NOTES 463


EDMUND SWYLAN
3 Vecsaules Street, Riga LV1004, Latvia
e-mail: Edmund.Swylan@ponymail.com

86.63 Generalising the Fibonacci sequence

The next stage in a possible generalisation of the fundamental Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ... is the so-called ‘tribonacci’ sequence 0, 1, 1, 2, 4, 7, 13, 24, 44, ..., which is generated by the recurrence relation

\[ u_{n+3} = u_{n+2} + u_{n+1} + u_n, \]

where the ratio of successive terms \( A_{n+1}/A_n \) tends to the real root \( \eta \) (say) of the cubic equation \( x^3 - x^2 - x - 1 = 0 \).

Then, if \( f(x) \) is the product of the general Fibonacci polynomial \( x^n - x^{n-1} - x^{n-2} - \ldots - x - 1 (n > 1) \) with the linear factor \( x - 1 \), we obtain the useful expression

\[ f(x) = x^{n+1} - 2x^n + 1 \quad (n > 2). \]

By Descartes' rule, the general Fibonacci polynomial has a single positive root \( \alpha_n \) (say), and we will show that the sequence \( \{\alpha_n\} \) converges to limit 2.

Now the derivative \( f'(x) = ((n + 1)x - 2n)x^{n-1} \) has the single positive root \( 2n/(n + 1) \), and \( f(2n/(n + 1)) < 0 \) (minimum) by induction or the use of \( f(1) = 0 \) and \( f'(1) = 1 - n < 0 \).

Thus \( f(2) = 1 \) leads to the inequality \( 2n/(n + 1) < \alpha_n < 2 \) whence \( \alpha_n \to 2 \) as \( n \to \infty \).

Note also that this convergent sequence is increasing since

\[ f(\alpha_{n+1}) = 1 - 1/\alpha_{n+1} > 0. \]

The scenario may be illustrated graphically by sketching the curves \( y_1 = x^n \), \( y_2 = 1/(2 - x) \) (say), and the roots \( \alpha_n \) may be found by using the first order iteration

\[ x_{r+1} = F(x_r) = 2 - x_r^n, \]

which meets the convergence requirement \( |F'(x)| < 1 \) near the root \( \alpha_n \).

Thus \( \alpha_2 = \frac{1}{2}(1 + \sqrt{5}) = 1.61803... \) (golden section),

\( \alpha_3 = \eta = 1.83928... \) (tribonacci number),

\( \alpha_4 = 1.92756... \), \( \alpha_5 = 1.96594, \ldots \)

In his *Gazette* article [1, p. 203], Sharp demonstrated the relationship between \( \eta \) and the golden section in the context of polyhedra, where the former number is associated with the snub cube and the latter with the dodecahedron and icosahedron. The following minor observation is also
submitted in response to the invitation to report other sightings of ‘that elusive number η’.

First note that the integrand in the routine exercise

\[ \int_0^\infty \frac{1}{x} \left( \frac{1}{1 + ax} - \frac{1}{1 + a^2x^2} \right) \, dx = 0 \quad (a > 0) \]

is negative for \( 0 < ax < 1 \) and positive for \( ax > 1 \). Furthermore, it is easy to demonstrate by calculus that there is a maximum point with coordinates \((\eta/a, (\eta - 1)/2\eta^3)\). The abscissa of the point of inflection \((\eta - 1)/2\eta^3\), however, is the real root of the quintic equation

\[ 3x^5 - 3x^4 - 8x^3 - 12x^2 - 3x - 1 = 0 \quad (a = 1) \]

and is not simply related to η.

Reference


J. A. SCOTT

1 Shiptons Lane, Great Somerford, Chippenham SN15 5EJ

86.64 Some new triples of integers and associated triangles

If \( m \) and \( n \) are two integers such that \( m > n > 0 \) then the triangle with sides,

\[ m^2 - n^2, \quad 2mn, \quad m^2 + n^2 \]

is right-angled. These Pythagorean triples are well-known. Here we study three new triples formed from two positive integers \( m, n \) as above and consider the angles of their associated triangles. We adopt the usual notations for the sides \( a, b, c \) and opposite angles \( A, B, C \) of a triangle.

Any lengths \( a, b, c \) can be taken for the sides of a triangle provided that they satisfy the inequalities

\[ b + c > a, \quad c + a > b \quad \text{and} \quad a + b > c. \quad (1) \]

These inequalities are most easily checked by looking at cosines. It is sufficient to check, using the cosine rule, whether any one of the results for \( \cos A, \cos B \) or \( \cos C \) is between \(-1\) and \(1\).

The first triple

Let \( m \) and \( n \) be two integers such that \( 0 < n < m < 2n \). Then

\[ a = n^2, \quad b = mn, \quad c = m^2 - n^2 \]

are the sides of a triangle \( ABC \) in which \( B = 2A \).

The cosine formulae gives

\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{m^2n^2 + (m^2 - n^2)^2 - n^4}{2mn(m^2 - n^2)} = \frac{m}{2n} \]