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Flow equivalence of subshifts of finite type

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Abstract. A complete set of computable invariants is given for deciding whether two irreducible subshifts of finite type have topologically equivalent suspension flows.

0. *Introduction*

One of the fundamental tools in the qualitative study of dynamical systems has been symbolic dynamics. Subshifts of finite type (see § 1 for a definition) play a central role, as they occur quite frequently in smooth dynamical systems. In many instances a smooth flow will possess a compact invariant set Λ which admits a cross section Σ_1, and the first return map under the flow $\rho_1:\Sigma_1\to\Sigma_1$ will be topologically conjugate to a subshift of finite type $\sigma_A:\Sigma_A\to\Sigma_A$ corresponding to a non-negative irreducible matrix A. The choice of a different cross-section Σ_2 with different return map $\rho_2:\Sigma_2\to\Sigma_2$ may result in a quite different subshift of finite type $\sigma_B:\Sigma_B\to\Sigma_B$. However in this case σ_A and σ_B or the matrices A and B are said to be *flow equivalent.* Alternatively the non-negative matrices A and B are flow equivalent when the corresponding subshifts σ_A and σ_B have topologically equivalent suspension flows (see § 1 for definitions). The case of primary interest is when the matrices are *irreducible* (this is equivalent to the existence of a dense forward orbit in the suspension flow). A special case of rather little interest occurs with permutation matrices whose corresponding flows consist of a single periodic orbit. We call this the *trivial flow equivalence class.*

The main result of this article is a simple, easily computed algebraic characterization of flow equivalence for irreducible matrices not in the trivial flow equivalence class.

THEOREM. *Suppose that A and B are non-negative irreducible integer matrices neither of which is in the trivial flow equivalence class. The matrices A and B are flow equivalent if and only if:*

$$\det(I_n-A)=\det(I_m-B)$$

and

$$\mathbb{Z}^n/(I_n-A)\mathbb{Z}^n\cong\mathbb{Z}^m/(I_m-B)\mathbb{Z}^m,$$

where n and m are the sizes of A and B respectively and I_n and I_m are identity matrices.

The necessity of these conditions was proved in [**3**] and [**1**]. Parry and Sullivan gave

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generators for flow equivalence in [**3**] and proved that $\det(I-A)$ is an invariant. In [**1**] it was shown that $\mathbb{Z}^n/(I-A)\mathbb{Z}^n$ is also an invariant. Notice that this invariant contains all of the information of $\det(I-A)$ except its sign, so that one could replace $\det(I-A)$ and $\det(I-B)$ in the theorem by $\operatorname{sgn}\det(I-A)$ and $\operatorname{sgn}\det(I-B)$.

The new content of this article is the converse direction. Namely if $\det(I-A)=\det(I-B)$ and $\mathbb{Z}^n/(I-A)\mathbb{Z}^n \cong \mathbb{Z}^m/(I-B)\mathbb{Z}^m$ then A and B are flow equivalent. In the process we give canonical forms for representatives of a flow equivalence class (theorem 3.3).

1. *Background and definitions*

If A is an n by n non-negative integer matrix we can form an oriented graph Γ with n vertices and a_{ij} edges joining vertex i to vertex j and oriented from i to j. It is important to know when any vertex can be joined to any other by an oriented path on the graph. This property is called irreducibility and it is not difficult to show that in terms of the matrix it is given by the following criterion.

(1.1) *Definition.* A non-negative square integer matrix A is called *irreducible* provided that for each i, j with $1 \le i \le n$ and $1 \le j \le n$, there is an $N > 0$ such that the ijth entry of A^N is not zero.

Given A and its graph Γ we can construct the corresponding subshift of finite type which we now define. Let $E=\{e_i\}$ be the set of edges of Γ. We give E the discrete topology and consider the compact zero dimensional space $\prod_{-\infty}^{\infty} E$ of sequences of elements of E indexed by the integers. We define a subset $\Sigma_A \subset \prod_{-\infty}^{\infty} E$ by saying $\underline{e}=(\ldots, e_{-1}, e_0, e_1, \ldots)$ is in Σ_A provided for each i, the oriented edge e_i ends at the vertex of Γ where e_{i+1} begins. (An edge of Γ is allowed to begin and end at the same vertex.) It is easy to see that Σ_A is a closed subset of $\prod_{-\infty}^{\infty} E$ and that it is invariant under the shift homeomorphism

$$\sigma: \prod_{-\infty}^{\infty} E \to \prod_{-\infty}^{\infty} E$$

given by $\sigma(\underline{e})=\underline{f}$ where $f_i = e_{i-1}$ for all i.

