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ON THE MONOTONE CONVERGENCE OF GENERAL NEWTON-LIKE METHODS

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This paper examines conditions for the monotone convergence of general Newton-like methods generated by point-to-point mappings. The speed of convergence of such mappings is also examined.

1. INTRODUCTION

This paper examines conditions for the monotone convergence of Newton-like methods. Using the famous Kantorovich lemma on monotone mappings (see for example Kantorovich and Akilov [**5**]) we derive several convergence results. The speed of convergence of these processes is also examined.

In particular, let us consider the Newton-like iterates

$$z_{k+1} = G_k(z_k)\ (k \geqslant 0), \tag{1}$$

where

$$G_k(z_k) = z_k - A_k(z_k, z_{k-1})^{-1} f_k(x_k)\ (k \geqslant 0). \tag{2}$$

Here $f_k, G_k \colon D \subseteq B \to B_1$ $(k \geqslant 0)$ are nonlinear mappings acting between two partially ordered linear topological spaces (POL-spaces), (for definitions see for example Krasnoselskii, [**6**]), whereas $A_k(u, v)(.) \colon D \to B_1 (k \geqslant 0)$ are invertible linear mappings. We provide sufficient conditions for the convergence of iteration (1) to 0. We may have this assumption without losing generality, since any solution x^* can be transformed into 0 by introducing the transformed mapping $g_k(x) = f_k(x + x^*) - x^*$ $(k \geqslant 0)$. Iterations of the above type are extremely important in solving optimisation problems, as well as linear and nonlinear equations. A very important field of such applications can also be found in solving equilibrium problems, in economy and in solving nonlinear input-output systems (see for example [**4, 6, 7, 8, 9, 12, 13**]). Our results can be reduced to the ones obtained earlier in [**1, 2, 3, 5, 6, 10, 11, 12**] when $f_k = f$ $(k \geqslant 0)$.

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2. Convergence theorems

Let us first define some special types of mappings between two POL-spaces. First we introduce some notation. If B and B_1 are two linear spaces then we denote by (B, B_1) the set of all mappings from B into B_1 and by $L(B, B_1)$ the set of all linear mappings from B into B_1. If B and B_1 are topological linear spaces then we denote by $LB(B, B_1)$ the set of all continuous linear mappings from B into B_1. For simplicity, spaces $L(B, B)$ and $LB(B, B)$ will be denoted by $L(B)$ and $LB(B)$. Now let B and B_1 be two POL-spaces and consider a mapping $G \in (B, B_1)$. G is called isotone (respectively antitone) if $x < y$ implies $G(x) \leqslant G(y)$ (respectively $G(x) \geqslant G(y)$). G is called nonnegative if $x > 0$ implies $G(x) > 0$. For linear mappings nonnegativity is clearly equivalent to isotony. Also, a linear mapping is inverse nonnegative if and only if it is invertible and its inverse is nonnegative. If G is a nonnegative mapping we write $G \geqslant 0$. If G and H are two mappings from B into B_1 such that $H - G$ is nonnegative then we write $G \leqslant H$. If Z is a linear space then we obviously have $I \geqslant 0$. Suppose that B and B_1 are two POL-spaces and consider the mappings $T \in L(B_1, B)$ and $T_1 \in L(B, B_1)$. If $T_1T \leqslant I_B$ (respectively if $T_1T \geqslant I_B$) then T_1 is called a left subinverse (respectively superinverse) of T, and T is called a right subinverse (respectively superinverse) of T_1. We say that T_1 is a subinverse of T if T_1 is a left as well as a right subinverse of T.

We assume that the following conditions hold:

(A) Consider mappings $f_k\colon D \subset B \to B_1$ where B is a regular POL-space and B_1 is a POL-space. Let x_0, y_0, y_{-1} be three points of D such that

$$x_0 \leqslant y_0 \leqslant y_{-1},\ \langle x_0, y_{-1}\rangle \subset D,\ f_0(x_0) \leqslant 0 \leqslant f_0(y_0), \tag{3}$$

and denote

$$S_1 = \{(x, y) \in B^2 \mid x_0 \leqslant x \leqslant y \leqslant y_0\},$$
$$S_2 = \{(u, y_{-1}) \in B^2 \mid x_0 \leqslant u \leqslant y_0\},$$
$$S_3 = S_1 \cup S_2.$$

Assume mappings $A_k(.,.)\colon S_3 \to LB(B, B_1)$ such that

$$f_k(y) - f_k(x) \leqslant A_k(w, z)(y - x) \text{ for all } k \geqslant 0,\ (x, y),\ (y, w) \in S_1,\ (w, z) \in S_3. \tag{4}$$

Suppose for any $(u, v) \in S_3$ the linear mappings $A_k(u, v)$ $(k \geqslant 0)$ have a continuous nonsingular nonnegative subinverse. Assume furthermore that

$$f_k(x) \leqslant f_{k-1}(x) \text{ for all } x \in \langle x_0, y_0\rangle\ (k \geqslant 1),\ f_{k-1}(x) \leqslant 0, \tag{5}$$

$$f_k(y) \geqslant f_{k-1}(y) \text{ for all } y \in \langle x_0, y_0\rangle\ (k \geqslant 1),\ f_{k-1}(y) \geqslant 0. \tag{6}$$

We can now formulate the main result.

