Can. J. Math. Vol. **46** (3), 1994 pp. 543–573.

REAL PROJECTIVE REPRESENTATIONS OF S_N AND A_N

JOHN Q. HUANG

ABSTRACT. Three main results are obtained in this paper: one generalizes the Atiyah-Bott-Shapiro periodicity equivalence on the category of real Clifford modules, (Theorem 2.2); another establishes the existence of two algebras for real projective representations of the symmetric group S_n and the alternating group A_n, (Theorem 3.2) and determines their structure, (Theorem 6.1); the third describes all the real projective representations of S_n and A_n except for some small numbers n, (Theorem 7.2).

Introduction. Let $\mathbb{F}$ be a division algebra and G be a group with a central element z of order 2. Assume further that G is equipped with a parity homomorphism $\sigma\colon G \longrightarrow \mathbb{Z}/2$ such that z is in the kernel. Let $M(\mathbb{F}[G])$ be the category of $\mathbb{F}[G]$-modules on which z acts as -1. In [6], P. Hoffman defined certain categories $Z_{\mathbb{C}}^{(n)}(G)$ of objects which are simultaneously $\mathbb{Z}/2$-graded G-modules and Clifford modules over the complex numbers, then proved an equivalence between $Z_{\mathbb{C}}^{(n)}(G)$ and $Z_{\mathbb{C}}^{(n+2)}(G)$, which generalized the complex Clifford module periodicity of Atiyah-Bott-Shapiro [3]. Moreover, $Z_{\mathbb{C}}^{(0)}(G)$ is equivalent to $M(\mathbb{C}[\ker\sigma])$ if σ is non-zero and $Z_{\mathbb{C}}^{(1)}(G)$ is equivalent to $M(\mathbb{C}[G])$. This allows one to re-obtain the work of P. Hoffman and J. Humphreys [8]; a revised and more readable version appears in Appendix 8 in their recently published book [10]. If we define $T_{\mathbb{F}}^{(n)}(G)$ to be the Grothendieck group of the category $Z_{\mathbb{F}}^{(n)}(G)$, then, in fact, Hoffman and Humphreys determined the structure of the $\mathbb{Z}/2\times\mathbb{N}$-graded algebra

$$T^* := \{T_{\mathbb{C}}^{(m)}(\tilde{S}_n) : m \in \mathbb{Z}/2, n \in \mathbb{N}\}$$

by generators and relations. As consequence, they gave a new formulation of I. Schur's original work on complex projective characters for S_n and A_n [19]. Here $\tilde{S}_n$ is one of the two essential double covers of S_n.

In this paper we shall extend all the work above to real numbers. We prove the analogous generalization over real numbers: $Z_{\mathbb{R}}^{(n)}(G)$ and $Z_{\mathbb{R}}^{(n+8)}(G)$ are equivalent. This will be done in Section 2. In Section 3, by using the 8-fold periodicity of $T_{\mathbb{R}}^{(n)}(G)$, one can prove that

$$T_{\mathbb{R}}^* := \{T_{\mathbb{R}}^{(m)}\hat{S}_n : m \in \mathbb{Z}/8, n \in \mathbb{N}\}$$

is a $\mathbb{Z}/8\times\mathbb{N}$-graded ring. In Section 6, we come to the main purpose of this paper: to determine the structure of the above ring by generators and relations. As a consequence,

This paper is a part of the doctoral thesis of the author at the University of Waterloo.
Received by the editors October 27, 1992.
AMS subject classification: Primary: 20C25, 20C15; secondary: 16A24, 20C30.

we describe all the real projective representations of S_n and A_n except for small n in Section 7. Based on the results of Schur [20] and Nazarov [17, 18], M. Makhool and A. Morris [16] constructed a complete set of real irreducible projective representations of S_n for two of the three non-trivial classes in its Schur multiplier. These are constructed by giving "negative" modules for $\tilde{S}_n$ and $\hat{S}_n$ for each n. Here $\hat{S}_n$ is another essential double cover of S_n.

The main method of proof is by converting T^* graded over $\mathbb{Z}/2$ to a $\mathbb{Z}/8$-graded ring $T^*_{\mathbb{C}}$, getting the structure of $T^*_{\mathbb{C}}$. This will be done in Section 4. We then use the classic techniques, *i.e.*, restrictions from $\mathbb{H}$ to $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{R}$, extensions from $\mathbb{R}$ to $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{H}$, and conjugation from $\mathbb{C}$ to $\mathbb{C}$, in relating the irreducible real and quaternionic representations to the irreducible complex representations [1] to obtain the structure of the real ring. In Section 5, we show how to extend these techniques to our $\mathbb{Z}/2$-graded modules.

ACKNOWLEDGEMENT. I would like to thank Professor P. Hoffman who suggested some of the questions studied here. This paper may be regarded as a sequel to [6] and [10, Appendix 8].

1. **Preliminaries.** We shall recall in this section the categories of graded representations, the tensor product and Hom as described in [6], then build a $T^*_{\mathbb{F}}\{1,z\}$-module $T^*_{\mathbb{F}}G$ for every $\mathcal{G}$-object G. Many of the definitions before Lemma 1.1 are either from [6] or are analogous to definitions there, but we shall repeat them here, since they are fundamental to the rest of this paper.

