

## OPTIMAL EXERCISE PRICE OF AMERICAN OPTIONS NEAR EXPIRY

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### Abstract

This paper investigates American puts on a dividend-paying underlying whose volatility is a function of both time and underlying asset price. The asymptotic behaviour of the critical price near expiry is deduced by means of singular perturbation methods. It turns out that if the underlying dividend is greater than the risk-free interest rate, the behaviour of the critical price is parabolic, otherwise an extra logarithmic factor appears, which is similar to the constant volatility case. The results of this paper complement numerical approaches used to calculate the option values and the optimal exercise price at times that are not close to expiry.

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### 1. Introduction

How to price an option remains one of the major challenges in today's finance industry. Pricing American options is especially challenging due to the nonlinearity introduced by the fact that they can be exercised at any time during their lifespan, which effectively makes the problem a free boundary problem.

In the past two decades many researchers have attempted to tackle the problem of pricing American options. Although analytical solutions for American puts were determined by Zhu [6], under a Black–Scholes framework with nondividend yield, and by Zhao [5], in local volatility models, numerical methods are still preferable for market practitioners as they are usually much faster and have acceptable accuracy. However, due to the fact that the critical price is singular at expiry, as is the case in a similar Stefan problem [4], it is difficult to maintain the same level of accuracy in approximating the optimal exercise price at the time near expiry by using numerical methods. For example, when using both lattice methods and the projected successive

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over-relaxation (SOR) method for a partial differential equation (PDE) system, a fine discretization of the time domain must be used near expiry, which is both expensive and of limited accuracy [3]. Therefore, it is quite helpful to determine the asymptotic behaviour of the critical price near expiry, and this asymptotic solution can be used as a complement to numerical approaches to calculate option values and the optimal exercise price at times that are not close to expiry.

Analyses of the asymptotic behaviour of the critical price near expiry have already been carried out. For instance, Barles *et al.* [1] derived the first term of the critical price by constructing a subsolution as well as a supersolution. Evans *et al.* [3] obtained a similar result by both the method of integral equations and the method of matched asymptotic expansions. It should be remarked that these results are only valid under the Black–Scholes model with constant volatility.

Empirical evidence, however, shows that when using observed option prices to determine the implied volatility (for which the theoretical prices fit with the observed prices) options with different strikes have different implied volatility, which violates the Black–Scholes assumption that the volatility is constant. One possible strategy to cope with the empirical facts is to use the local volatility model, where the volatility is a deterministic function of both the underlying and the time such as the hyperbolic sine model, the constant elasticity of variance (CEV) model, and so on. The CEV model further nests the Brownian motion and the Ornstein–Uhlenbeck process as special cases. The asymptotic behaviour of the critical price near expiry in the local volatility model remains unclear. Recent progress was made by Chevalier [2] in extending the previous results under the constant volatility framework to a stock-price-dependent volatility. However, it must be pointed out that the results in [2] cannot be representative, as stock-price-dependent volatility is only a special case of the local volatility model.

In this paper, an explicit analytical expression for the critical price near expiry is presented under the local volatility model. The expression was found by means of the method of matched asymptotic expansions, which generates a sequence of problems for local behaviour near expiry. The results show that if the underlying dividend is greater than the market interest rate, the behaviour of the critical price is parabolic, otherwise an extra logarithmic factor appears, which agrees with the constant volatility case.

The paper is organized as follows. In Section 2, we introduce the PDE system that the price of an American put option must satisfy in the local volatility model. In Section 3, we first deduce the asymptotic behaviour of the critical price near expiry under the assumption that the volatility is stock-price-dependent, and then extend the analysis to the general case, where volatility depends on both the underlying and time. Concluding remarks are given in Section 4.

## 2. American puts under general diffusion process

This paper considers a general diffusion process for the underlying under the risk-neutral measure. Specifically, the underlying  $S_t$ , as a function of time, is assumed to

follow a diffusion process:

$$dS_t = (r - D) dt + \sigma_F dW_t, \tag{2.1}$$

where the constants  $r \geq 0$  and  $D \geq 0$  denote the risk-free interest rate and the dividend yield respectively, and the deterministic function  $\sigma_F$  represents the local volatility. In this paper, two cases related to the different forms of  $\sigma_F$  are discussed separately: in the first,  $\sigma_F$  is a function of  $S_t$  only, that is,  $\sigma_F = \sigma_F(S_t)$ ; whereas in the second,  $\sigma_F$  is a function of both  $S_t$  and  $t$ , that is,  $\sigma_F = \sigma_F(S_t, t)$ . The assumptions for the two cases differ as:

- for  $\sigma_F = \sigma_F(S_t)$ , it is assumed that  $\sigma_F(S_t)$  is at least second-order differentiable in the vicinity of  $S_t = K$ ;
- for  $\sigma_F = \sigma_F(S_t, t)$ , it is assumed that  $\sigma_F(S_t, t)$  is at least second-order differentiable in the vicinity of  $S_t = K, t = T_E$ .

In fact, these assumptions are in line with almost all the commonly used local volatility models, such as the CEV model, the hyperbolic sine model, and so on.

Let  $P_A(S, t)$  be the price of an American put option, with  $S$  being the underlying and  $t$  being the time. Then, under the proposed diffusion process (2.1), it can easily be shown that the valuation of an American put option can be formulated as a free boundary problem, with  $P_A(S, t)$  satisfying

$$\begin{cases} \frac{\partial P_A}{\partial t} + \frac{1}{2}\sigma_F^2 S^2 \frac{\partial^2 P_A}{\partial S^2} + (r - D)S \frac{\partial P_A}{\partial S} - r P_A = 0, \\ P_A(x, T_E) = \max(K - S, 0), \quad P_A(S_f(t), t) = K - S_f(t), \\ \frac{\partial P_A}{\partial S}(S_f(t), t) = -1, \quad \lim_{S \rightarrow \infty} P_A(S, t) = 0. \end{cases} \tag{2.2}$$