Theorem 1. *Assume Condition (A) is satisfied.*

Then there exist two sequences $\{x\}$, $\{y_k\}$ $(k \geqslant 0)$ *and points* x^*, y^*, x_1^*, y_1^* *such that for all* $k \geqslant 0$;

(7) $$f_k(y_k) + A_k(y_k, y_{k-1})(y_{k+1} - y_k) = 0,$$

(8) $$f_k(x_k) + A_k(y_k, y_{k-1})(x_{k+1} - x_k) = 0,$$

(9) $$f_k(x_k) \leqslant f_{k-1}(x_{k-1}) \leqslant 0 \leqslant f_{k-1}(y_{k-1}) \leqslant f_k(y_k)\ (k \geqslant 1)$$

(10) $$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_k \leqslant x_{k+1} \leqslant y_{k+1} \leqslant y_k \leqslant \ldots \leqslant y_1 \leqslant y_0$$

(11) $$x_k \to x^*,\ y_k \to y^* \text{ as } k \to \infty,\ x^* \leqslant y^*$$

and

(12) $$f_k(x_k) \to x_1^*,\ f_k(y_k) \to y_1^* \text{ as } k \to \infty, \text{ with } x_1^* \leqslant 0 \leqslant y_1^*.$$

Proof: Let L_0 be a continuous nonsingular nonnegative left subinverse of $A_0(y_0, y_{-1}) \equiv A_0$ and consider the mapping $P\colon \langle 0, y_0 - x_0\rangle \to B$ defined by

$$P(x) = x - L_0(f_0(x_0) + A_0(x)),$$

where $A_0(x)$ denotes the image of x with respect to the mapping $A_0 = A_0(y_0, y_{-1})$. It is easy to see that P is isotone and continuous. We also have

$$\begin{aligned} P(0) &= -L_0(f_0(x_0)) \geqslant 0,\\ P(y_0 - x_0) &= y_0 - x_0 - L_0(f_0(y_0)) + L_0(f_0(y_0) - f_0(x_0) - A_0(y_0 - x_0))\\ &\leqslant y_0 - x_0 - L_0(f_0(y_0)) \leqslant y_0 - x_0. \end{aligned}$$

According to the famous Kantorovich lemma (see for example [5]) mapping P has a fixed point $w \in \langle 0, y_0 - x_0\rangle$. Taking $x_1 = x_0 + w$, we have

$$f_0(x_0) + A_0(x_1 - x_0) = 0,\ x_0 \leqslant x_1 \leqslant y_0.$$

Using (4), (5) and the above relation we get

$$f_1(x_1) \leqslant f_0(x_1) = f_0(x_1) - f_0(x_0) + A_0(x_0 - x_1) \leqslant 0.$$

Consider now the mapping $Q\colon \langle 0, y_0 - x_1\rangle \to B$ given by

$$Q(x) = x + L_0(f_0(y_0) - A_0(x)).$$

Q is clearly continuous, isotone, and

$$\begin{aligned} Q(0) &= L_0 f_0(y_0) \geqslant 0,\\ Q(y_0 - x_1) &= y_0 - x_1 + L_0 f_0(x_1) + L_0(f_0(y_0) - f_0(x_1) - A_0(y_0 - x_1))\\ &\leqslant y_0 - x_1 + L_0 f_0(x_1) \leqslant y_0 - x_1. \end{aligned}$$

Applying the Kantorovich lemma again, we deduce the existence of a point $z \in \langle 0, y_0 - x_1 \rangle$ such that $Q(z) = z$. Taking $y_1 = y_0 - z$,

$$f_0(y_0) + A_0(y_1 - y_0) = 0, \; x_1 \leqslant y_1 \leqslant y_0.$$

Using the above relations and conditions (4), (6) we obtain

$$f_1(y_1) \leqslant f_0(y_1) = f_0(y_1) - f_0(y_0) + A_0(y_0 - y_1) \geqslant 0.$$

By induction it is easy to show there exist four sequences $\{x_k\}$, $\{y_k\}$, $\{f_k(x_k)\}$, $\{f_k(y_k)\} (k \geqslant 0)$, satisfying (7)–(10). Since space B is regular, from (9) and (10) we know that there eixst x^*, y^*, x_1^*, $y_1^* \in B$ satisfying (11)–(12), which completes the proof. ∎

In the next part we give some natural conditions which guarantee that points x^*, y^* are common solutions of equations $f_k(x) = 0 \; (k \geqslant 0)$.