In this paper we will write $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively, for the sets of non-negative integers, integers, rationals, reals, complex numbers and quaternions.

Let $\mathbb{F}$ be a division ring and G a group. By the term G-module, we mean any left module over the group algebra $\mathbb{F}[G]$, which is finite dimensional over $\mathbb{F}$.

DEFINITION 1.1. Let $\mathcal{G}$ be the class of triples (G,z,σ), where G is a group, z is an element of order 2 in the center of G, and σ is a homomorphism from G to $\mathbb{Z}/2$ with $\sigma(z)=0$. When z and σ are unambiguous, we often denote (G,z,σ) simply as G. We say $\phi\colon G\longrightarrow G'$ is a $\mathcal{G}$-map, if ϕ is a group homomorphism such that $\phi(z)=z'$ and $\sigma'\phi(g)=\sigma(g)$ for all $g\in G$.

EXAMPLE 1. Define $\hat{S}_n$ to be the group with generators $\{z,t_1,\ldots,t_{n-1}\}$ and relations

$$z^2 = 1 = [z,t_j];$$
$$t_j^2 = 1 = (t_kt_{k+1})^3,\quad \text{for } 1\le j\le n-1;\ 1\le k\le n-2;$$
$$t_jt_k = zt_kt_j,\quad \text{for } |j-k|>1 \text{ and } 1\le j,\ k\le n-1.$$

So $(\hat{S}_n,z,\sigma)$ is a canonical example of an object in $\mathcal{G}$, where σ is the composite $s\circ\pi$, where $(-1)^s$ is the sign homomorphism on S_n, and π is the map $\hat{S}_n\longrightarrow S_n$ defined by sending t_i to the transposition $(ii+1)$. In the literature, $\hat{S}_n$ is a complex representation group of S_n when $n>3$ and is called a *double covering group* of S_n, since $\hat{S}_n$ has order $2(n!)$.

Let G be an object in $\mathcal{G}$ (z and σ being understood). Define $M(\mathbb{F}[G])$ to be the category of $\mathbb{F}[G]$-modules on which z acts as -1. Morphisms in $M(\mathbb{F}[G])$ are module homomorphisms.

Let $\mathcal{Z}_{\mathbb{F}}^{(0)}(G)$ be the category of all triples $(V^{(0)}+V^{(1)}, V^{(0)}, V^{(1)})$ where:

(a) $V^{(0)}+V^{(1)}$ is in $M(\mathbb{F}[G])$;

(b) $V^{(0)}\bigcap V^{(1)}=\{0\}$ (*i.e.* the sum is direct as vector spaces); and

(c) If $g\in G$, then $gV^{(i)}=V^{(i+\sigma(g))}$.

Such triples are called $\mathbb{Z}/2$-*graded negative representations*, and are usually denoted just $(V^{(0)}, V^{(1)})$. A morphism in $\mathcal{Z}_{\mathbb{F}}^{(0)}(G)$ from $(V^{(0)}, V^{(1)})$ to $(W^{(0)}, W^{(1)})$ is a module homomorphism

$$V^{(0)}+V^{(1)}\longrightarrow W^{(0)}+W^{(1)}$$

mapping $V^{(i)}$ to $W^{(i)}$ for both i.

For non-negative integers p and q, let $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$ be the category of all sequences

$$(V, V', \eta_1,\ldots,\eta_p,\xi_1,\ldots,\xi_q)$$

such that:

(i) (V, V') is an object in $\mathcal{Z}_{\mathbb{F}}^{(0)}(G)$;

(ii) $\eta_i, \xi_j\colon V+V'\longrightarrow V+V'$ are $\mathbb{F}[G]$-module homomorphisms mapping V to V' and V' to V;

(iii) $\eta_i^2=\mathrm{id}$, $\xi_j^2=-\mathrm{id}$, and each pair of distinct elements from $\{\eta_1,\ldots,\eta_p,\xi_1,\ldots,\xi_q\}$ anticommutes.

A morphism in $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$, $(V, V', \eta_1,\ldots,\eta_p,\xi_1,\ldots,\xi_q)\xrightarrow{\theta}(W, W', \eta_1',\ldots,\eta_p', \xi_1'\ldots,\xi_q')$, is a morphism $(V, V')\longrightarrow(W, W')$ in $\mathcal{Z}_{\mathbb{F}}^{(0)}(G)$ such that $\eta_i'\theta=\theta\eta_i$, and $\xi_j'\theta=\theta\xi_j$.

When $p=0$, we denote $\mathcal{Z}_{\mathbb{F}}^{(0,q)}(G)$ as $\mathcal{Z}_{\mathbb{F}}^{(-q)}(G)$; when $q=0$, denote $\mathcal{Z}_{\mathbb{F}}^{(p,0)}(G)$ as $\mathcal{Z}_{\mathbb{F}}^{(p)}(G)$.

NOTE 1. Each category $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$ admits a $\oplus$ bifunctor with the obvious definition (taking the direct sums of everything in sight). Let $T_{\mathbb{F}}^{(p,q)}(G)$ (resp. $T_{\mathbb{F}}^{(0)}(G)$) be the Grothendieck group under $\oplus$ of $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$ (resp. $\mathcal{Z}_{\mathbb{F}}^{(0)}(G)$).