This PDE system is defined on  $S \in [S_f(t), +\infty)$  and  $t \in [0, T_E]$ . Moreover, the critical price  $S_f$  at expiry  $T_E$  in the local volatility model is found in [5] to be

$$S_f(T_E) = \min\left(\frac{r}{D}K, K\right).$$

### 3. Matched asymptotic analysis for the optimal exercise price near expiry

**3.1.  $\sigma$  is a function of  $S$  only** For convenience we use dimensionless variables

$$\begin{aligned} S &= Ke^x, \quad P = \frac{P_A e^{\rho\tau}}{K} + e^{\rho\tau}(e^x - 1), \quad S_f = Ke^{x_f}, \\ \tau &= \frac{\sigma_F^2(K)}{2}(T_E - t), \quad \sigma(x) = \sigma_F(Ke^x), \quad a(x) = \frac{\sigma^2(x)}{\sigma^2(0)}. \end{aligned}$$

The parameters  $\rho$  and  $v$  are defined as

$$\rho = \frac{2r}{\sigma^2(0)}, \quad v = \frac{2D}{\sigma^2(0)},$$

respectively. Then (2.2) can be written in the dimensionless form

$$\begin{cases} \frac{\partial P}{\partial \tau} = a(x) \frac{\partial^2 P}{\partial x^2} + (\rho - v - a(x)) \frac{\partial P}{\partial x} + e^{\rho\tau} (ve^x - \rho), \\ P(x, 0) = \max(e^x - 1, 0), \quad P(x_f, \tau) = 0, \\ \frac{\partial P}{\partial x}(x_f, \tau) = 0, \quad \lim_{x \rightarrow \infty} P(x, \tau) = e^{\rho\tau} (e^x - 1), \end{cases} \quad (3.1)$$

and

$$x_f(0) = \begin{cases} 0, & v \leq \rho, \\ -\log\left(\frac{v}{\rho}\right), & v > \rho. \end{cases}$$

Now, we shall use matched asymptotic analysis to construct the small- $\tau$  behaviour of  $x_f(\tau)$  for the PDE system (3.1).

First, we consider the case in which  $D \leq r$ , that is,  $v \leq \rho$ . By setting  $\tau = \epsilon T$ , where  $T = \mathcal{O}(1)$  and  $\epsilon$  is an artificial small parameter, we obtain the PDE system for  $P(x, T)$ :

$$\begin{cases} \frac{\partial P}{\partial T} = \epsilon \left( a(x) \frac{\partial^2 P}{\partial x^2} + (\rho - v - a(x)) \frac{\partial P}{\partial x} + e^{\epsilon\rho T} (ve^x - \rho) \right), \\ P(x, 0) = \max(e^x - 1, 0), \quad P(x_f, T) = 0, \\ \frac{\partial P}{\partial x}(x_f, T) = 0, \quad \lim_{x \rightarrow \infty} P(x, T) = e^{\epsilon\rho T} (e^x - 1). \end{cases} \quad (3.2)$$

By assuming that the solution of (3.2) can be expanded in powers of  $\epsilon$ , we obtain the outer solution, which is only valid for  $x > 0$ :

$$P(x, T) = e^x - 1 + \rho T \epsilon (e^x - 1) + \mathcal{O}(\epsilon^2).$$

Since the outer expansion breaks down at  $x_f(0) = 0$ , we need to perform a local analysis in the vicinity of  $x = 0$ . By using the stretched variable

$$X = \frac{x}{\sqrt{\epsilon}}, \quad (3.3)$$

and substituting (3.3) into (3.2), we have

$$\frac{\partial P}{\partial T} = a(\sqrt{\epsilon}X) \frac{\partial^2 P}{\partial X^2} + (\rho - v - a(\sqrt{\epsilon}X)) \frac{\partial P}{\partial X} + \epsilon e^{\epsilon\rho T} (ve^x - \rho). \quad (3.4)$$

Since the boundary conditions all have a factor  $\sqrt{\epsilon}$  in common, we rescale the problem by defining

$$P = \sqrt{\epsilon} p. \quad (3.5)$$

On the other hand, assuming that the coefficients have the Taylor expansions to second order at  $x = 0$ , we can expand  $a$ , written in the local variable  $X$ , as

$$a(\sqrt{\epsilon}X) = a(0) + \sqrt{\epsilon}Xa'(0) + \frac{\epsilon X^2}{2}a''(0) + \mathcal{O}(\epsilon^{3/2}). \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain the leading-order PDE system

$$\begin{cases} \frac{\partial p_0}{\partial T} = \frac{\partial^2 p_0}{\partial X^2}, \\ p_0(X, 0) = \max(X, 0), \\ \lim_{X \rightarrow \infty} p_0(X, T) = X. \end{cases}$$

The solution of this PDE system can easily be found by using similarity solution techniques. It is

$$p_0(X, T) = \frac{\sqrt{T}}{\sqrt{\pi}}e^{-X^2/4T} + \frac{X}{2}\operatorname{erfc}\left(-\frac{X}{2T}\right).$$

The following lemma states that the location of the free boundary  $x_f(\tau)$  is outside the layer near  $x = 0$ , which, on the other hand, implies that another layer exists near  $x_f$ . Let  $U(a, \delta)$  denote the neighborhood of a point  $a$ , that is,

$$U(a, \delta) = \{x \mid 0 \leq |x - a| < \delta\}.$$

**LEMMA 3.1.** *When  $v \leq \rho$ , we have  $x_f(\tau) \notin U(0, \sqrt{\epsilon})$ , where  $\tau = \epsilon T$ , and  $T = \mathcal{O}(1)$ .*

**PROOF.** We shall use the method of *reductio ad absurdum* to prove this lemma. Assuming that  $x_f(\tau) \in U(0, \sqrt{\epsilon})$ , we have  $\lim_{\epsilon \rightarrow 0}(x_f(\tau)/\sqrt{\epsilon}) = X_0$ , where  $X_0$  is finite for  $T = \mathcal{O}(1)$ . Therefore, we can rescale  $x_f(\tau)$  and expand it in terms of  $\sqrt{\epsilon}$ , that is,

$$X_f = \frac{x_f}{\sqrt{\epsilon}} = X_0 + \sqrt{\epsilon}X_1 + \mathcal{O}(\epsilon).$$