THEOREM 2. *Under the hypotheses of Theorem 1, assume furthermore that*

(i) *there exists $u \in B$ such that $x_0 \leqslant u \leqslant y_0$ and $f_k(u) = 0 \; (k \geqslant 0)$;*
(ii) *linear mappings $A_k(w, z) \; (k \geqslant 0)$, $(w, z) \in S_3$ are inverse nonnegative.*

Then

$$x_k \leqslant u \leqslant y_k \; (k \geqslant 0)$$

and

$$x^* \leqslant u \leqslant y^*.$$

Moreover if $x^* = y^*$, then $x^* = u = y^*$.

PROOF: Using (i),

$$\begin{aligned} A_0(y_1 - u) &= A_0(y_0) - f_0(y_0) - A_0(u) \\ &= A_0(y_0 - u) - (f_0(y_0) - f_0(u)) \geqslant 0 \end{aligned}$$

and

$$\begin{aligned} A_0(x_1 - u) &= A_0(x_0) - f_0(x_0) - A_0(u) \\ &= A_0(x_0 - u) - (f_0(x_0) - f_0(u)) \leqslant 0. \end{aligned}$$

By(ii) it follows that $x_1 \leqslant u \leqslant y_1$. By induction it is easy to show that $x_k \leqslant u \leqslant y_k$ for all $k \geqslant 0$. Hence, $x^* \leqslant u \leqslant y^*$. Moreover if $x^* = y^*$, then $x^* = u = y^*$, which completes the proof. ∎

Moreover we can show:

THEOREM 3. *Under the hypotheses of Theorem 2, assume that either*

(i) *B is normal and there exists a mapping $L\colon B \to B_1$ $(L(0) = 0)$ which has an isotone inverse continuous at the origin and $A_k(y_k, y_{k-1}) \leqslant L$ for all sufficiently large $k \geqslant 0$;*

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or

(ii) *B_1 is normal and there exists a mapping $T\colon B \to B_1$ $(T(0) = 0)$ continuous at the origin and $A_k(y_k, y_{k-1}) \leqslant T$ for sufficiently large $k \geqslant 0$;*

or

(iii) *Mappings $A_k(y_k, y_{k-1})$ $(k \geqslant 0)$ are equicontinuous.*

Then $f_k(x_k) \to 0$, $f_k(y_k) \to 0$ as $k \to \infty$.

PROOF: (i) Using relations (6)–(10) we get

$$0 \geqslant f_k(x_k) = A_k(y_k, y_{k-1})(x_k - x_{k+1}) \geqslant L(x_k - x_{k+1}),$$
$$0 \leqslant f_k(y_k) = A_k(y_k, y_{k-1})(y_k - y_{k+1}) \leqslant L(y_k - y_{k+1}).$$

Hence $$0 \geqslant L^{-1} f_k(x_k) \geqslant x_k - x_{k+1},\ 0 \leqslant L^{-1} f_k(x_k) \leqslant y_k - y_{k+1}.$$

Since B is normal and both $x_k - x_{k+1}$ and $y_k - y_{k+1}$ converge to zero, $L^{-1} f_k(x_k) \to 0$, $f_k(y_k) \to 0$ as $k \to \infty$, from which the result follows.

(ii) Using relations (7)–(10) we have

$$0 \geqslant f_k(x_k) = A_k(y_k, y_{k-1})(x_k - x_{k+1}) \geqslant T(x_k - x_{k+1}),$$
$$0 \leqslant f_k(y_k) = A_k(y_k, y_{k-1})(y_k - y_{k+1}) \leqslant T(y_k - y_{k+1}).$$

By letting $k \to \infty$ we obtain the result.

(iii) From equicontinuity of mappings $A_k(y_k, y_{k-1})$ it follows that $A_k(y_k, y_{k-1})(z_k) \to 0$ whenever $z_k \to 0$. In particular, we have

$$A_k(y_k, y_{k-1})(x_k - x_{k+1}) \to 0,\ A_k(y_k, y_{k-1})(y_k - y_{k+1}) \to 0 \text{ as } k \to \infty.$$

By (7) and (8) and above estimate the result follows. □

The uniqueness of a common solution of equations $f_k(x) = 0$ $(k \geqslant 0)$ in $\langle x_0, y_0 \rangle$ can be proven assuming a condition which is complementary to (4). More precisely we can prove the following:

THEOREM 4. *Let B and B_1 be two POL-spaces. Let $f_k(.)\colon D \subset B \to B_1$ be nonlinear mappings and suppose there exist two points $x_0, y_0 \in D$ such that $x_0 \leqslant y_0$ and $\langle x_0, y_0 \rangle \subset D$. Denote by $S_1 = \{(x, y) \in B^2; x_0 \leqslant x \leqslant y \leqslant y_0\}$ and assume there exist mappings $L_k(.,.)\colon S_1 \to L(B, B_1)$ such that $L_k(x, y)$ has a nonnegative left superinverse for each $(x, y) \in S_1$ and*

$$f_k(y) - f_k(x) \geqslant L_k(x, y)(y - x) \text{ for all } (x, y) \in S_1.$$

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Under these assumptions if $(x^*, y^*) \in S_1$ *and* $f_k(x^*) = f_k(y^*)$, *then* $x^* = y^*$.