NOTE 2. When $G=\{1,z\}$ and $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, we can interpret $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$ as the category of negative $C(Q_{p,q})$ modules, where $C(Q_{p,q})$ is the Clifford algebra of the quadratic form $Q_{p,q}$ on $\mathbb{F}^{p+q}$, with

$$Q_{p,q}(x_1,\ldots,x_{p+q})=\sum_1^p x_i^2-\sum_{p+1}^{p+q}x_i^2$$

for $x_i\in\mathbb{R}$ or $\mathbb{C}$, and $1\leq i\leq p+q$.

NOTE 3. For an object (G, z, σ_G) in $\mathcal{G}$, where G is a finite group, define for each pair of integers $p, q\geq 0$ a new object $(G_{p,q}, z, \sigma)$ as follows: $G_{p,q}$ is the group generated by

$$\{g, \eta_1,\ldots,\eta_p,\xi_1\ldots,\xi_q \mid g\in G\}$$

subject to the relations in G and

$$[g,\eta_i]=[g,\xi_k]=1=\eta_i^2,\quad [\eta_i,\eta_j]=[\xi_k,\xi_l]=[\eta_i,\xi_k]=z=\xi_k^2,$$
$$\sigma|_G=\sigma_G \quad\text{and}\quad \sigma(\eta_i)=\sigma(\xi_j)=1.$$

So $G_{p.q}$ is finite and what is more, $Z_{\mathbb{F}}^{(p.q)}(G)$ and $T_{\mathbb{F}}^{(0)}(G_{p.q})$ are equivalent. Therefore, since $Z_{\mathbb{F}}^{(0)}(G_{p.q})$ and $M(\mathbb{F}[\ker\sigma])$ are equivalent (see 2.1, 2.2 [5]; the proofs there for $\mathbb{C}$ apply to any division ring of characteristic 0) if σ is not zero, $Z_{\mathbb{F}}^{(p.q)}(G)$ and $M(\mathbb{F}[\ker\sigma])$ are equivalent as categories. In fact, we have functors:

$$P\colon Z_{\mathbb{F}}^{(p.q)}(G) \longrightarrow Z_{\mathbb{F}}^{(0)}(G_{p.q}) \quad \text{and} \quad Q\colon Z_{\mathbb{F}}^{(0)}(G_{p.q}) \longrightarrow Z_{\mathbb{F}}^{(p.q)}(G).$$

Here $P(V, V', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q)$ is (V, V'), the $G_{p.q}$-action is induced by G, η_i and ξ_j in $Z_{\mathbb{F}}^{(p.q)}(G)$. Similarly we can define Q with PQ and QP being naturally isomorphic to identity functors. Hence any result about $Z_{\mathbb{F}}^{(p.q)}(G)$ can be translated into a result concerning the more familiar category $M(\mathbb{F}[\ker\sigma])$. When $p = 0$, we denote $G_{0.q}$ as G_{-q}; when $q = 0$, denote $G_{p.0}$ as G_p.

Two objects (G, z_G, σ_G) and (H, z_H, σ_H) produce a new object $(G\hat{Y}H, z, \sigma)$ as in [10, Chapter 3]. Let

$$G\hat{Y}H := (G\hat{\times}H)/Z$$

where $G\hat{\times}H$ is the cartesian set product of G and H, with group operation

$$(a, b)(c, d) = (z^{\sigma_H(b)\sigma_G(c)}ac, bd) \quad \text{and} \quad Z := \{(1, 1), (z_G, z_H)\}.$$

Denote elements of $G\hat{Y}H$ as ordered pairs, and define

$$z = (z_G, 1) = (1, z_H) \quad \text{and} \quad \sigma(g, h) = \sigma_G(g) + \sigma_H(h).$$

There are natural isomorphisms of groups

$$G \cong \{1, z\}\hat{Y}G, \quad G \cong G\hat{Y}\{1, z\}, \quad G\hat{Y}H \cong H\hat{Y}G$$

by sending

$$g \longmapsto (1, g), \quad g \longmapsto (g, 1), \quad (g, h) \overset{T}{\longmapsto} z^{|g|\,|h|}(h, g),$$

respectively.

EXAMPLE 2. Let k, l and n be positive numbers with $n \geq k + l$. We regard $\hat{S}_k$ as the subgroup of $\hat{S}_n$ which is the double cover of the symmetric group on $\{1, \ldots, k\}$, and $\hat{S}_l$ as the double cover of the symmetric group on $\{k+1, \ldots, k+l\}$. Thus the generators for $\hat{S}_l$ are now denoted $t_{k+1}, \ldots, t_{k+l-1}$. As subgroups of $\hat{S}_n$, we may form $\langle \hat{S}_k, \hat{S}_l \rangle$, the subgroup generated by $\hat{S}_k \cup \hat{S}_l$. This is a group with generators $z, t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{k+l-1}$, with z a central element of order 2, and relations

$$\begin{gathered} t_i^2 = 1, \quad 1 \leq i \leq k + l - 1,\ i \neq k; \\ (t_i t_{i+1})^3 = 1, \quad \text{for } 1 \leq i \leq k - 2 \text{ or } k + 1 \leq i \leq k + l - 2; \\ t_i t_j = z t_j t_i,\ 1 \leq i, j \leq k + l - 1,\ |i - j| > 2,\ i \neq k, \quad j \neq k. \end{gathered}$$

On the other hand, $\hat{S}_k$ and $\hat{S}_l$ are both objects in $\mathcal{G}$, and so we can form $\hat{S}_k\hat{Y}\hat{S}_l$. Just as in [10, Chapter 3], one can prove that the map $\phi_{k.l}$ taking $(t_i, 1)Z$ to t_i for $1 \leq i \leq k - 1$, and $(1, t_i)Z$ to t_i for $k + 1 \leq i \leq k + l - 1$, is a $\mathcal{G}$-isomorphism between $\hat{S}_k\hat{Y}\hat{S}_l$ and $\langle \hat{S}_k, \hat{S}_l \rangle$.