(1.2) *Definition.* The homeomorphism $\sigma_A : \Sigma_A \to \Sigma_A$ given by $\sigma_A = \sigma|_{\Sigma_A}$ is called the *subshift of finite type* associated to the matrix A.

For further details on this and subsequent definitions in this section see chapter 3 and appendix A of [**2**].

An important question, answered by Williams [**4**], is when do two matrices give rise to subshifts which are topologically conjugate.

(1.3) THEOREM (Williams). *Suppose A and B are non-negative square matrices. The corresponding subshifts of finite type σ_A and σ_B are topologically conjugate if and only if there are (not necessarily square) matrices R_i, S_i, $1 \le i \le n$ such that*

$$A = R_1 S_1, \quad B = S_n R_n, \quad \textit{and} \quad S_i R_i = R_{i+1} S_{i+1}$$

for all $1 \le i \le n-1$.

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Two matrices A and B which satisfy this condition are said to be *strong shift equivalent.*

We are concerned here with flows admitting a cross-section whose return map is a subshift of finite type. To construct (abstractly) such a flow we use the following construction.

(1.4) *Definition.* If $h: X \to X$ is a homeomorphism then its *suspension flow* (also called its *mapping torus flow*) is defined on the identification space

$$Y = X \times \mathbb{R}/(x, s+1) \sim (h(x), s),$$

and is defined to be the flow on Y induced by the flow ϕ_t on $X \times \mathbb{R}$ given by $\phi_t(x, s) = (x, s+t)$.

Recall that two flows ϕ_t on X and ϕ'_t on X' are said to be *topologically equivalent* provided there is a homeomorphism $h: X \to X'$ which carries orbits of ϕ_t onto orbits of ϕ'_t and preserves their orientation.

(1.5) *Definition.* If A and B are non-negative square integer matrices they are *flow equivalent* provided the suspension flows of the subshifts of finite type σ_A and σ_B are topologically equivalent.

The following result of Parry and Sullivan gives a characterization of this equivalence relation on matrices.

(1.6) THEOREM (Parry & Sullivan [**3**]). *The equivalence relation of flow equivalence is generated by a strong shift equivalence and the relation*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ a_{11} & 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

This means that if A and B are flow equivalent there exist matrices $A_1 = A$, $A_2, \ldots, A_m = B$ such that for each i either A_i is strong shift equivalent to A_{i+1} or else A_i has the form of one of the matrices above and A_{i+1} has the form of the other.

In subsequent sections we want to alter a matrix to a canonical form while keeping it in the same flow equivalence class. One operation we shall use is to change between two matrices of the form indicated in theorem (1.6) and another comes from an observation of Williams.

(1.7) PROPOSITION (Williams). *The following two matrices are strong shift equivalent:*

$$\begin{pmatrix} a_1 & a_1 & a_2 & \cdots & a_n \\ b_1 & b_1 & b_2 & \cdots & b_n \\ c_1 & c_1 & c_2 & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} a_1+b_1 & a_2+b_2 & \cdots & a_n+b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}.$$

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Proof. Let R be the $(n+1)$ by n matrix

$$\begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ c_1 & \cdots & c_n \\ \cdots & \cdots & \cdots \end{pmatrix}$$

and let S be the n by $(n+1)$ matrix

$$\left(\begin{array}{c|c} 1 & \\ 0 & \\ \vdots & I \\ 0 & \end{array}\right).$$

Then RS is the first of the matrices above and SR is the second. □

The operation of replacing the right hand matrix in (1.7) with the left hand one will be referred to as *splitting* the first row and *replicating* the first column. Of course we could just as well split row j and replicate column j. In fact permuting the rows and then doing the *same* permutation to the columns corresponds to renaming the vertices of the graph and does not change flow equivalence class. Likewise since the transpose matrix corresponds to the inverse flow we can split a column and replicate the corresponding row.