PROOF: Let $T_k(x^*, y^*)$ denote a nonnegative left superinverse of $L_k(x^*, y^*)$ for all $k \geqslant 0$. We have

$$0 \leqslant y^* - x^* \leqslant T_k(x^*, y^*)L_k(x^*, y^*)(y^* - x^*)$$
$$\leqslant T_k(x^*, y^*)(f_k(y^*) - f_k(x^*)) = 0.$$

Hence $x^* = y^*$, which completes the proof. ∎

REMARK 1. The conclusions of Theorem 1 hold if iteration (7)–(8) is modified as

$$f_k(y_k) + A_k(y_k, y_{k+1})(y_{k+1} - y_k) = 0,$$
$$f_k(x_k) + A_k(y_{k+1}, y_k)(x_{k+1} - x_k) = 0 \ (k \geqslant 0).$$

This modification seems to be advantageous (see for example Slugin, [**11**]) in many applications.

REMARK 2. Conditions (5) and (6) of Theorem 1 are very natural and they hold in many interesting problems in numerical analysis. See for example, Krasnoselskii, [**6**]. Let us consider equations $f_k(x) = (k+1)(k+2)^{-1}x$, $k \geqslant 0$ on $[-1, 1] = [x_0, y_0] \in \mathbb{R}$, where $\mathbb{R}$ is ordered with the usual ordering of real numbers. Then for any x, y with $x \in [x_0, 0]$ and $y \in [0, y_0]$, conditions (5) and (6) are satisfied. We note that when $f_k = f$ $(k \geqslant 0)$, the same conditions are satisfied as equalities.

REMARK 3. The regularity of space B which is assumed in Theorem 1, is a rather restrictive condition. This condition was essentially used in proving that the iterative procedure (7)–(8) is well defined (that is, there exist sequences $\{x_k\}$, $\{y_k\}$, $\{f_k(x_k)\}$, $\{f_k(y_k)\}$ $(k \geqslant 0)$ satisfying (7)–(10) and they are convergent. Next, we present a method to avoid this regularity assumption. Consider now the following explicit method:

$$y_{k+1} = y_k - A_k^1(y_k, y_{k-1})f_k(y_k) \ (k \geqslant 0) \tag{13}$$

$$x_{k+1} = x_k - A_k^2(y_k, y_{k-1})f_k(x_k) \ (k \geqslant 0), \tag{14}$$

where $A_k^1(y_k, y_{k-1})$ and $A_k^2(y_k, y_{k-1})$ are nonnegative subinverses of $A_k(y_k, y_{k-1})$ $(k \geqslant 1)$. Without the regularity it is impossible to prove that sequences $\{x_k\}$, $\{y_k\}$, $\{f_k(x_k)\}$, $\{f_k(y_k)\}$ $(k \geqslant 0)$ produced by (13)–(14) are convergent. However, we can verify that for any common solution $u \in \langle x_0, y_0\rangle$ of the equations $f_k(x) = 0$ $(k \geqslant 0)$,

$$x_k \leqslant x_{k+1} \leqslant u \leqslant y_{k+1} \leqslant y_k \ (k \geqslant 0).$$

This result becomes important when the existence of the solution is proven by other methods, but it has to be enclosed monotonically (see the next section).

THEOREM 5. *Consider mappings $f_k\colon D \subset B \to B_1 (k \geqslant 0)$, where B and B_1 are two POL-spaces and let x_0, y_0, y_{-1} be three points of B for which condition (3) holds. Define S_1, S_2, S_3 as in Theorem 1 and assume that there exist mappings $A_k(.)\colon S_3 \to L(B, B_1)$ $(k \geqslant 0)$, satisfying conditions (4)–(6) and such that $A_k(u, v)$ has a nonnegative subinverse for any $(u, v) \in S_3$.*

Then, iteration (13)–(14) defines four sequences $\{x_k\}$, $\{y_k\}$, $\{f_k(x_k)\}$, $\{f_k(y_k)\}$ $(k \geqslant 0)$ and they satisfy properties (9)–(10).