DEFINITION 1.2. Define a tensor product

$$Z_{\mathbb{F}}^{(p,q)}(G) \times Z_{\mathbb{F}}^{(r,s)}(H) \longrightarrow Z_{\mathbb{F}}^{(p+r,q+s)}(G\hat{Y}H)$$
$$(\mathcal{V}, \mathcal{W}) \longmapsto \mathcal{V} \otimes \mathcal{W}$$

by

$$\begin{aligned}\mathcal{V} \otimes \mathcal{W} &= (V, V', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q) \otimes (W, W', \zeta_1, \ldots, \zeta_r, \chi_1, \ldots, \chi_s)\\ &:= \big((V, V') \otimes (W, W'), 1 \otimes \zeta_1, \ldots, 1 \otimes \zeta_r, \eta_1 \widehat{\otimes} 1, \ldots, \eta_p \widehat{\otimes} 1, 1 \otimes \chi_1, \ldots, 1\\ &\quad \otimes \chi_s, \xi_1 \widehat{\otimes} 1, \ldots, \xi_q \widehat{\otimes} 1\big),\end{aligned}$$

where $\otimes$ inside the right side is the $\mathbb{Z}/2$-graded tensor product, *i.e.*,

$$(V, V') \otimes (W, W') := (V \otimes W + V' \otimes W', V \otimes W' + V' \otimes W)$$

with action, for v in $V^{(k)}$, w in $W^{(j)}$, g in G and h in H, given by

$$(g, h)(v \otimes w) := (-1)^{k\sigma(h)} gv \otimes hw$$

and where

$$(\alpha \widehat{\otimes} 1)(v \otimes w) := (-1)^{|w|} \alpha(v) \otimes w.$$

Here and below, $|w|$ (= 0 or 1) is the grading of w, and $|g|$ denotes $\sigma(g)$ for an element g of an object G in $\mathcal{G}$.

DEFINITION 1.3. For $\mathcal{V} \in Z_{\mathbb{F}}^{(p,q)}(G)$, $\mathcal{W} \in Z_{\mathbb{F}}^{(r,s)}(G)$. Define $\mathrm{Hom}_{\mathbb{F}[G]}(\mathcal{V}, \mathcal{W})$ to be the vector space of linear maps $\phi\colon V+V' \longrightarrow W+W'$, where $\mathcal{V} = (V, V', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q)$ and $\mathcal{W} = (W, W', \zeta_1, \ldots, \zeta_r, \chi_1, \ldots, \chi_s)$, such that

(a) $\phi(V) \subset W$ and $\phi(V') \subset W'$;
(b) if $g \in G$, then $\phi(gv) = g(\phi v)$;
(c) if $1 \leq i \leq \min\{p, r\}$ and $1 \leq j \leq \min\{q, s\}$, then $\phi \circ \eta_i = \zeta_i \circ \phi$ and $\phi \circ \xi_j = \chi_j \circ \phi$.

Now let us define when objects are irreducible and inequivalent in $Z_{\mathbb{F}}^{(p,q)}(G)$.

DEFINITION 1.4. A subobject of $(V, V', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q)$ in $Z_{\mathbb{F}}^{(p,q)}(G)$ is any object $(W, W', \eta'_1, \ldots, \eta'_p, \xi'_1, \ldots, \xi'_q)$ with W, W' subspaces of V, V' respectively, the action being the restrictions (so $W + W'$ is G-invariant), and with $\eta'_i := \eta_i|_{W+W'}$, $\xi'_j = \xi_j|_{W+W'}$. An object is said to be *irreducible* iff it has no non-zero proper subobjects, and is itself non-zero. Two objects $\mathcal{U}$ and $\mathcal{V}$ are said to be *equivalent or isomorphic* if there exist morphisms $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ and $\psi\colon \mathcal{V} \longrightarrow \mathcal{U}$ with $\varphi\psi = \mathrm{id}_{\mathcal{V}}$, $\psi\varphi = \mathrm{id}_{\mathcal{U}}$, such φ and ψ are called *isomorphisms* as usual. Otherwise, $\mathcal{U}$ and $\mathcal{V}$ are inequivalent.