In order to satisfy the moving boundary conditions, the leading-order term should satisfy

$$p_0(X_0) = \frac{\partial p_0}{\partial X}(X_0, T) = 0,$$

which yields

$$\frac{\sqrt{T}}{\sqrt{\pi}}e^{-X_0^2/4T} + \frac{X_0}{2}\operatorname{erfc}\left(-\frac{X_0}{2T}\right) = 0 \quad \text{and} \quad \operatorname{erfc}\left(-\frac{X_0}{2T}\right) = 0. \tag{3.7}$$

By solving (3.7), we obtain  $X_0 = \infty$ , in contrast to our assumption that  $x_f(\tau) \in U(0, \sqrt{\epsilon})$ . Therefore, the location of the free boundary should be outside the  $\mathcal{O}(\sqrt{\epsilon})$  layer near  $x = 0$ , and thus  $\lim_{\epsilon \rightarrow 0}(x_f(\tau)/\sqrt{\epsilon}) = -\infty$ , a contradiction. This completes the proof.  $\square$

On the other hand, in order to satisfy the free boundary conditions, we use the stretched variable

$$z = \frac{x - x_f}{\epsilon}, \tag{3.8}$$

where  $z = \mathcal{O}(1)$ . Substituting (3.8) into the governing equation contained in (3.2),

$$\left\{ \begin{aligned} \epsilon \frac{\partial P}{\partial T} - \frac{x_f}{\partial T} \frac{\partial P}{\partial z} &= a(\epsilon z + x_f) \frac{\partial^2 P}{\partial z^2} + \epsilon(\rho - v - a(\epsilon z + x_f)) \frac{\partial P}{\partial z} \\ &\quad + \epsilon^2 e^{\epsilon \rho T} (v e^{\epsilon z + x_f} - \rho), \\ P(0, T) &= 0, \quad \frac{\partial P}{\partial z}(0, T) = 0. \end{aligned} \right. \tag{3.9}$$

Again, an expansion in regular powers of  $\epsilon$  gives the solution of (3.9) as

$$P = \mathcal{O}(\epsilon^2). \tag{3.10}$$

In order to match with the solution near  $x_f$ , we need the solution in the  $\mathcal{O}(\sqrt{\epsilon})$  layer near  $x = 0$ . Assuming that

$$p = p_0 + \sqrt{\epsilon} p_1 + \epsilon^{3/2} p_2 + \mathcal{O}(\epsilon^2), \tag{3.11}$$

we obtain the following sequence of PDE systems:

$$\left\{ \begin{aligned} \frac{\partial p_0}{\partial T} &= \frac{\partial^2 p_0}{\partial X^2}, \\ p_0(X, 0) &= \max(X, 0), \\ \lim_{X \rightarrow \infty} p_0(X, T) &= X, \quad \lim_{X \rightarrow -\infty} p_0(X, T) = 0, \end{aligned} \right. \tag{3.12}$$

$$\left\{ \begin{aligned} \frac{\partial p_1}{\partial T} &= \frac{\partial^2 p_1}{\partial X^2} + a'(0)X \frac{\partial^2 p_0}{\partial X^2} + (\rho - v - 1) \frac{\partial p_0}{\partial X} + v - \rho, \\ p_1(X, 0) &= \max\left(\frac{1}{2}X^2, 0\right), \\ \lim_{X \rightarrow \infty} p_1(X, T) &= \frac{1}{2}X^2, \quad \lim_{X \rightarrow -\infty} \frac{\partial p_1}{\partial X}(X, T) = 0, \end{aligned} \right. \tag{3.13}$$

$$\left\{ \begin{aligned} \frac{\partial p_2}{\partial T} &= \frac{\partial^2 p_2}{\partial X^2} + a'(0)X \frac{\partial^2 p_1}{\partial X^2} + \frac{1}{2}a''(0)X^2 \frac{\partial^2 p_0}{\partial X^2} + (\rho - v - 1) \frac{\partial p_1}{\partial X} \\ &\quad - a'(0)X \frac{\partial p_0}{\partial X} + vX, \\ p_2(X, 0) &= \max\left(\frac{1}{6}X^3, 0\right), \\ \lim_{X \rightarrow \infty} p_2(X, T) &= \frac{1}{6}X^3 + \rho XT, \quad \lim_{X \rightarrow -\infty} \frac{\partial^2 p_2}{\partial X^2}(X, T) = 0. \end{aligned} \right. \tag{3.14}$$

One should notice that, in the above PDE systems, the boundary conditions as  $X \rightarrow +\infty$  are obtained by matching with the outer expansion; whereas the ones as  $X \rightarrow -\infty$  are required to properly close those PDE systems.

The solutions of the PDE systems (3.12)–(3.14) can be found by using similarity solution techniques. The details of the derivation are provided in Appendix A. The asymptotic behaviours for  $h_0(\xi)$ ,  $h_1(\xi)$  and  $h_2(\xi)$  as  $\xi \rightarrow -\infty$  can be derived as

$$h_0(\xi) = \frac{1}{2\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi^2} + \mathcal{O}\left(\frac{e^{-\xi^2}}{\xi^4}\right),$$

$$h_1(\xi) = v - \rho + \mathcal{O}\left(\xi e^{-\xi^2}\right), \quad h_2(\xi) = 2v\xi + \mathcal{O}\left(\xi^4 e^{-\xi^2}\right).$$