Moreover for any common solution $u \in \langle x_0, y_0 \rangle$ of equations $f_k(x) = 0$ $(k \geqslant 0)$,

$$x_k \leqslant u \leqslant y_k \ (k \geqslant 0). \tag{15}$$

PROOF: For $k = 0$, by denoting $A_0^1(0; y_0, y_{-1}) = A_0^1$ and $A_0^2(0; y_0, y_{-1}) = A_0^2$ we have

$$\begin{gathered} x_0 \leqslant y_0,\ f_0(x_0) \leqslant 0 \leqslant f_0(y_0),\ A_0^1 \geqslant 0,\ A_0^2 \geqslant 0,\ I \geqslant A_0 A_0^2, \\ I \geqslant A_0^1 A_0,\ I \geqslant A_0 A_0^2 \text{ and } A_0^2 A_0. \end{gathered} \tag{16}$$

Therefore

$$\begin{aligned} y_0 - y_1 &= A_0^1 f_0(y_0) \geqslant 0, \\ y_1 - x_0 &= y_0 - x_0 - A_0^1 f_0(y_0) \geqslant y_0 - x_0 - A_0^1(f_0(y_0) - f_0(x_0)) \\ &\geqslant A_0^1(A_0(y_0 - x_0) - (f_0(y_0) - f_0(x_0))) \geqslant 0, \end{aligned} \tag{17}$$

$$\begin{aligned} x_1 - x_0 &= -A_0^2 f_0(x_0) \geqslant 0, \\ y_0 - x_1 &= y_0 - x_0 + A_0^2 f_0(x_0) \geqslant y_0 - x_0 - A_0^2(f_0(y_0) - f_0(x_0)) \\ &\geqslant A_0^2(A_0(y_0 - x_0) - (f_0(y_0) - f_0(x_0))) \geqslant 0. \end{aligned} \tag{18}$$

Hence both x_1 and y_1 belong to the interval $\langle x_0, y_0 \rangle$.

From (4)–(6), (13), (14) and (16) we get

$$\begin{aligned} f_1(y_1) \geqslant f_0(y_1) &= f_0(y_1) + A_0\big(y_0 - y_{-1} - A_0^1 f_0(y_0)\big) \\ &= f_0(y_1) - A_0 A_0^1 f_0(y_0) + A_0(y_0 - y_1) \\ &\geqslant f_0(y_1) - f_0(y_0) + A_0(y_0 - y_1) \geqslant 0, \\ f_1(x_1) \leqslant f_0(x_1) &= f_0(x_1) - A_0(y_0, y_{-1})\big(x_1 - x_0 + A_0^2 f_0(x_0)\big) \\ &= f_0(x_1) - A_0 A_0^2 f_0(x_0) - A_0(x_1 - x_0) \\ &\leqslant f_0(x_1) - f_0(x_0) - A_0(x_1 - x_0) \leqslant 0, \end{aligned}$$

and

$$\begin{aligned} y_1 - x_1 \geqslant y_1 - x_1 + A_0^1 f_1(x_1) &= y_0 - x_1 + A_0^1(f_1(y_0) - f_1(x_1)) \\ &\geqslant A_0^1[A_0(y_0 - x_1) - (f_1(y_0) - f_1(x_1))] \geqslant 0. \end{aligned} \tag{19}$$

Thus, we have proved $x_0 \leqslant x_1 \leqslant y_1 \leqslant y_0$ and

$$f_1(x_1) \leqslant f_0(x_1) \leqslant 0 \leqslant f_0(y_1) \leqslant f_1(y_1).$$

By induction we can easily obtain (9) and (10). Consider now $u \in [x_0, y_0]$ such that $f_k(u) = 0$ $(k \geqslant 0)$. We have

$$\begin{aligned} y_1 - u &= y_0 - u - A_0^1 f_0(y_0) + A_0^1 f_0(u) \geqslant A_0^1[A_0(y_0 - u) - (f_0(y_0) - f_0(u))] \geqslant 0, \\ u - x_1 &= u - x_0 + A_0^2 f_0(x_0) - A_0^2 f_0(u) \\ &\geqslant A_0^2[A_0(u - x_0) - (f_0(u) - f_0(x_0))] \geqslant 0. \end{aligned}$$

Hence, $x_1 \leqslant u \leqslant y_1$. By induction it follows that $x_k \leqslant u \leqslant y_k$, which completes the proof. □

If the space B is regular then from (9) and (10) it follows that the sequences $\{x_k\}$, $\{y_k\}$, $\{f_k(x_k)\}$, $\{f_k(y_k)\}$ $(k \geqslant 0)$ are convergent. In some cases the convergence of these sequences can follow from other conditions than regularity.

In the following theorem we provide some sufficient conditions for the convergence of iterations $\{f_k(x_k)\}$, $\{f_k(y_k)\}$ $(k \geqslant 0)$.

