_p$ as a polar coordinate we find by Proposition 2 that the strip Δ has an infinite area. Now join $\gamma_0(t)$ and $\gamma_1(t)$ by a geodesic segment ρ_t, and consider the triangle $\Delta_t = \Delta(p, \gamma_0(t), \gamma_1(t))$. Clearly they satisfy $\Delta_t \subset \Delta_{t'} \subset \Delta$ if $t < t'$. By the Gauss–Bonnet Theorem the set of their areas $\{\mathrm{Area}(\Delta_t)\}_t$ is bounded, therefore the distance $d(p, \rho_t)$ from p to ρ_t is bounded with respect to t. We hence obtain that $\{\rho_t\}$ converges to a geodesic ρ with $d(p, \rho) < \infty$ as t goes to infinity (c.f. p. 370 of [9]). Since $\rho(\infty) = \rho(-\infty) = y$, this is a contradiction.

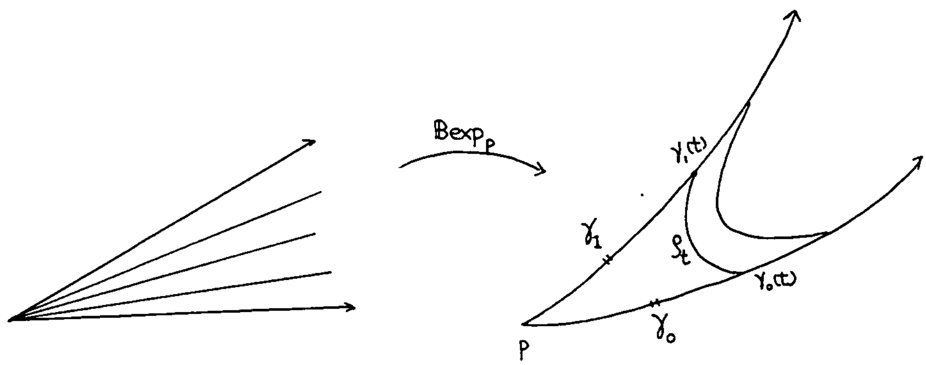

FIGURE 1

We define maps $\Phi_p^+ : U_pM \to M(\infty)$ and $\Phi_p^- : U_pM \to M(\infty)$ by

$$\Phi_p^+(u) = \gamma_u(\infty) \quad \text{and} \quad \Phi_p^-(u) = \gamma_u(-\infty),$$

and set $\Phi^+ : UM \to M(\infty)$ and $\Phi^- : UM \to M(\infty)$ by $\Phi^+(u) = \Phi^+_{\tau(u)}(u)$ and $\Phi^-(u) = \Phi^-_{\tau(u)}(u)$. The above Corollary implies that Φ_p^+ and Φ_p^- are homeomorphism with respect to the cone topology on $M(\infty)$.

We now consider normal trajectories joining two point on the ideal boundary. On a hyperbolic plane $H^2(-\alpha)$ of curvature $-\alpha$, the asymptotic behaviour of normal trajectories γ for $k \cdot Vol_{H^2(-\alpha)}$ are well known;

(1) when $k = \pm\sqrt{\alpha}$, they are horocyclic, hence $\gamma(\infty) = \gamma(-\infty)$,

(2) when $|k| < \pm\sqrt{\alpha}$, for every distinct points x, y on the ideal boundary there is a unique normal trajectory γ with $\gamma(\infty) = x$, $\gamma(-\infty) = y$.

We show the second property holds for general Hadamard surfaces of bounded negative curvature.

Theorem 4. *Let M be a Hadamard surface of bounded negative curvature $K_M \leq -\alpha \leq 0$. Consider the uniform magnetic field $k \cdot Vol_M$ with $|k| < \sqrt{\alpha}$. For distinct points $x, y \in M(\infty)$ we have a unique normal trajectory γ with $\gamma(\infty) = x$ and $\gamma(-\infty) = y$.*