(1.8) *Remark.* If we are given non-negative integer matrices R and S, then, by (1.3), $A=RS$ and $B=SR$ are strong shift equivalent (and hence flow equivalent). However even though A is strictly positive it may happen that B has an entire row (or column) of zeros; e.g.

$$A=(1 \quad 2 \quad 3)\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}=(7),$$

but the product in the reverse order is

$$B=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 4 & 6 \end{pmatrix}.$$

The fact that the second row has all entries zero means that in the graph for B there are no edges emanating from vertex 2. Thus no edge ending in vertex 2 can ever appear in one of the sequences in Σ_B. It follows that σ_B is the same as $\sigma_{B'}$, where

$$B'=\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

is obtained by removing the row of zeros from B and the corresponding column.

2. *Row and column operations*

In this section we develop some basic operations on matrices which preserve flow equivalence and will be used to achieve canonical forms for flow equivalence.

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(2.1) LEMMA. *If $A=(a_{ij})$ is a non-negative square integer matrix and $a_{12}>0$, then A is flow equivalent to*

$$\begin{pmatrix} a_{11}+a_{21} & a_{12}+a_{22}-1 & a_{13}+a_{23} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Proof. Starting with A we split the first row and replicate the first column (see (1.7) and the remarks following its proof) to obtain

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \cdots \\ a_{11} & a_{11} & a_{12}-1 & a_{13} & \cdots \\ a_{21} & a_{21} & a_{22} & a_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Next, using the same sort of operation on this matrix, we split the third column and replicate the third row to obtain

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots \\ a_{11} & a_{11} & 0 & a_{12}-1 & a_{13} & \cdots \\ a_{21} & a_{21} & 0 & a_{22} & a_{23} & \cdots \\ a_{21} & a_{21} & 0 & a_{22} & a_{23} & \cdots \\ a_{31} & a_{31} & 0 & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

After switching rows and columns 2 and 3 we can apply the Parry–Sullivan operation from (1.6) to remove the first row and second column to obtain

$$\begin{pmatrix} a_{21} & a_{21} & a_{22} & a_{23} & \cdots \\ a_{11} & a_{11} & a_{12}-1 & a_{13} & \cdots \\ a_{21} & a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Finally we notice that this matrix is exactly the result of taking the matrix

$$\begin{pmatrix} a_{11}+a_{21} & a_{12}-1+a_{22} & a_{13}+a_{23} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and splitting the first row and replicating the first column. The only thing to check is that at all stages the matrices were non-negative and this is the case since $a_{12}>0$. □

(2.2) COROLLARY. *If $A=(a_{ij})$ is a non-negative square integer matrix and $a_{pq}>0$ $p\neq q$ then A is flow equivalent to the matrix obtained by subtracting 1 from a_{pq} and*

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adding row q to row p. Likewise A is flow equivalent to the matrix obtained by subtracting 1 from a_{pq} and adding column p to column q.

Proof. The first assertion is proved exactly as in (2.1). (Alternatively one could conjugate both matrices in the conclusion of (2.1) by a permutation matrix P which moves row and column 1 to p and row and column 2 to q since A and $P^{-1}AP$ are strong shift equivalent.)

The second assertion follows because the transpose A^t corresponds to the inverse flow to the flow to which A corresponds. Hence two matrices are flow equivalent if and only if their transposes are. □

(2.3) COROLLARY. *If $A=(a_{ij})$ is a non-negative square integer matrix then A is flow equivalent to a non-negative integer matrix A' all of whose diagonal entries are non-zero.*

Proof. Suppose A has a zero diagonal entry; without loss of generality we can assume it is a_{11}. We will show that A is flow equivalent to a matrix whose size is smaller than that of A. If this new matrix has a zero diagonal entry the process can be repeated to get a still smaller matrix. We repeat, several times if necessary, until all diagonal entries are non-zero.