THEOREM 6. *Under the hypotheses of Theorem 2, assume:*

(i) *B_1 is a POL-space and B is a normal POL-space;*
(ii) *$x_k \to x^*$ and $y_k \to y^*$ as $k \to \infty$,*
(iii) *there are two continuous nonsingular nonnegative mappings A^1 and A^2 such that $A_k^1(y_k, y_{k-1}) \geqslant A^1$ and $A_k^2(y_k, y_{k-1}) \geqslant A^2$ for sufficiently large k.*

Then

$$f_k(x_k) \to 0 \text{ and } f_k(y_k) \to 0 \text{ as } k \to \infty.$$

PROOF: Note first that

$$\begin{aligned} 0 \leqslant A^1 f_k(x_k) \leqslant A_k^1(y_k, y_{k-1}) f_k(x_k) = y_k - y_{k+1}, \\ x_k - x_{k+1} = A_k^2(y_k, y_{k-1}) f_k(x_k) \leqslant A_k^2(y_k, y_{k-1}) f_k(x_k) \leqslant 0 \end{aligned}$$

for sufficiently large k. The normality of B implies that

$$A^1 f_k(x_k) \to 0 \text{ and } A^2 f_k(y_k) \to 0 \text{ as } k \to \infty,$$

from which the result follows.

REMARK 1. Instead of the algorithm (13), (14) we may consider, more generally, an iteration scheme of the form

$$y_{k+1} = y_k - A_k^1(y_k, y_{k-1})z_k^1 \ (k \geqslant 0),$$
$$x_{k+1} = x_k - A_k^2(y_k, y_{k-1})z_k^2 \ (k \geqslant 0),$$

where z_k^1, z_k^2 are arbitrary elements satisfying the inequality

$$f_k(x_k) \leqslant z_k^2 \leqslant 0 \leqslant z_k^1 \leqslant f_k(y_k) \ (k \geqslant 0).$$

Similar to the previous results it can be shown that under the hypotheses of Theorem 2 this iteration is well defined and the resulting sequences satisfy (9) and (10). This shows, roughly speaking, that if $f_k(x_k)$ is approximated from "below" and $f_k(x_k)$ is approximated from "above" then monotone convergence is preserved.

This observation is important in many practical computations.

REMARK 2. In Theorem 2, we assumed that $A_k(u, v)$ $(k \geqslant 0)$, have nonnegative subinverses for $(u, v) \in S_3$. If we make the stronger assumption that $A_k(u; v)$ is inverse nonnegative for $(u, v) \in S_3$ then in iteration (13)–(14) $A_k^1(y_k, y_{k-1})$ and $A_k^2(y_k, y_{k-1})$ can be taken as any *nonnegative right subinverses* of $A_k(y_k, y_{k-1})$ $(k \geqslant 0)$. Note that the property that it is a left subinverse was used only in proving inequalities (17)–(19). Observing that

$$\begin{aligned} A_0(y_1 - x_0) &= A_0\big(y_0 - x_0 - A_0^1 f_0(y_0)\big) \\ &\geqslant A_0(y_0 - x_0) - f_0(y_0) \\ &\geqslant A_0(y_0 - x_0) - (f_0(y_0) - f_0(x_0)) \geqslant 0 \end{aligned}$$

and using the inverse nonnegativity of A_0 we deduce that $x_0 \leqslant y_1$. The inequalities $x_1 \leqslant y_0$ and $x_1 \leqslant y_1$ can be proved analogously.

REMARK 3. Note that replacing condition (4) by the milder condition

$$(20) \qquad f_k(y) - f_k(x) \leqslant A_k(y, z)(y - x),\ k \geqslant 0,\ (x, y) \in S_1,\ (y, z) \in S_3$$

we can still prove that iteration (7) is well defined and that iteration sequence satisfies $y_k \downarrow y^* \geqslant x_0$ whereas $f_k(y_k) \to 0$ as $k \to \infty$. However, assumption (20) does not imply these properties. However by replacing (4) by (20), we can only prove that sequence (13) satisfies $x_0 \leqslant y_{k+1} \leqslant y_k \leqslant y_0$ $(k \geqslant 0)$.

As the conclusion of this section we will now give some examples which satisfy conditions (4) and indicate how the general results of this section can be applied to obtain monotone convergence theorems for Newton's and secant methods.

Let us consider mappings $f_k\colon D \subset B \to B_1 (k \geqslant 0)$, where B and B_1 are POL-spaces. We recall that f_k is called order-convex on an interval $\langle x_0, y_0\rangle \subset D$ if

$$f_k(\lambda x + (1-\lambda x)y) \leqslant \lambda f_k(x) + (1-\lambda) f_k(y)\ (k \geqslant 0) \tag{21}$$

for all comparable $x, y \in \langle x_0, y_0\rangle$ and $\lambda \in [0, 1]$. If B and B_1 are POL-spaces and if f_k $(k \geqslant 0)$ has a linear G-derivative $f_k'(x)$ at each point $x \in \langle x_0, y_0\rangle$ then (21) holds if and only if

$$f_k(x)(y-x) \leqslant f_k(y) - f_k(x) \leqslant f_k'(y)(y-x)\ (k \geqslant 0) \text{ for } x_0 \leqslant x \leqslant y \leqslant y_0.$$

Thus, for order-convex G-differentiable mappings, (20) is satisfied with $A_k(u, v) = f_k'(u)$. In the unidimensional case (21) is equivalent with isotony of the mapping $x \to f_k'(x)$ but in general the latter property is stronger. Assuming isotony of the mapping $x \to f_k'(x)$, we have

$$f_k(y) - f_k(x) \leqslant f_k'(w)\ (y-x)(k \geqslant 0) \text{ for } x_0 \leqslant x \leqslant y \leqslant w \leqslant y_0$$

so, in this case condition (4) is satisfied for $A_k(w, z) = f_k'(w)$ $(k \geqslant 0)$.