We now match the values of  $P_\tau$  in the two different regions, as suggested by Keller [3], to complete the analysis. This is accomplished by taking the limit of  $X \rightarrow -\infty$  ( $\xi \rightarrow -\infty$  or  $x \rightarrow x_f$ ) of  $P_\tau$  given by (3.11) and (3.10). The leading-order term forms the following transcendental equations:

$$\frac{1}{2\sqrt{\pi\tau}} e^{-x_f^2/4\tau} + v - \rho = 0, \quad v < \rho,$$

$$\frac{1}{2\sqrt{\pi\tau}} e^{-x_f^2/4\tau} + vx_f = 0, \quad v = \rho,$$

which have the solutions

$$x_f(\tau) = \begin{cases} -2\sqrt{\tau} \left[ \ln \frac{1}{2(\rho - v)\sqrt{\pi\tau}} \right]^{1/2}, & v < \rho, \\ -2\sqrt{\tau} \left[ \ln \frac{1}{4\sqrt{\pi}v\tau} \right]^{1/2}, & v = \rho, \end{cases}$$

respectively. Therefore, recalling that  $S_f(t) = K e^{x_f(t)}$ , for  $D < r$ ,

$$S_f(t) = K - K\sigma(K) \sqrt{(T_E - t) \ln \frac{\sigma^2(K)}{8\pi(T_E - t)(r - D)^2}} + o\left(\sqrt{(T_E - t) \ln \frac{1}{\sqrt{T_E - t}}}\right), \tag{3.15}$$

and for  $D = r$ ,

$$S_f(t) = K - K\sigma(K) \sqrt{2(T_E - t) \ln \frac{1}{4\sqrt{\pi}D(T_E - t)}} + o\left(\sqrt{(T_E - t) \ln \frac{1}{T_E - t}}\right). \tag{3.16}$$

We now consider the case where  $D > r$ , that is,  $v > \rho$ . Here, we assume that  $x_0 = -\log(v/\rho) \ll -\sqrt{\epsilon}$ . The procedure in deriving the outer expansion is quite

similar to the case where  $D \leq r$ , and the outer solution is

$$P(x, T) = \begin{cases} e^x - 1 + \rho(e^x - 1)T\epsilon + \mathcal{O}(\epsilon^2), & x \gg \sqrt{\epsilon}, \\ (ve^x - \rho)T\epsilon + \mathcal{O}(\epsilon^2), & x \ll -\sqrt{\epsilon}. \end{cases}$$

One should notice that the leading-order solution  $P_0(x, T) = \max(e^x - 1, 0)$  is continuous but not differentiable at  $x = 0$ . Thus, we expect that there is a corner layer at  $x = 0$ , the thickness of which is  $\mathcal{O}(\sqrt{\epsilon})$ . (The interested reader may refer to [Appendix B](#) for the derivation of the solution in this layer.) However, based on the assumption that  $x_0 = -\log(v/\rho) \ll -\sqrt{\epsilon}$ , the free boundary is expected to be located outside the corner layer. Therefore, for future analysis, we only need the outer solution which is valid for  $x \ll -\sqrt{\epsilon}$ .

Since the outer expansion fails to satisfy the free boundary conditions, we perform a local analysis in the vicinity of  $x_0$  by rescaling as follows:

$$X = \frac{x - x_0}{\sqrt{\epsilon}}, \quad p = \frac{P}{\epsilon^{3/2}}, \quad X_f = \frac{x_f - x_0}{\sqrt{\epsilon}}. \tag{3.17}$$

Substituting (3.17) into (3.2), the governing equation becomes

$$\frac{\partial p}{\partial T} = a(\sqrt{\epsilon}X + x_0) \frac{\partial^2 p}{\partial X^2} + (\rho - v - a(\sqrt{\epsilon}X + x_0))\sqrt{\epsilon} \frac{\partial p}{\partial X} + e^{\rho\epsilon T} \rho \frac{(e^{\sqrt{\epsilon}X} - 1)}{\sqrt{\epsilon}}. \tag{3.18}$$

On the other hand, assume that  $a(x)$  has Taylor expansions to first order at  $x = x_0$ , that is,

$$a(\sqrt{\epsilon}X + x_0) = a(x_0) + a'(x_0)\sqrt{\epsilon}X + \mathcal{O}(\epsilon). \tag{3.19}$$

Substituting (3.19) into (3.18), the leading-order PDE system is found to be

$$\begin{cases} \frac{\partial p_0}{\partial T} = a(x_0) \frac{\partial^2 p_0}{\partial X^2} + \rho X, \\ p_0(X, 0) = 0, \\ \lim_{X \rightarrow \infty} p_0(X, T) = \rho XT. \end{cases} \tag{3.20}$$

Observe that the boundary condition as  $X \rightarrow +\infty$  is obtained by matching with one branch of the outer expansion ( $x \ll -\sqrt{\epsilon}$ ). It is straightforward to derive the solution of (3.20) by using similarity solution techniques. This has the structure

$$p_0(X, T) = T^{3/2}h(\xi),$$

where

$$\xi = \frac{X}{2\sqrt{a(x_0)T}}, \quad h(\xi) = 2\rho\xi + C \left[ (\xi^2 + 1)e^{-\xi^2} - (2\xi^3 + 3\xi) \int_{\xi}^{+\infty} e^{-t^2} dt \right],$$



with  $C$  constant. Now, assuming that the free boundary is located inside the layer near  $x_0$ , just as we did in analyzing the previous case, the rescaled free boundary can thus be expanded in powers of  $\sqrt{\epsilon}$ , that is,

$$X_f = X_1 + \sqrt{\epsilon}X_2 + \mathcal{O}(\epsilon).$$

It is clear that  $p_0(X, T)$  should also satisfy

$$p_0(X_1, T) = \frac{\partial p_0}{\partial X}(X_1, T) = 0, \tag{3.21}$$

which is equivalent to  $h(\xi_1) = h'(\xi_1) = 0$ , where  $\xi_1 = X_1/2\sqrt{a(x_0)T}$ . Consequently, we obtain

$$\begin{aligned} 2\rho\xi_1 + C \left[ (\xi_1^2 + 1)e^{-\xi_1^2} - (2\xi_1^3 + 3\xi_1) \int_{\xi_1}^{+\infty} e^{-t^2} dt \right] &= 0, \\ 2\rho + C \left[ 3\xi_1 e^{-\xi_1^2} - (6\xi_1^2 + 3) \int_{\xi_1}^{+\infty} e^{-t^2} dt \right] &= 0, \end{aligned}$$

from which the transcendental equation for  $\xi_1$  can be derived as

$$-\xi_1^3 e^{\xi_1^2} \int_{\xi_1}^{+\infty} e^{-t^2} dt = \frac{1}{4}(1 - 2\xi_1^2). \tag{3.22}$$