We show that A is flow equivalent to a smaller matrix (under the assumption that $a_{11}=0$) by altering it within its flow equivalence class until the first column is zero. We do this as follows: if $a_{21}>0$ we use (2.2) to subtract 1 from it and add row 1 to row 2. The 21 entry of this new matrix is $a_{21}-1$ since $a_{11}=0$. We repeat this process until the 21 entry is 0 and then proceed to do likewise to all other non-zero entries in the first column. The resulting matrix, since its first column is zero, represents the same subshift of finite type as the matrix obtained by deleting the first row and column (see (1.8)). □

In view of this corollary we will begin to restrict our attention largely to matrices with non-zero diagonal entries. The advantage of such matrices is that they can be written in the form $B+I$, where I is the identity matrix of the appropriate size and B is non-negative. We come now to the major tool which will be used to establish canonical forms. This is the first place we make essential use of irreducibility of the matrices we consider.

(2.4) THEOREM. *Suppose B is a non-negative square integer matrix and $A=B+I$ is irreducible. Then if B' is obtained from B by adding any row to a different row or adding any column to a different column, the matrix A is flow equivalent to $B'+I$.*

Proof. Consider the case of adding row q to row p. All rows except the pth will be identical for A and $B'+I$. Row p of A is

$$(a_{p1}, a_{p2}, \ldots, a_{pn}) = (b_{p1}, \ldots, b_{pp}+1, \ldots, b_{pq}, \ldots, b_{pn}).$$

Row p of $B'+I$ is

$$\begin{aligned}&(b_{p1}+b_{q1}, \ldots, b_{pp}+b_{qp}+1, \ldots, b_{pq}+b_{qq}, \ldots, b_{pn}+b_{qn})\\ &=(a_{p1}+a_{q1}, \ldots, a_{pp}+a_{qp}, \ldots, a_{pq}+a_{qq}-1, \ldots, a_{pn}+a_{qn}),\end{aligned}$$

since $a_{pp}=b_{pp}+1$ and $a_{qq}-1=b_{qq}$.

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But now (2.1) and (2.2) say that provided a_{pq} was non-zero, A and $B'+I$ are flow equivalent. Thus whenever b_{pq} (which equals a_{pq}) is non-zero we can add row q of B to row p forming B' and $(B+I)$ will be flow equivalent to $(B'+I)$. Saying this another way: if we are given B' we can subtract row q from row p to form B and provided B is non-negative and $b_{pq}>0$ (after the subtraction) we will have $(B+I)$ flow equivalent to $(B'+I)$.

In the case $a_{pq}=0$ we use the irreducibility of A as follows. There is a non-zero element of row p, say a_{pk}. Since A is irreducible there is a sequence $i_0=p, i_1, \ldots, i_m=q$ such that for $0\le j<m$, the $i_{j+1}i_j$ entry of A is non-zero and all the i_j are distinct. We choose the sequence $\{i_j\}$ to be of minimal length. That such sequences exist follows easily from the criterion for irreducibility in terms of the graph Γ of A mentioned at the beginning of § 1. Note that this implies A is irreducible if and only if B is.

Now since the i_1 element of row p is non-zero we can add row i_1 to row p. The new row will be non-zero in the i_2 position since the i_2 position in row i_1 was non-zero. Hence we can now add row i_2 to row p which will make it non-zero in the i_3 position etc. None of the i_j except i_0 will equal p since the sequence was chosen to have minimal length. Since $i_m=q$ we will end up with a non-zero entry in position q of row p. We can then add row $i_m=q$ to row p and begin to subtract the auxiliary rows we added on. First subtract row i_{m-1}; the i_{m-1} entry of row p is still non-zero. We then subtract row i_{m-2}, then i_{m-3} etc.

Since we subtract these rows in the reverse order to that in which we added them, the appropriate entry of row p is non-zero when we need it to apply (2.2). Hence at each stage we do not change the flow equivalence class of the matrix plus I. When we have subtracted row i_1 the net effect is having formed B' by adding row q of B to row p.

Since two matrices are flow equivalent if and only if their transposes are, the analogous result for columns also follows. □

(2.5) COROLLARY. *Let B be an n by n non-negative irreducible matrix with $n>1$. If $A=B+I$ and $N>0$, there is an n by n matrix A', each entry of which is greater than N, and which is flow equivalent to A.*

Proof. For some $1\le j\le n$ there is an i with $b_{ij}>0$. If we add row i of B to each row except i of B and repeat this N times we will have each entry except i of column j of B greater than N. Now adding one row other than i to row i makes every entry of column j greater than N.