The above observations show that our results can be applied for the Newton method. Iteration (7)–(8) becomes

$$f_k(y_k) + f_k'(y_k)(y_{k+1} - y_k) = 0, \tag{22}$$

$$f_k(x_k) + f_k'(y_k)(x_{k+1} - x_k) = 0, \tag{23}$$

whereas iteration (13)–(14) becomes

$$y_{k+1} = y_k - f_k'(y_k)^{-1} f_k(y_k), \tag{24}$$

$$x_{k+1} = x_k - f_k'(y_k)^{-1} f_k(x_k). \tag{25}$$

Moreover, if in addition

$$\left\| f_k'(z)^{-1}(f_k'(x) - f_k'(y)) \right\| \leqslant \gamma \|x - y\|,\ \text{for } x, y, z \in \langle x_0, y_0\rangle \tag{26}$$

then
$$\|y_{k+1} - x_{k+1}\| \leqslant .5\gamma \|y_k - x_k\|^2\ (k \geqslant 0),$$
$$\|y_{k+1} - y^*\| \leqslant .5\gamma \|y_k - y^*\|^2\ (k \geqslant 0)$$
and
$$\|x_{k+1} - x^*\| \leqslant .5\gamma \|x_k - x^*\|^2\ (k \geqslant 0).$$

These results follow immediately by using (24)–(26), since

$$\begin{aligned}\|y_{k+1} - x_{k+1}\| &= \left\| y_k - x_k - f_k'(y_k)^{-1}(f_k(y_k) - f_k(x_k)) \right\| \\ &\left\| f_k'(y_k)^{-1}[f_k'(y_k)(y_k - x_k) - (f_k(y_k) - f_k(x_k))] \right\| \\ &\leqslant .5\gamma \|y_k - x_k\|^2\ (k \geqslant 0).\end{aligned}$$

Note that iteration (7)–(8) with $f_k = f$ $(k \geqslant 0)$ and $A_k(u, v) = f'(u)$ is exactly the same algorithm which was proposed by Fourier in 1918, (see for example [5]) in the unidimensional case and was extended by Baluev, [2]) in the general case.

If $f_k \colon [a, b] \to \mathbb{R}$ is a real mapping of a real variable then f_k $(k \geqslant 0)$, is order convex if and only if

$$(f_k(x) - f_k(y))(x - y)^{-1} \leqslant (f_k(u) - f_k(v))(u - v)^{-1}$$

for all $x, y, u, v \in [a, b]$ such that $x \leqslant u$ and $y \leqslant v$. This fact motivates the notion of convexity with respect to a divided difference discussed earlier for the case $f_k = f$ $(k \geqslant 0)$.

Let $f_k \colon D \subset B \to B_1$ be nonlinear mappings between two linear spaces B and B_1. A mapping $\delta f_k(.,.) \colon D \times D \to L(B, B_1)$ is called a divided difference of f_k $(k \geqslant 0)$ on D if

$$\delta f_k(u, v)(u - v) = f_k(u) - f_k(v) \; (k \geqslant 0), \; u, v \in D.$$

If B and B_1 are topological linear spaces then the linear mapping $\delta f_k(u, v)$ is supposed continuous (that is, $\delta f_k(u, v) \in LB(B, B_1)$). Now suppose B, B_1 are two POL-spaces and assume the nonlinear mapping $f_k(.) \colon D \subset B \to B_1$ $(k \geqslant 0)$ has a divided difference δf_k on D $(k \geqslant 0)$. Then f_k $(k \geqslant 0)$ is called convex with respect to the divided difference $\delta f_k(.)$ on D if

$$\delta f_k(x, y) \leqslant \delta f_k(u, v) \; (k \geqslant 0), \text{ for all } x, y, u, v \in D, \tag{27}$$

with $x \leqslant y$ and $y \leqslant v$. Moreover, the mapping $\delta f_k(.,.) : D \times D \to L(B, B_1)$ $(k \geqslant 0)$ satisfying

$$\delta f_k(u, v)(u - v) \geqslant f_k(u) - f_k(v) \; (k \geqslant 0) \text{ for all comparable } u, v \in D \tag{28}$$