The solution of (3.22) is  $\xi_1 = 0.4517$ . Therefore

$$\begin{aligned} x_f(\tau) &= x_0 - 2\xi_1\sqrt{a(x_0)T} + \mathcal{O}(\tau) \\ &= x_0 - \sqrt{2}\xi_1\sigma\left(\frac{r}{D}K\right)\sqrt{T_E - t} + \mathcal{O}(T_E - t) \end{aligned}$$

and

$$S_f(t) = Ke^{x_f} = \frac{r}{D}K \left[ 1 - \sigma\left(\frac{r}{D}K\right)\xi_1\sqrt{2(T_E - t)} \right] + \mathcal{O}(T_E - t).$$

Remarkably, the leading-order terms of the critical price derived in this section appear to be reasonable, since they agree with those derived in [2], and they degenerate to the results of Evans *et al.* when  $\sigma(S)$  is independent of  $S$ .

**3.2.  $\sigma$  is a function of both  $S$  and  $t$**  Here, we apply singular perturbation techniques to derive the explicit analytical expression for the optimal exercise price near expiry in the local volatility model where  $\sigma$  is a function of both  $S$  and  $t$ . For convenience, we shall also make PDE system (2.2) dimensionless. This is achieved by adopting the new variables:

$$\begin{aligned} S &= Ke^x, \quad P = \frac{P_A e^{\rho\tau}}{K} + e^{\rho\tau}(e^x - 1), \quad S_f = Ke^{x_f}, \quad \tau = \frac{\sigma_F^2(K, T_E)}{2}(T_E - t), \\ \sigma(x, \tau) &= \sigma_F\left(K e^x, T_E - \frac{2}{\sigma_F^2(K, T_E)}\tau\right), \quad a(x, \tau) = \frac{\sigma^2(x, \tau)}{\sigma^2(0, 0)}. \end{aligned}$$

The parameters  $\rho$  and  $v$  are defined as

$$\rho = \frac{2r}{\sigma^2(0, 0)}, \quad v = \frac{2D}{\sigma^2(0, 0)}.$$

Then, (2.2) can be written in the dimensionless form

$$\begin{cases} \frac{\partial P}{\partial \tau} = a(x, \tau) \frac{\partial^2 P}{\partial x^2} + (\rho - v - a(x, \tau)) \frac{\partial P}{\partial x} + e^{\rho\tau}(ve^x - \rho), \\ P(x, 0) = \max(e^x - 1, 0), \quad P(x_f, \tau) = 0, \\ \frac{\partial P}{\partial x}(x_f, \tau) = 0, \quad \lim_{x \rightarrow \infty} P(x, \tau) = e^{\rho\tau}(e^x - 1) \end{cases} \quad (3.23)$$

and

$$x_f(0) = \begin{cases} 0, & v \leq \rho, \\ -\log\left(\frac{v}{\rho}\right), & v > \rho. \end{cases}$$

When  $D \leq r$ , that is,  $v \leq \rho$ , the construction of the asymptotic expansions uses an  $\mathcal{O}(\sqrt{\epsilon})$  layer at  $x = 0$ , and the free boundary, in which  $P = \mathcal{O}(\tau^2)$ , is located outside this  $\mathcal{O}(\sqrt{\epsilon})$  interior layer. The analysis proceeds similarly to the previous case in which  $\sigma$  is a function of  $S$ . Thus, we shall confine ourselves to describing the results.

For  $x \geq \sqrt{\epsilon}$ ,  $P(x, T)$  has the outer expansion

$$P(x, T) = (e^x - 1) + \rho T \epsilon (e^x - 1) + \mathcal{O}(\epsilon^2).$$

For  $x = \mathcal{O}(\sqrt{\epsilon})$ , substituting  $X = x/\sqrt{\epsilon}$  and  $p = P/\sqrt{\epsilon}$  into PDE system (3.23), we obtain

$$\frac{\partial p}{\partial T} = a(\sqrt{\epsilon}X, \epsilon T) \frac{\partial^2 p}{\partial X^2} + (\rho - v - a(\sqrt{\epsilon}X, \epsilon T)) \frac{\partial p}{\partial X} + \sqrt{\epsilon} e^{\epsilon\rho T} (ve^x - \rho).$$

By assuming that  $a(x, \tau)$  has Taylor expansions to second order, that is,

$$a(\sqrt{\epsilon}X, \epsilon T) = a(0, 0) + a_x(0, 0)\sqrt{\epsilon}X + a_\tau(0, 0)\epsilon T + \frac{1}{2}a_{xx}(0, 0)\epsilon X^2 + \mathcal{O}(\epsilon^{3/2}),$$

and  $p$  can be expanded in powers of  $\sqrt{\epsilon}$ ,

$$p(X, T) = p_0(X, T) + \sqrt{\epsilon} p_1(X, T) + \epsilon p_2(X, T) + \mathcal{O}(\epsilon^{3/2}),$$

we obtain the sequence of PDE systems:

$$\begin{cases} \frac{\partial p_0}{\partial T} = \frac{\partial^2 p_0}{\partial X^2}, \\ p_0(X, 0) = \max(X, 0), \\ \lim_{X \rightarrow \infty} p_0(X, T) = X, \quad \lim_{X \rightarrow -\infty} p_0(X, T) = 0, \end{cases} \tag{3.24}$$

$$\begin{cases} \frac{\partial p_1}{\partial T} = \frac{\partial^2 p_1}{\partial X^2} + a_x(0, 0)X \frac{\partial^2 p_0}{\partial X^2} + (\rho - v - 1) \frac{\partial p_1}{\partial X} + v - \rho, \\ p_1(X, 0) = \max\left(\frac{1}{2}X^2, 0\right), \\ \lim_{X \rightarrow \infty} p_1(X, T) = \frac{1}{2}X^2, \quad \lim_{X \rightarrow -\infty} \frac{\partial p_1}{\partial X}(X, T) = 0, \end{cases} \tag{3.25}$$