If we form B' by adding the new column j to every other column then every entry of B' is greater than N. By (2.4) $A'=B'+I$ is flow equivalent to $A=B+I$. □

(2.6) COROLLARY. *Let A be a non-negative irreducible matrix which is not in the trivial flow equivalence class. There exists $N>0$ such that for any $n>N$, there is a strictly positive n by n matrix B' with $(B'+I)$ flow equivalent to A.*

Proof. By (2.3) A is flow equivalent to a matrix A_0 all of whose diagonal entries are non-zero. The hypothesis that A is not in the trivial flow equivalence class

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guarantees that A_0 is not the 1 by 1 matrix (1). Hence we can write $A_0=B_0+I$ where B_0 is non-negative and irreducible. Applying (2.5) we know that $A_0=B_0+I$ is flow equivalent to $A_1=B_1+I$, where every entry of B_1 is $\geq 2^n$. Thus applying (1.7) we can split the first row and replicate the first column of A_1 and repeat the process a sufficient number of times to obtain a matrix A' of size n by n with the desired properties. □

Note that if the matrix A in the above corollary is the 1 by 1 matrix (1) the conclusion is false. This is the reason for the requirement of the main theorem that A should correspond to a non-trivial flow.

We describe now one further operation which we will use to alter matrices within their flow equivalence class. It may be that in a matrix $B=(b_{ij})$ we have $b_{ps}<b_{qs}$ and we would like to subtract row p from row q to obtain $b_{qs}-b_{ps}$ in position qs. We can only do this if each entry of row q is greater than or equal to the corresponding entry of row p since otherwise we would have negative entries. However if no negative entries result from the subtraction and the resulting matrix B' is irreducible then (2.4) asserts that $(B+I)$ and $(B'+I)$ are flow equivalent. To deal with the problem that an entry of row q, say b_{qr}, is less than the corresponding entry b_{pr} of row p we can do the following:

Choose m such that $m(b_{qs}-b_{ps})>b_{pr}-b_{qr}$. Now add column s of B to column r, m times. The new entry in position qr is $b_{qr}+mb_{qs}$, while the entry in position pr is $b_{pr}+mb_{ps}$ which is strictly smaller.

If we do this for all the columns where it is necessary we will be able to subtract (the new) row p from (the new) row q to obtain $b_{qs}-b_{ps}$ in position qs. Notice that since we have created no negative entries and in fact no new zero entries at any stage, the matrix at every stage is irreducible. We summarize this process in the following definition.

(2.7) *Definition.* If B is a non-negative irreducible matrix with $b_{qs}>b_{ps}$ we will call the following process a *column augmentation and row subtraction.* First add column s to every column whose qth element is $\leq$ its pth element. Do this a sufficient number of times such that each element of row q is greater than the corresponding element of row p. Then subtract row p from row q. A *row augmentation and column subtraction* is defined similarly.

The remarks prior to this definition together with (2.4) make the following lemma immediate.

(2.8) LEMMA. *If B' is the result of performing a column augmentation and row subtraction on an irreducible matrix B then $(B+I)$ is flow equivalent to $(B'+I)$. The analogous result for row augmentation and column subtraction is also valid.*

As an application of this operation we will obtain the following result which is the first step in getting canonical forms.

(2.9) PROPOSITION. *Let $A=B+I$ be an n by n integer matrix with B strictly positive. Given j with $1\leq j\leq n$, there is a strictly positive n by n matrix B' such that $(B'+I)$*

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is flow equivalent to A and every entry of the j'th column of B' equals d, where d is the greatest common divisor of the entries of B and the greatest common divisor of the entries of B'.

Proof. We give an algorithm in terms of the previously defined operations for producing the desired matrix B'.

Step 1. Choose the smallest element of the current matrix (B to begin with). Suppose it is in position pq, and has value d_0.

Step 2. For the first element of column q perform a column augmentation and row subtraction (see (2.7)), subtracting row p from the row 1, decreasing the entry in the $1q$ position by d_0. (If $p=1$ start with 2.) Repeat this until the entry in position $1q$ is $\leq d_0$. If this entry or any other is $<d_0$ return to step 1 and start again with the new matrix.

Step 3. Repeat step 2 for each entry of column q except the pth.