It is easy to see that kernels and images of morphisms in $Z_{\mathbb{F}}^{(p,q)}(G)$ yield subobjects when $\mathbb{F}$ is a field, so the usual proof of Schur's lemma gives:

LEMMA 1.1. *If $\mathcal{V}$ and $\mathcal{W}$ are irreducibles in $Z_{\mathbb{F}}^{(p,q)}(G)$, then we have*
1. $\mathrm{Hom}_{\mathbb{F}[G]}(\mathcal{V}, \mathcal{W})$ *is a division algebra with $\mathbb{F}$ in its center if $\mathcal{V}$ and $\mathcal{W}$ are isomorphic;*

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

2. $\mathrm{Hom}_{\mathbb{F}[G]}(\mathcal{V}, \mathcal{W}) \cong \mathbb{F}$ *if* $\mathbb{F}$ *is algebraically closed and* $\mathcal{V}$ *and* $\mathcal{W}$ *are isomorphic;*
3. $\mathrm{Hom}_{\mathbb{F}[G]}(\mathcal{V}, \mathcal{W}) \cong 0$ *if* $\mathcal{V}$ *and* $\mathcal{W}$ *are not isomorphic.*

When G is finite and $\mathbb{F}$ is a field with characteristic 0, the usual averaging trick applies to morphisms in $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$ yielding the analogue of Maschke's theorem: *Each object is isomorphic to a direct sum of irreducibles*. Since $\mathrm{Hom}_{\mathbb{F}[G]}(\mathcal{V}, \mathcal{W})$ is easily seen to be bi-additive with respect to $\oplus$, it follows from Lemma 1.1 that the multiset of isomorphism classes of irreducibles (occurring in a decomposition of a given object) is unique, *i.e.*, independent of the decomposition.

Let $\phi\colon G \longrightarrow G'$ be a $\mathcal{G}$-map. If V' is a G'-module, we denote by $\phi^* V'$ the G-module obtained by "restricting V' along ϕ", that is, $\phi^* V'$ is V' as a vector space, with G-action by $gv' := \phi(g)v'$. Similarly, if $\mathcal{V}' = (V_0', V_1', \eta_1, \ldots, \xi_q)$ is an object in $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G')$, then $\phi^*\mathcal{V}'$ is just $(V_0', V_1', \eta_1, \ldots, \xi_q)$, an object in $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$ with G-action $gv' := \phi(g)v'$ and the same η_i, ξ_j. Now suppose $\phi\colon G \longrightarrow G'$ is an injective $\mathcal{G}$-map and let $\mathcal{V}$ be an object in $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$. Then one can "induce along ϕ" to produce an object in $\mathcal{Z}_{\mathbb{F}}^{(p,q)}(G')$, called $\phi_*\mathcal{V}$. This is done in exact analogy with the ungraded case. For this classical treatment, we refer to [21].

Let $1 := (\mathbb{F}, 0)$ and $1^* := (0, \mathbb{F})$, which are elements of $\mathcal{Z}_{\mathbb{F}}^{(0)}(\{1, z\})$. Then immediately we have, using the product in Definition 1.2,

$$1\mathcal{V} \cong \mathcal{V} \cong \mathcal{V}1;$$

after identifying $\{1, z\}\hat{Y}G$ and $G\hat{Y}\{1, z\}$ with G. We shall supress this identification in the paper. Also it is easy to see that $(1^*)^2 \cong 1$. Now if $\mathcal{V} = (V, V', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q) \in \mathcal{Z}_{\mathbb{F}}^{(p,q)}(G)$, then we have following three results.

(i) $1^*\mathcal{V} \cong \big((V', V, *), \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q\big)$, where $g * v = (-1)^{|g|} gv$;
(ii) $\mathcal{V}1^* \cong (V', V, -\eta_1, \ldots, -\eta_p, -\xi_1, \ldots, -\xi_q)$;
(iii) $1^*\mathcal{V} \cong \mathcal{V}1^*$.

The proofs of the above results are straightforward and we omit them. Note that 1^* has the same meaning as ρ in [6].

DEFINITION 1.5. An irreducible object $\mathcal{V}$ of $T_{\mathbb{F}}^{(p,q)}(G)$ is said to be a *special* one if and only if $1^*\mathcal{V}$ is not isomorphic to $\mathcal{V}$.

For each object G in $\mathcal{G}$, define $T_{\mathbb{F}}^* G$ to be the $\mathbb{N}$-graded abelian group $\sum_{n=0}^{\infty} \oplus T_{\mathbb{F}}^{(n)}(G)$. The tensor product is associative (for example see Proposition 2.1 in [6]), so $T_{\mathbb{F}}^*\{1, z\}$ is a graded ring, with the multiplication as the tensor product defined above. Using the same tensor product and identifying $\{1, z\}\hat{Y}G$ with G as we did before, we make $T_{\mathbb{F}}^* G$ into a left module over the ring $T_{\mathbb{F}}^*\{1, z\}$ with 1 as the identity and 1^* as an involution.

From now on we assume $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ till the end of the paper except for Section 5.

LEMMA 1.2. *We have*

$$T_{\mathbb{F}}^{(1)}(\{1, z\}_1) \cong \mathbb{Z}b \oplus \mathbb{Z}1^*b.$$

Here the group $\{1, z\}_1$ *is defined in Note 3 and b is a special irreducible of dimension* 2 *in* $\mathcal{Z}_{\mathbb{F}}^{(1)}(\{1, z\}_1)$.