$$\begin{cases} \frac{\partial p_2}{\partial T} = \frac{\partial^2 p_2}{\partial X^2} + a_x(0, 0)X \frac{\partial^2 p_1}{\partial X^2} + \left(a_\tau(0, 0)T + \frac{a_{xx}(0, 0)}{2}X^2\right) \frac{\partial^2 p_0}{\partial X^2} \\ \quad + (\rho - v - 1) \frac{\partial p_1}{\partial X} - a_x(0, 0)X \frac{\partial p_0}{\partial X} + vX, \\ p_2(X, 0) = \max\left(\frac{1}{6}X^3, 0\right), \\ \lim_{X \rightarrow \infty} p_2(X, T) = \frac{1}{6}X^3 + \rho XT, \quad \lim_{X \rightarrow -\infty} \frac{\partial^2 p_2}{\partial X^2}(X, T) = 0. \end{cases} \tag{3.26}$$

The solutions of the above PDE systems are derived in [Appendix C](#).

Notice that though the option prices  $p_0$ ,  $p_1$  and  $p_2$  are much more complicated than the corresponding ones in Subsection 3.1, they fortunately have the same asymptotic behaviours as  $\xi \rightarrow -\infty$ . Next, with the utilization of the same matching procedures as adopted in the previous case, we obtain the transcendental equations

$$\begin{aligned} \frac{1}{2\sqrt{\pi\tau}} e^{-x_f^2/4\tau} + v - \rho &= 0, \quad v < \rho, \\ \frac{1}{2\sqrt{\pi\tau}} e^{-x_f^2/4\tau} + vx_f &= 0, \quad v = \rho, \end{aligned}$$

from which the asymptotic behaviour of the optimal exercise price near expiry can be derived as  $S_f(t) = Ke^{x_f(t)}$  which is equal to the right-hand side of (3.15) ((3.16)) for  $D < r$  ( $D = r$ ) with  $\sigma(K)$  replaced by  $\sigma(K, T_E)$ .

When  $D > r$ , that is,  $v > \rho$ , we assume that  $x_0 = -\log(v/\rho) \ll -\sqrt{\epsilon}$ . The construction of the asymptotic expansion uses one  $\mathcal{O}(\sqrt{\epsilon})$  corner layer at  $x = 0$ , and another  $\mathcal{O}(\sqrt{\epsilon})$  inner layer at  $x_0$ . The free boundary is located inside the inner layer. The analysis is also similar to that of the last case. For simplicity, we shall briefly describe the difference and list the results.

The outer expansion is valid for  $x \gg \sqrt{\epsilon}$  and  $x \ll -\sqrt{\epsilon}$ , specifically,

$$P(x, T) = \begin{cases} e^x - 1 + \rho(e^x - 1)T\epsilon + \mathcal{O}(\epsilon^2) & x \gg \sqrt{\epsilon}, \\ (ve^x - \rho)T\epsilon + \mathcal{O}(\epsilon^2) & x \ll -\sqrt{\epsilon}. \end{cases}$$

As mentioned in the previous section, for the matching process to be discussed later, we only need one branch of the outer solution which is valid for  $x \ll -\sqrt{\epsilon}$ , and thus we omit the derivation of the solutions in the corner layer. For  $x \in U(x_0, \sqrt{\epsilon})$ , a local analysis is performed by rescaling as follows:

$$X = \frac{x - x_0}{\sqrt{\epsilon}}, \quad p = \frac{P}{\epsilon^{3/2}}, \quad X_f = \frac{x_f - x_0}{\sqrt{\epsilon}}. \tag{3.27}$$

Again, assume that  $a(x, \tau)$  has Taylor expansions to first order at  $x = x_0$  and  $\tau = 0$ , that is,

$$a(\sqrt{\epsilon}X + x_0, \epsilon T) = a(x_0, 0) + a_x(x_0, 0)\sqrt{\epsilon}X + a_\tau(x_0, 0)\epsilon T + \mathcal{O}(\epsilon). \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.23), the leading-order PDE system is

$$\begin{cases} \frac{\partial p_0}{\partial T} = a(x_0, 0) \frac{\partial^2 p_0}{\partial X^2} + \rho X, \\ p_0(X, 0) = 0, \quad \lim_{X \rightarrow \infty} p_0(X, T) = \rho XT, \end{cases}$$

which has solution

$$p_0(X, T) = T^{3/2}h(\xi),$$

where

$$\xi = \frac{X}{2\sqrt{a(x_0, 0)T}}, \quad h(\xi) = 2\rho\xi + C \left[ (\xi^2 + 1)e^{-\xi^2} - (2\xi^3 + 3\xi) \int_{\xi}^{+\infty} e^{-t^2} dt \right],$$

with  $C$  a constant. Then, by using (3.21) on the free boundary, we obtain

$$\begin{aligned} 2\rho\xi_1 + C \left[ (\xi_1^2 + 1)e^{-\xi_1^2} - (2\xi_1^3 + 3\xi_1) \int_{\xi_1}^{+\infty} e^{-t^2} dt \right] &= 0, \\ 2\rho + C \left[ 3\xi_1 e^{-\xi_1^2} - (6\xi_1^2 + 3) \int_{\xi_1}^{+\infty} e^{-t^2} dt \right] &= 0, \end{aligned}$$

from which, the transcendental equation for  $\xi_1$  can be derived as

$$-\xi_1^3 e^{\xi_1^2} \int_{\xi_1}^{+\infty} e^{-t^2} dt = \frac{1}{4}(1 - 2\xi_1^2). \tag{3.29}$$

Here,  $\xi_1 = X_1/2\sqrt{a(x_0, 0)T}$  and  $X_1$  is the leading-order term of  $X_f$ . The solution of (3.29) is  $\xi_1 = 0.4517$ . Therefore,

$$\begin{aligned} x_f(\tau) &= x_0 - 2\xi_1\sqrt{a(x_0, 0)T} + \mathcal{O}(\tau) \\ &= x_0 - \sqrt{2}\xi_1\sigma\left(\frac{r}{D}K, T_E\right)\sqrt{T_E - t} + \mathcal{O}(T_E - t), \end{aligned}$$

and thus

$$S_f(t) = Ke^{x_f} = \frac{r}{D}K\left[1 - \sigma\left(\frac{r}{D}K, T_E\right)\xi_1\sqrt{2(T_E - t)}\right] + \mathcal{O}(T_E - t).$$

It is clear that if  $\sigma(S, t)$  is independent of both  $S$  and  $t$ , our results again degenerate to those derived in [3].