Step 4. When this step is reached the matrix will have a column (called q) each entry of which is d, where $d>0$ is the minimal entry of the matrix. If d is the g.c.d. of the entries of the matrix proceed to step 5. Otherwise there is an entry, say in position rs, which is greater than d but not a multiple of d. By means of a row augmentation and column subtraction, subtracting column q from s we can decrease the rs entry by d. Repeating this as often as necessary we end up with an entry $d'>0$ which is $<d$. Now return to step 1 and start again.

Step 5. When this stage is reached the matrix will contain a column (called q) each element of which is d, the g.c.d. of all the elements of the matrix. Conjugating by a permutation matrix move column q to position j. This is the matrix B'.

Notice that the algorithm is finite and must terminate since the maximal number of times it is possible to start again with step 1 is the original $d_0=$ minimal element of B. Since all changes made in going from B to B' were of the type considered in lemma (2.8) or conjugating by a permutation matrix we have that $(B+I)$ is flow equivalent to $(B'+I)$. Finally, since the changes were also all ultimately combinations of ordinary elementary row and column operations, the g.c.d. of the entries of B equals the g.c.d. of the entries of B'. □

3. *Canonical forms*

We recall from elementary algebra that any n by n integer matrix A can, by standard row and column operations, be diagonalized. This diagonal form can be arranged so that for each i the i'th diagonal entry d_i divides the $(i+1)$'st, d_{i+1}. The d_i so constructed are called the *elementary divisors* of A and d_1 is the g.c.d. of the entries of A. Also the group $\mathbb{Z}^n/A(\mathbb{Z}^n)$ is isomorphic to $\bigoplus_{i=1}^n \mathbb{Z}/(d_i)$.

(3.1) PROPOSITION. *Suppose B is a strictly positive* 2 *by* 2 *matrix with elementary divisors* d_1 *and* d_2. *Then* $A=B+I$ *is flow equivalent to the* 2 *by* 2 *matrix* $(I+B')$, *where*

$$B'=\begin{pmatrix} 0 & d_2 \\ d_1 & 0 \end{pmatrix} \qquad \textit{if } \det(I-A)<0,$$

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and

$$B' = \begin{pmatrix} d_1 & d_1 \\ d_1 & d_1 + d_2 \end{pmatrix} \qquad \textit{if } \det(I - A) \geq 0.$$

Proof. Let d_1 = g.c.d. of the entries of B. By (2.9) A is flow equivalent to a matrix $I + B_0$ where

$$B_0 = \begin{pmatrix} d_1 & x \\ d_1 & y \end{pmatrix},$$

and $x, y > 0$.

If $\det(B_0) = \det(I - A) < 0$, then $x > y$. Hence we can apply (2.4) to conclude A is flow equivalent to $(I + B_1)$ where B_1 is obtained by subtracting row 2 of B_0 from row 1. Thus

$$B_1 = \begin{pmatrix} 0 & x - y \\ d_1 & y \end{pmatrix}.$$

Since d_1 divides y, we can now form

$$B_2 = \begin{pmatrix} 0 & z \\ d_1 & 0 \end{pmatrix}$$

by repeatedly subtracting column 1 from column 2. Clearly $z = x - y$ is an elementary divisor of B_2 and hence of B, so $z = d_2$ and $B' = B_2$.

If $\det(B_0) = \det(I - A) \geq 0$, we note that $y \geq x$. Since d_1 divides x and $d_1 \leq x$ we can form

$$B_1 = \begin{pmatrix} d_1 & d_1 \\ d_1 & z \end{pmatrix}$$

by repeatedly subtracting column 1 from column 2. By (2.4) A is flow equivalent to $(I + B_1)$ and using standard row and column operations to diagonalize B_1 shows $z = d_1 + d_2$. Thus $B' = B_1$ is the desired matrix. □

(3.2) PROPOSITION. *Suppose B is a strictly positive n by n matrix of rank 1. Then $(B + I)$ is flow equivalent to the n by n matrix $(B' + I)$ where B' is the matrix each entry of which equals d, the g.c.d. of the entries of B.*

Proof. Applying (2.9) we can assume that each entry of the first column of B is d. Since B has rank one and is strictly positive, every other column is a positive multiple. Hence repeated subtraction of the first column from the others will transform B to the desired B'. By (2.4) the matrices $(B + I)$ and $(B' + I)$ will be flow equivalent. □

We can now give essentially canonical forms for flow equivalence classes of n by n matrices. There are three different forms depending on whether $\det(I - A)$ is positive, negative or zero.