PROOF. Let

$$G = \{1, z\}_1 \quad \text{and} \quad H = (\{1, z\}_1)_1.$$

Then

$$G = \{1, z, \zeta, z\zeta \mid \zeta^2 = 1 = [z, \zeta]\} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

by notation in Note 3. And similarly

$$H = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Let H^0 be the subgroup consisting of all the elements with even degree. Then

$$H^0 = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

By Note 3 and [5, 2.1 and 2,2], we have

$$Z_{\mathbb{F}}^{(1)}(G) \simeq Z_{\mathbb{F}}^{(0)}(H) \simeq M(\mathbb{F}[\mathbb{Z}/2 \oplus \mathbb{Z}/2]).$$

Here and below $\simeq$ means category equivalence. It is quite easy to see that there are only two (associated) irreducibles in $M(\mathbb{F}[\mathbb{Z}/2 \oplus \mathbb{Z}/2])$ and both are of dimension 1. This completes our proof. ∎

Define

$$\kappa := (\mathbb{F}^{(0)}, \mathbb{F}^{(1)}, \xi) \quad \text{with} \quad \mathbb{F}^{(0)} = \mathbb{F}^{(1)} = \mathbb{F}, \quad \xi(f_0, f_1) = (f_1, f_0),$$

which is an element of $Z_{\mathbb{F}}^{(1)}(\{1, z\})$. The following lemma will play an important role in finding the basic special irreducible Clifford modules naturally in Sections 3 and 6.

LEMMA 1.3. *If κ is non-special and $\mathcal{V}$ is an object in $Z_{\mathbb{F}}^{(0)}(\{1, z\}_1)$ of dimension* 2. *Then*

$$\kappa\mathcal{V} \cong b \oplus 1^*b.$$

PROOF. By Lemma 1.2 and a dimension count, we have

$$\kappa\mathcal{V} \cong 2b \quad \text{or} \quad 2(1^*b) \quad \text{or} \quad b + 1^*b.$$

That $1^*\kappa \cong \kappa$ and $1^*b \ncong b$ implies $\kappa\mathcal{V} \cong b + 1^*b$ as required. ∎

2. **Periodicity equivalence theorem.** The well known results in part 1 of [3] give periodicity equivalences between the following pairs of categories:

$$Z_{\mathbb{C}}^{(n)}(\{1, z\}) \text{ and } Z_{\mathbb{C}}^{(n+2)}(\{1, z\}), \quad \text{for all } n;$$

and

$$Z_{\mathbb{R}}^{(-n)}(\{1, z\}) \text{ and } Z_{\mathbb{R}}^{(-n-8)}(\{1, z\}) \quad \text{for all } n \geq 0.$$

Recently, P. Hoffman generalized this when $\mathbb{F} = \mathbb{C}$:

$$Z_{\mathbb{C}}^{(n)}(G) \simeq Z_{\mathbb{C}}^{(n+2)}(G),$$

see [6, Theorem 1.1].

In this section we shall prove the analogous generalization for the case $\mathbb{F} = \mathbb{R}$: $Z_{\mathbb{R}}^{(n)}(G)$ is of period 8 for $n \in \mathbb{Z}$. First of all, we have following mixed equivalence.

THEOREM 2.1.

$$Z_{\mathbb{R}}^{(p,q)}(G) \simeq Z_{\mathbb{R}}^{(p-1,q-1)}(G)$$

for $p, q > 0$.

PROOF. Define a functor

$$Z_{\mathbb{R}}^{(p,q)} \xrightarrow{\Gamma} Z_{\mathbb{R}}^{(p-1,q-1)}(G)$$

by

$$(W, W', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q) \stackrel{\Gamma}{\longmapsto} (U, U', \zeta_1, \ldots, \zeta_{p-1}, \chi_1, \ldots, \chi_{q-1})$$

where

$$U := W \bigcap \{1\text{-eigenspace of } \eta_p \xi_q\}, \quad U' := W' \bigcap \{1\text{-eigenspace of } \eta_p \xi_q\}$$
$$\zeta_i := \eta_i|_{U+U'}, \quad \chi_j := \xi_j|_{U+U'}$$

for $1 \leq i \leq p-1$, $1 \leq j \leq q-1$. Since η_i, ξ_j commute with $\eta_p \xi_q$, it maps U to U' and U' to U, this gives an element of $Z_{\mathbb{R}}^{(p-1,q-1)}(G)$. Also, if τ is a morphism in $Z_{\mathbb{R}}^{(p,q)}(G)$ whose domain is the above object $(W, W', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q)$, then $\Gamma\tau$, the restriction of τ on $U + U'$, yields a morphism in $Z_{\mathbb{R}}^{(p-1,q-1)}(G)$.