### 4. Conclusion

In this paper the asymptotic behaviour of the optimal exercise price for an American put option is investigated in the local volatility model. Based on singular perturbation methods, the leading-order term of the optimal exercise price is derived, which is expected to be complementary to numerical methods. The result derived in this paper is believed to be quite reasonable, since the leading-order term of the optimal exercise price in the stock-price-dependent volatility model agrees with those in the literature, and it degenerates to the result of Evans *et al.* if the volatility function is assumed to be a constant. As the singular perturbation method is not limited to one-dimensional problems, a further task will be to consider its application to American options on an underlying described by a multi-factor model.

### Appendix A. Solutions of the PDE systems (3.12)–(3.14)

To find the solution of PDE system (3.12), we assume that it can be written as

$$p_0(X, T) = \sqrt{T}h_0(\xi) \quad \text{where } \xi = \frac{X}{2\sqrt{T}}. \tag{A.1}$$

By substituting (A.1) into (3.12), we obtain the following ordinary differential equation (ODE) system for  $h_0(\xi)$ :

$$\begin{cases} h_0''(\xi) + 2\xi h_0'(\xi) - 2h_0(\xi) = 0, \\ \lim_{\xi \rightarrow \infty} h_0(\xi) = 2\xi, \quad \lim_{\xi \rightarrow -\infty} h_0(\xi) = 0. \end{cases}$$

The analytical solution of this ODE system can be readily found to be

$$h_0(\xi) = \frac{1}{\sqrt{\pi}}e^{-\xi^2} + \xi \operatorname{erfc}(-\xi).$$

Similarly, by assuming that the solution of PDE system (3.13) is in the form

$$p_1(X, T) = Th_1(\xi),$$

we have

$$\begin{cases} h_1''(\xi) + 2\xi h_1'(\xi) - 4h_1(\xi) = 2(1 - \rho + v)\operatorname{erfc}(-\xi) - \frac{4a'(0)}{\sqrt{\pi}}\xi e^{-\xi^2} + 4(\rho - v), \\ \lim_{\xi \rightarrow \infty} h_1(\xi) = 2\xi^2, \quad \lim_{\xi \rightarrow -\infty} h_1'(\xi) = 0. \end{cases} \quad (\text{A.2})$$

Suppose that the solution  $h_1(\xi)$  has the structure

$$h_1(\xi) = f(\xi)e^{-\xi^2} + g(\xi) \int_{-\infty}^{\xi} e^{-t^2} dt + m(\xi), \quad (\text{A.3})$$

where  $f(\xi)$ ,  $g(\xi)$  and  $m(\xi)$  are polynomials in  $\xi$ . By substituting (A.3) into (A.2), we obtain

$$\begin{aligned} & (f'' - 2\xi f' + 2g' - 6f) e^{-\xi^2} + (g'' + 2\xi g' - 4g) \int_{-\infty}^{\xi} e^{-t^2} dt + m'' + 2\xi m' - 4m \\ &= \frac{4(1 - \rho + v)}{\sqrt{\pi}} \int_{-\infty}^{\xi} e^{-t^2} dt - \frac{4a'(0)}{\sqrt{\pi}} \xi e^{-\xi^2} + 4(\rho - v), \end{aligned}$$

which, combined with the boundary conditions at  $\xi = \pm\infty$ , yields

$$\begin{cases} m'' + 2\xi m' - 4m = 4(\rho - v), \\ \lim_{\xi \rightarrow -\infty} m' = 0, \end{cases} \quad (\text{A.4})$$

$$\begin{cases} g'' + 2\xi g' - 4g = \frac{4(1 - \rho + v)}{\sqrt{\pi}}, \\ \lim_{\xi \rightarrow \infty} \sqrt{\pi}g + m = 2\xi^2, \end{cases} \quad (\text{A.5})$$

$$f'' - 2\xi f' + 2g' - 6f = -\frac{4a'(0)}{\sqrt{\pi}}\xi. \quad (\text{A.6})$$

The polynomial solutions of (A.4)–(A.6) can be readily found:

$$m(\xi) = v - \rho, \quad g(\xi) = \frac{2}{\sqrt{\pi}}\xi^2 + \frac{\rho - v}{\sqrt{\pi}}, \quad f(\xi) = \frac{1}{\sqrt{\pi}} \left( 1 + \frac{a'(0)}{2} \right) \xi.$$

Therefore,

$$h_1(\xi) = \frac{1}{\sqrt{\pi}} \left( 1 + \frac{a'(0)}{2} \right) \xi e^{-\xi^2} + \left( \frac{2}{\sqrt{\pi}}\xi^2 + \frac{\rho - v}{\sqrt{\pi}} \right) \int_{-\infty}^{\xi} e^{-t^2} dt + v - \rho.$$

Using the above solution technique, it is not hard to find the solution of (3.14), though the solution is in quite a complicated form. By assuming that

$$p_2(X, T) = T^{3/2}h_2(\xi),$$

and substituting it to (3.14), we obtain the following ODE system for  $h_2(\xi)$ :

$$\begin{cases} h_2''(\xi) + 2\xi h_2'(\xi) - 6h_2(\xi) = (C\xi^4 + A\xi^2 + B)e^{-\xi^2} \\ \quad + \frac{8(1+v-\rho)}{\sqrt{\pi}}\xi \int_{-\infty}^{\xi} e^{-t^2} dt - 8v\xi, \\ \lim_{\xi \rightarrow \infty} h_2(\xi) = \frac{4}{3}\xi^3 + 2\rho\xi, \quad \lim_{\xi \rightarrow -\infty} h_2''(\xi) = 0, \end{cases}$$

where

$$A = -\frac{4a''(0)}{\sqrt{\pi}} + \frac{6(a'(0))^2}{\sqrt{\pi}} - \frac{4a'(0)}{\sqrt{\pi}} - \frac{6a'(0)(v-\rho)}{\sqrt{\pi}},$$

$$B = \frac{\rho-v-1}{\sqrt{\pi}}(-2-a'(0)+2v-2\rho), \quad C = -\frac{4(a''(0))^2}{\sqrt{\pi}}.$$