(3.3) THEOREM. *Suppose that B is an n by n, $n > 1$, strictly positive matrix with elementary divisors $d_1, \ldots, d_n$, each d_i a factor of d_{i+1}. If $A = I + B$ then A is flow* *equivalent to* $(I+B')$ *where*

$$B' = \begin{pmatrix} 0 & & \cdots & 0 & d_n \\ d_1 & 0 & & & 0 \\ 0 & d_2 & & & \\ & & \ddots & & \vdots \\ 0 & \cdots & & d_{n-1} & 0 \end{pmatrix} \quad \text{if } \det(I-A)<0,$$

$$B' = \begin{pmatrix} 0 & \cdots & 0 & d_{n-1} & d_{n-1} \\ d_1 & 0 & & 0 & 0 \\ 0 & d_2 & & \vdots & \vdots \\ & & \ddots & & \\ & & & 0 & 0 \\ 0 & \cdots & 0 & d_{n-1} & d_{n-1}+d_n \end{pmatrix} \quad \text{if } \det(I-A)>0,$$

and

$$B' = \left(\begin{array}{cccc|c} 0 & \cdots & & 0 & d_k \cdots d_k \\ d_1 & 0 & & & 0 \cdots 0 \\ 0 & d_2 & & & \\ & & \ddots & & \\ & & & d_{k-1} & \\ \hline & & 0 & & D \end{array}\right) \quad \text{if } k = \operatorname{rank}(I-A) \text{ is less than } n.$$

The matrix D *is the* $(n-k)$ *by* $(n-k+1)$ *matrix each entry of which is* d_k.

Proof. Our proof is by induction on n. We suppose inductively that any matrix B of size less than or equal to n can be put in the desired form using only addition (and subtraction) of rows and columns without ever leaving the class of non-negative irreducible matrices (and hence by (2.4) not changing the flow equivalence class of $(B+I)$). By (3.1) and its proof, this is true for matrices of size 2 by 2, so this starts the induction.

We now consider a matrix B of size $n+1$ and suppose first that

$$(-1)^{n+1}\det(B) = \det(I-A) < 0.$$

By (2.9) we can alter B using only row and column addition and subtractions and never leaving strictly positive matrices to obtain a matrix B_1 whose first column has every entry d_1. We would like to achieve a zero entry in the first column leaving all other first column entries d_1 and the rest of the matrix strictly positive. To do this we note that since the rank of the matrix is >1 there is a column, say s, with two distinct entries, say entry ps is $>$ entry qs. We now add column s to every other column, except 1, a sufficient number of times such that every entry of row p except the first is greater than the corresponding entry of row q. If we now subtract row q from row p we have achieved a matrix with 0 in position $p1$, d_1 in every other entry of the first column and strictly positive elsewhere.

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We would like to have column 1 consist of d_1 in position 21 and zeros elsewhere. Suppose $p \neq 2$; then if row j is any row other than 2 or p, we can add row p to row j a sufficient number of times such that each entry except the first of row j is greater than the corresponding entry of row 2. The first entries of both row j and row 2 are d_1; hence if we subtract row 2 from row j we will have a 0 in the jth entry of the first column. Repeating this for all choices of j results in a matrix with d_1 in position 21, zeros in the rest of the first column and strictly positive elsewhere. If p is 2 we use row 3 instead of 2 (there are at least 3 rows!) then add row 3 to row 2; add row 1 to row 3 a sufficient number of times to be able to subtract row 2 from row 3 and again we have d_1 in position 21 and zeros elsewhere in the first column.