Conversely, define a functor

$$Z_{\mathbb{R}}^{(p-1,q-1)}(G) \xrightarrow{\Lambda} Z_{\mathbb{R}}^{(p,q)}(G)$$
$$\mathcal{V} \longmapsto \mathcal{E} \otimes \mathcal{V}$$

where

$$\mathcal{V} = (V, V', \eta_1, \ldots, \eta_{p-1}, \xi_1, \ldots, \xi_{q-1})$$

and $\mathcal{E}$ is the element of $Z_{\mathbb{R}}^{(1,1)}(\{1, z\})$ defined by

$$\mathcal{E} := (E, E', \eta, \xi), \quad E = \mathbb{R} = E', \quad \eta(a, b) = (b, a), \quad \xi(a, b) = (b, -a)$$

and $\mathcal{E} \otimes \mathcal{V} = (E \otimes V + E' \otimes V', E' \otimes V + E \otimes V', 1 \otimes \eta_1, \ldots, 1 \otimes \eta_{p-1}, \eta \widehat{\otimes} 1, 1 \otimes \xi_1, \ldots, 1 \otimes \xi_{q-1}, \xi \widehat{\otimes} 1)$ as defined in Section 1. We identify $Z_{\mathbb{R}}^{(p,q)}(G)$ with $Z_{\mathbb{R}}^{(p,q)}(\{1, z\} \hat{Y} G)$ again under the isomorphism $G \cong \{1, z\} \hat{Y} G \left(g \leftrightarrow (1, g)\right)$, so $\mathcal{E} \otimes \mathcal{V}$ is a well-defined element of $Z_{\mathbb{R}}^{(p,q)}(G)$. Let Λ send a morphism, say θ, in $Z_{\mathbb{R}}^{(p-1,q-1)}(G)$ to the function $\Lambda\theta$: $e \otimes v \longmapsto e \otimes \theta v$. The latter is a morphism in $Z_{\mathbb{R}}^{(p,q)}(G)$ by a straightforward check.

It remains to show that both $\Gamma\Lambda$ and $\Lambda\Gamma$ are naturally isomorphic to identity functors. Clearly the 1-eigenspace of $\eta\xi$ is spanned by $b = (1, 0)$. A natural isomorphism from $(V, V', \eta_1, \ldots, \eta_{p-1}, \xi_1, \ldots, \xi_{q-1})$ to $\Gamma\Lambda(V, V', \eta_1, \ldots, \eta_{p-1}, \xi_1, \ldots, \xi_{q-1})$ is given by

$$v \longmapsto b \otimes v.$$

A natural isomorphism from $\Lambda\Gamma(W, W', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q)$ to $(W, W', \eta_1, \ldots, \eta_p, \xi_1, \ldots, \xi_q)$ is given by

$$(e \otimes w + e' \otimes w', e_1' \otimes w_1 + e_1 \otimes w_1') \overset{\beta}{\longmapsto} (ew - e'\eta_p w', e_1'\xi_q w_1 + e_1 w_1')$$

where $w_i \in W$, $w_i' \in W'$ and $(e, e'), (e_1, e_1') \in E \oplus E'$. ∎

Before proving the periodicity theorem, let us recall the structure of real Clifford algebras which has been explained in an elegant manner in [3]. Let C_k, C_k' and $C_{p,q}$ be the real Clifford algebras of quadratic forms $Q_{0,k}$, $Q_{k,0}$ and $Q_{p,q}$ respectively. Then Propositions 1.6 and 4.2 in [3] give following isomorphisms among graded Clifford algebras.

$$C_{p+q} \cong C_p \otimes C_q, \quad C'_{p+q} \cong C'_p \otimes C'_q, \quad C_{p,q} \cong C'_p \otimes C_q, \quad C_4 \cong C'_4,$$

therefore

$$C_{k+8} \cong C_4 \otimes C_{k+4} \cong C_{4,k+4}, \quad C'_{k+8} \cong C'_{k+4} \otimes C'_4 \cong C_{k+4,4}.$$

The symbol $\otimes$ denotes the $\mathbb{Z}/2$-graded tensor product of $\mathbb{Z}/2$-graded algebras, so that, for two garded algebras $A = \sum_{\alpha=0,1} A^\alpha$, $B = \sum_{\alpha=0,1} B^\alpha$, the module $A^i \otimes B^j$ is one of the two summands of $(A \otimes B)^{i+j}$, and with multiplication defined by:

$$(a \otimes b_i)(a_j \otimes b) = (-1)^{ij} a a_j \otimes b_i b$$

if $a_j \in A^j$ and $b_i \in B^i$.

THEOREM 2.2.

$$Z_{\mathbb{R}}^{(n)}(G) \simeq Z_{\mathbb{R}}^{(n+8)}(G)$$

for all $n \in \mathbb{Z}$.

PROOF. If $n \geq 0$, then by using Note 2 in Section 1 and Theorem 2.1, also by identifying C_{n+8} with $C_{4,n+4}$ and C'_{n+8} with $C_{n+4,4}$, we have

$$Z_{\mathbb{R}}^{(n+8)}(G) \simeq Z_{\mathbb{R}}^{(n+4,4)}(G) \simeq Z_{\mathbb{R}}^{(n,0)}(G) = Z_{\mathbb{R}}^{(n)}(G).$$
$$Z_{\mathbb{R}}^{(-n-8)}(G) = Z_{\mathbb{R}}^{(0,n+8)}(G) \simeq Z_{\mathbb{R}}^{(4,n+4)}(G) \simeq Z_{\mathbb{R}}^{(0,n)}(G) = Z_{\mathbb{R}}^{(-n)}(G).$$

Therefore the category equivalence holds for $n \geq 0$ and for $n + 8 \leq 0$. Now let us do the cases when $n + 8 > 0$ and $n < 0$. By the same reasons as above, we get

$$Z_{\mathbb{R}}^{(-7)}(G) = Z_{\mathbb{R}}^{(0,7)}(G) \simeq Z_{\mathbb{R}}^{(4,3)}(G) \simeq Z_{\mathbb{R}}^{(1,0)}(G) = Z_{\mathbb{R}}^{(1)}(G).$$