Proof. We first show the existence of such a trajectory. Since $|k| < \sqrt{\alpha}$, we can find a positive ε satisfying the following; if a $k \cdot Vol_M$-trajectory γ and a geodesic ρ satisfies $\gamma(0) = \rho(0) = p$ and $\gamma(\infty) = \rho(\infty)$ then the angle $\angle(\dot\gamma(0), \dot\rho(0))$ of these curves at p is not greater than $\pi - \varepsilon$. Now choose a point $p \in M$ so that the angle $\angle(\dot\mu_1(0), \dot\mu_2(0))$ of the geodesics μ_1, μ_2 with $\mu_i(0) = p$, $\mu_1(\infty) = x$ and $\mu_2(\infty) = y$ at p is smaller than ε (see the following figure). We choose normal trajectories γ_1, γ_2 for $k \cdot Vol_M$ so that $\gamma_1(0) = \gamma_2(0) = p, \gamma_1(\infty) = x$ and $\gamma_2(-\infty) = y$. Then the angle $\angle(\dot{\gamma}_1(0), \dot{\gamma}_2(0))$ of these trajectories at p is smaller than π. Let $\sigma_t, \rho_t (t > 0)$ be the normal trajectory and the geodesic from $\gamma_2(-t)$ to $\gamma_1(t)$ (see the figure). The trajectory σ_t does not cross to γ_i except at these points. We shall show that the distance $d(p, \sigma_t)$ from p to σ_t is bounded with respect to t. Since the angle $\angle(\dot{\gamma}_1(0), \dot{\gamma}_2(0))$ is smaller than π, we find that $d(p, \sigma_t) \le d(p, \rho_t)$. Each geodesic ρ_t crosses to the geodesic μ_i at a point $q_{i,t}$. When one of the sets $\{q_{1,t}\}, \{q_{2,t}\}$ is bounded, it is clear that $d(p, \sigma_t)$ is bounded. When both sets are unbounded, ρ_t converges to the geodesic ρ with $\rho(\infty) = x$ and $\rho(-\infty) = y$. Hence $d(p, \sigma_t)(< d(p, \rho_t) < d(p, \rho))$ is bounded. We therefore find that the sequence of tangent vectors $\{v_n\}$ to σ_n at the base point of the perpendicular from p to σ_n possesses an accumulation point v. The trajectory γ with $\dot{\gamma}(0) = v$ satisfies $\gamma(\infty) = x$ and $\gamma(-\infty) = y$.

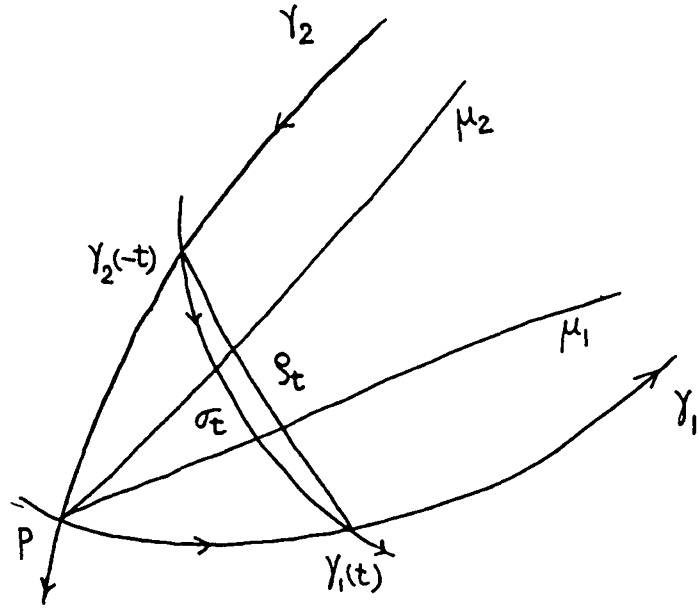

FIGURE 2

Secondly we show uniqueness. Suppose there exist two normal trajectories $\gamma, \hat{\gamma}$ satisfying $\gamma(\infty) = \hat{\gamma}(\infty) = x$ and $\gamma(-\infty) = \hat{\gamma}(-\infty) = y$. By the above Corollary we find that the strip between these trajectories is constructed by a family of normal trajectories. By Proposition 2 we get that this strip has infinite area. By the same argument as the Corollary we can conclude it is a contradiction.

Remark. Let M be a Hadamard surface of bounded negative curvature $K \le -\alpha < 0$. Consider the uniform magnetic field $\mathbb{B} = k \cdot Vol_M$ with $|k| < \sqrt{\alpha}$. For given $u_0 \in UM$ and positive ε, we find positive δ such that if $d(u, u_0) < \delta$ then the following hold;

(1) there is $v(u) \in UM$ with $d(v(u), u_0) < \varepsilon$, $\Phi^+(v(u)) = \Phi^+(u_0)$ and $\Phi^-(v(u)) = \Phi^-(u)$,

(2) there is $w(u) \in UM$ with $d(w(u), -u_0) < \varepsilon$, $\Phi^-(w(u)) = \Phi^-(u_0)$ and $\Phi^+(w(u)) = \Phi^+(u)$.

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