We would now like to get zeros in all entries of row 2 except the first which should remain d_1. But this is easy since d_1 is still the g.c.d. of all entries of our current matrix so each element of row 2 is a multiple of d_1. Our matrix looks like

$$\begin{pmatrix} 0 & x_{11} & \cdots & x_{1n} \\ d_1 & m_1 d_1 & \cdots & m_n d_1 \\ 0 & x_{21} & \cdots & x_{2n} \\ & \cdots & \cdots & \\ 0 & x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

where each of the x's is strictly positive. Hence subtracting column 1 from each other column an appropriate number of times results in

$$B_2 = \begin{pmatrix} 0 & x_{11} & \cdots & x_{1n} \\ d_1 & 0 & \cdots & 0 \\ 0 & x_{21} & \cdots & x_{2n} \\ & \cdots & \cdots & \\ 0 & x_{n1} & \cdots & x_{nn} \end{pmatrix}.$$

Notice that at each stage the matrix remained irreducible.

If we now apply the induction hypothesis to the n by n matrix obtained by deleting row 2 and column 1 (i.e. the matrix of x's), it follows that it can be put in the form

$$\begin{pmatrix} 0 & \cdots & 0 & d_{n+1} \\ d_2 & & & 0 \\ 0 & & & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & d_n & 0 \end{pmatrix}$$

by row addition and subtraction operations which leave it at all times non-negative and irreducible. If we do precisely these operations to the corresponding rows and columns of B_2 (never using or changing column 1 or row 2 of B_2) we will achieve

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the desired form B'. Notice that the fact that the n by n submatrix remains irreducible implies that the $(n+1)$ by $(n+1)$ matrix does too (compare graphs). Hence we have that $(B+I)$ is flow equivalent to $(B'+I)$. By construction each d_i factors d_{i+1}. Since $d_1, \ldots, d_{n+1}$ are clearly the elementary divisors of B', they are also the elementary divisors of B. This completes the case $\det(I-A)<0$.

The case $\det(I-A)>0$ is proved in precisely the same way.

For the case that $\operatorname{rank}(B) = \operatorname{rank}(I-A) = k < n$, we perform the induction on k. The case $k=1$ was done in (3.2). As in the previous cases we construct a matrix

$$B_2 = \begin{pmatrix} 0 & * & \cdots & * \\ d_1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ & \cdots & & \\ 0 & * & \cdots & * \end{pmatrix}$$

with B_2+I flow equivalent to $B+I$, where all the $*$'s represent positive entries. The n by n submatrix (the matrix of $*$'s) has rank $k-1$ so the inductive hypothesis applies to it and the proof is completed as before. □

(3.4) *Proof of main theorem.* The fact that $\det(I-A)$ is an invariant of the flow equivalence class of A is proved in [**3**]. The fact that $\mathbb{Z}^n/(I-A)\mathbb{Z}^n$ is also a flow equivalence invariant is proved in [**1**].

To prove the converse we observe that by (2.6) A and B are flow equivalent to A_1 and B_1 respectively where A_1 and B_1 are the same size, say n by n, and $A_1 = C_1 + I$, $B_1 = D_1 + I$ with C_1 and D_1 strictly positive. Let $\{c_i\}_{i=1}^n$ and $\{d_i\}_{i=1}^n$ be the elementary divisors of C_1 and D_1 respectively, arranged so that c_i factors c_{i+1} and d_i factors d_{i+1} for all i.

Now by the fundamental theorem of abelian groups and the fact that

$$\bigoplus_{i=1}^{n} \mathbb{Z}/(c_i) \cong \mathbb{Z}^n/(A_1-I)\mathbb{Z}^n \cong \mathbb{Z}^n/(B_1-I)\mathbb{Z}^n \cong \bigoplus_{i=1}^{n} \mathbb{Z}/(d_i),$$

it follows that $d_i = c_i$ for all i. Since $\det(I-A_1) = \det(I-B_1)$ we conclude from (3.3) that A_1 and B_1 are flow equivalent to the same canonical form. Hence A is flow equivalent to B. □

Two immediate corollaries are worth noting:

(3.5) Corollary. *If A is a non-negative irreducible matrix then A is flow equivalent to its transpose, which corresponds to the inverse flow.*

(3.6) Corollary. *If A and B are non-negative irreducible matrices with* $\det(I-A) = \det(I-B) = m$, *a non-zero square-free integer, then A and B are flow equivalent.*

This last follows since $\mathbb{Z}^n/(I-A)\mathbb{Z}^n$ and the corresponding group for B both have order $|m|$. Since m is square-free there is a unique abelian group of order $|m|$.

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