is called the generalised divided difference of f_k $(k \geqslant 0)$ on D. If both conditions (27) and (28) are satisfied, then we say $f_k(k \geqslant 0)$, is convex with respect to the generalised divided difference $\delta f_k(k \geqslant 0)$. It is easily seen that if (27) and (28) are satisfied on $D = \langle x_0, y_{-1} \rangle$ then condition (4) is satisfied with $A_k(u, v) = \delta f_k(u, v)$ $(k \geqslant 0)$. Indeed, for $x_0 \leqslant x \leqslant y \leqslant w \leqslant z \leqslant y_{-1}$, we have

$$\begin{aligned} \delta f_k(x, y)(y - x) \leqslant f_k(y) - f_k(x) &\leqslant \delta f_k(y, x)(y - x) \\ &\leqslant \delta f_k(w, z)(y - x). \end{aligned}$$

That is, our results can be applied also for secant method.

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3. A lattice theoretical fixed point theorem

In this section we reformulate two fixed point theorems which hold in arbitrary complete lattices. These theorems are due to Tarski, [**13**].

The first theorem provides sufficient conditions for the existence of a fixed point of a mapping $f: S \to S$ where S is a nonempty set. The second theorem provides sufficient conditions for the existence of a common fixed point x^* of a sequence $f_k: S \to S$ $(k \geqslant 0)$ of mappings.

We shall need some definitions:

Definition 1: By a *lattice* we mean a system $Q = \{S, \leqslant\}$ formed by a nonempty set S and a binary relation $\leqslant$; it is assumed that $\leqslant$ establishes a partion order in S and that for any two elements $a, b \in S$ there is a least upper bound (join) $a \cup b$ and a greatest lower bound (meet) $a \cap b$. The relations $\geqslant$, $<$, and $>$ are defined in the usual way in terms of $\leqslant$.

Definition 2: The lattice $Q = \{S, \leqslant\}$ is called *complete*, if every subset S_1 of S has a least upper bound $\cup S_1$ and a greatest lower bound $\cap S_1$. Such a lattice has in particular two elements 0 and 1 defined by the formulas

$$0 = \cap S \text{ and } 1 = \cup S.$$

Given any two elements $a, b \in S$ with $a \leqslant b$, we denote by $[a, b]$ the interval with the end points a and b, that is the set of all elements $x \in S$ for which $a \leqslant x \leqslant b$; in symbols $[a, b] = E_x[x \in S \text{ and } a \leqslant x \leqslant b]$. System $\{[a, b], \leqslant\}$ is clearly a lattice; it is complete if Q is complete.

We consider functions f on S to S and, more generally on a subset S_1 of S to another subset S_2 of S. Such a function f is called increasing if, for any elements $x, y \in S_1$, $x \leqslant y$ implies $f(x) \leqslant f(y)$. Note that this assumption is the same as isotony.

We can now present the following theorem whose proof can be found for example in Tarski, [**13**].

Assume that

(B_1) $Q = \{S, \leqslant\}$ is a complete lattice;
(B_2) f is an increasing function on S to S;
(B_3) P is the set of all fixed points of f.

Theorem 7. *Assume conditions (B_1)–(B_3) are satisfied.*

Then the set P is not empty and the system $\{P, \leqslant\}$ is a complete lattice. In

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particular,

$$\cup P = \cup E_x[f(x) \geqslant x] \in P$$

and

$$\cap P = \cap E_x[f(x) \leqslant x] \in P.$$

By the above theorem, the existence of a fixed point for every increasing function is a necessary condition for the completeness of a lattice. The question arises as to whether this condition is also sufficient. It has been shown that the answer to this question is affirmative (see, [**13**]).

A set W of functions is called commutative if

(i) All functions of W have a common domain, say, S_1 and the ranges of all functions of W are subsets of S_1;

(ii) For any $f, g \in W$,

$$f(g(x)) = g(f(x)) \text{ for all } x \in S_1.$$

Assume that

(C_1) $Q = \langle S, \leqslant \rangle$ is a complete lattice;

(C_2) W is any commutative set of increasing functions on S to S;

(C_3) P is the set of all common fixed points of all functions $f \in W$.

We can provide the following.

Theorem 8. *Assume conditions (C_1)–(C_3) are satisfied.*

Then P is not empty and the system $\{P, \leqslant\}$ is a complete lattice. In particular, we have

$$\cup P = \cup E_x[f(x) \geqslant x \text{ for every } f \in W] \in P$$

and

$$\cap P = \cap E_x[f(x) \leqslant x \text{ for every } f \in W] \in P.$$

The proof of this theorem is found also in Tarski, [**13**], and it can be used in connection with the theorems of the previous section. In particular all monotone convergence methods introduced in the previous sections can be used to approximate fixed points x^* of mappings f_k $(k \geqslant 0)$, whose existence is guaranteed under the hypotheses of the above theorems.

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