Suppose that  $h_2(\xi)$  can be written as

$$h_2(\xi) = f(\xi)e^{-\xi^2} + g(\xi) \int_{-\infty}^{\xi} e^{-t^2} dt + m(\xi),$$

where  $f(\xi)$ ,  $g(\xi)$  and  $m(\xi)$  are polynomials in  $\xi$ . By using the same procedure as in deriving  $h_1$ , the ODE systems for  $f(\xi)$ ,  $g(\xi)$  and  $m(\xi)$  can easily be found to be

$$\begin{cases} m'' + 2\xi m' - 6m = -8v\xi, \\ \lim_{\xi \rightarrow -\infty} m'' = 0, \end{cases} \tag{A.7}$$

$$\begin{cases} g'' + 2\xi g' - 4g = \frac{8(1+v-\rho)}{\sqrt{\pi}}, \\ \lim_{\xi \rightarrow \infty} \sqrt{\pi}g + m = \frac{4\xi^2}{3} + 2\rho\xi, \end{cases} \tag{A.8}$$

$$f'' - 2\xi f' - 8f + 2g' = C\xi^4 + A\xi^2 + B. \tag{A.9}$$

The polynomial solutions of (A.7)–(A.9) are

$$m(\xi) = 2v\xi, \quad g(\xi) = \frac{4\xi^3}{3\sqrt{\pi}} + \frac{2(\rho-v)\xi}{\sqrt{\pi}},$$

$$f(\xi) = -\frac{C}{16}\xi^4 - \left(\frac{C}{16} + \frac{\tilde{A}}{12}\right)\xi^2 - \frac{C}{64} - \frac{\tilde{A}}{48} - \frac{\tilde{B}}{8},$$

where

$$\tilde{A} = A - \frac{8}{\sqrt{\pi}}, \quad \tilde{B} = B - \frac{4(\rho-v)}{\sqrt{\pi}}.$$





$$\begin{cases} h_2''(\xi) + 2\xi h_2'(\xi) - 6h_2(\xi) \\ = (C\xi^4 + A\xi^2 + B)e^{-\xi^2} + \frac{8(1+v-\rho)}{\sqrt{\pi}} \int_{-\infty}^{\xi} e^{-t^2} dt - 8v\xi, \\ \lim_{\xi \rightarrow \infty} h_2(\xi) = \frac{4}{3}\xi^3 + 2\rho\xi, \quad \lim_{\xi \rightarrow -\infty} h_2''(\xi) = 0, \end{cases}$$

where

$$A = -\frac{2a_x(0, 0)(2 - 3a_x(0, 0) + 2v - 2\rho)}{\sqrt{\pi}} - \frac{2a_x(0, 0)(1 + v - \rho)}{\sqrt{\pi}} - \frac{4a_{xx}(0, 0)}{\sqrt{\pi}},$$

$$B = -\frac{(1 + v - \rho)(-2 - a_x(0, 0) + 2v - 2\rho)}{\sqrt{\pi}} - \frac{2a_\tau(0, 0)}{\sqrt{\pi}}, \quad C = -\frac{4a_x^2(0, 0)}{\sqrt{\pi}}.$$

By using the solution techniques as introduced in **Appendix A**, we obtain

$$h_0(\xi) = \frac{1}{\sqrt{\pi}}e^{-\xi^2} + \xi \operatorname{erfc}(-\xi),$$

$$h_1(\xi) = \frac{1}{\sqrt{\pi}}\left(1 + \frac{a_x(0, 0)}{2}\right)\xi e^{-\xi^2} + \left(\frac{2}{\sqrt{\pi}}\xi^2 + \frac{\rho - v}{\sqrt{\pi}}\right) \int_{-\infty}^{\xi} e^{-t^2} dt + v - \rho,$$

$$h_2(\xi) = \left[-\frac{C}{16}\xi^4 - \left(\frac{C}{16} + \frac{\tilde{A}}{12}\right)\xi^2 - \frac{C}{64} - \frac{\tilde{A}}{48} - \frac{B}{8}\right]e^{-\xi^2}$$

$$+ \left(\frac{4\xi^3}{3\sqrt{\pi}} + \frac{2(\rho - v)\xi}{\sqrt{\pi}}\right) \int_{-\infty}^{\xi} e^{-t^2} dt + 2v\xi,$$

where

$$\tilde{A} = A - \frac{8}{\sqrt{\pi}}, \quad \tilde{B} = B - \frac{4(\rho - v)}{\sqrt{\pi}}.$$

### References

- [1] G. Barles, J. Burdeau, M. Romano and N. Samscen, “Critical stock price near expiration”, *Math. Finance* **5** (1995) 77–95.
- [2] E. Chevalier, “Critical price near maturity for an American option on a dividend-paying stock in a local volatility model”, *Math. Finance* **15** (2005) 439–463.
- [3] J. D. Evans, R. Kuske and J. B. Keller, “American options on assets with dividends near expiry”, *Math. Finance* **12** (2002) 219–237.
- [4] A. Kirsch, *Introduction to the mathematical theory of inverse problems* (Springer, New York, 1996).
- [5] J. Zhao and H. Y. Wong, “A closed-form solution to American options under general diffusion processes”, *Quant. Finance*, to appear, doi:10.1080/14697680903193405.
- [6] S. P. Zhu, “An exact and explicit solution for the valuation of American put options”, *Quant. Finance* **6** (2006) 229–242.