Similarly for $2 \leq n + 8 \leq 7$, we have $Z_{\mathbb{R}}^{(n+8)}(G) \simeq Z_{\mathbb{R}}^{(n)}(G)$. ∎

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

NOTE 1. We get the following table about C'_k-modules, which is not explicit in [3], but is similar to their table of C_k-modules.

k	C'_k	$T^{(k)}_{\mathbb{R}}(\{1,z\})$	a'_k	$T^{(k)}_{\mathbb{C}}(\{1,z\})$	$a'^{\mathbb{C}}_k$
0	$\mathbb{R}$	$\mathbb{Z}\oplus\mathbb{Z}$	1	$\mathbb{Z}\oplus\mathbb{Z}$	1
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{Z}$	2	$\mathbb{Z}$	2
2	$\mathbb{R}(2)$	$\mathbb{Z}$	4	$\mathbb{Z}\oplus\mathbb{Z}$	2
3	$\mathbb{C}(2)$	$\mathbb{Z}$	8	$\mathbb{Z}$	4
4	$\mathbb{H}(2)$	$\mathbb{Z}\oplus\mathbb{Z}$	8	$\mathbb{Z}\oplus\mathbb{Z}$	4
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{Z}$	16	$\mathbb{Z}$	8
6	$\mathbb{H}(4)$	$\mathbb{Z}$	16	$\mathbb{Z}\oplus\mathbb{Z}$	8
7	$\mathbb{C}(8)$	$\mathbb{Z}$	16	$\mathbb{Z}$	16

$$T^{(k+8)}_{\mathbb{R}}(\{1,z\}) \cong T^{(k)}_{\mathbb{R}}(\{1,z\}), \quad a'_{k+8} = 16a'_k, \quad a'^{\mathbb{C}}_{k+2} = 2a'^{\mathbb{C}}_k,$$

where $a'_k(a'^{\mathbb{C}}_k)$ is the $\mathbb{R}$ ($\mathbb{C}$)-dimension of an irreducible in $T^{(k)}_{\mathbb{R}}(\{1,z\})$ ($T^{(k)}_{\mathbb{C}}(\{1,z\})$).

NOTE 2. After we finished this paper, P. Hoffman [7] generalized the above equivalence to a wider case, both for complex numbers and real numbers.

There are only 8 non-equivalent categories among $T^{(n)}_{\mathbb{R}}(G)$, $n \in \mathbb{Z}$ for every $\mathcal{G}$-object G. If we re-define $T^*_{\mathbb{R}}G := \sum_{n=0}^{7} \oplus T^{(n)}_{\mathbb{R}}(G)$, then $T^*_{\mathbb{R}}G$ is a $T^*_{\mathbb{R}}\{1,z\}$-module in above sense. We will determine the ring structure of $T^*_{\mathbb{R}}\{1,z\}$ in Section 3.

Finally let us show how to get an irreducible representation of $\hat{S}_n$ by using Clifford modules, analogous to that in [10, Appendix 6]. Let e'_j, $1 \leq j \leq n$ be a basis of $\mathbb{R}^n$, and let $W := \{\sum \lambda_j e'_j : \sum \lambda_j = 0\}$, a subspace of $\mathbb{R}^n$. Choose $\{t'_1, \ldots, t'_{n-1}\}$ with $t'_j = 2^{-1/2}(e'_j - e'_{j+1})$ as a basis of W. The Clifford algebra of $(Q_{n,0})|_W$ over W is denoted B'_{n-1}. Then $\tilde{S}_n$ is isomorphic to the group generated by these $t'_j s$ under the map which sends the generators t_j to t'_j and z to -1. So an irreducible representation of B'_{n-1} will restrict to an irreducible representation of $\tilde{S}_n$, since $\{t'_1, \ldots, t'_{n-1}\}$ generates B_{n-1} as algebra. We shall use this comment in defining the real basic special irreducible Clifford modules in Section 6.

3. A $K_{\mathbb{R}}$-algebra $\sum_{n=0}^{\infty} \oplus T^*_{\mathbb{R}}\hat{S}_n$. In 1986, P. Hoffman and J. Humphreys [8] gave a new formulation of the Schur's results [19] on complex projective representations for the symmetric groups and for the alternating groups in the spirit of the well known "induction algebra" approach to the linear representations of S_n. To do this, they built an algebra in the language of this paper as following:

The operation $(\mathcal{V}, \mathcal{W}) \longmapsto \mathcal{V} \otimes \mathcal{W}$ gives rise to a pairing

$$T^{(k)}_{\mathbb{C}}(\{1,z\}) \otimes T^{(l)}_{\mathbb{C}}(\{1,z\}) \longrightarrow T^{(k+l)}_{\mathbb{C}}(\{1,z\})$$

and induces an $\mathbb{N}$-graded ring structure on the direct sum

$$T^*(\{1,z\}) := \sum_{n=0}^{\infty} \oplus T^{(n)}_{\mathbb{C}}(\{1,z\}).$$

If we view equivalent categories as the same category, there are, by periodicity, only 2 different non-equivalent ones among $\{T_{\mathbb{C}}^{(n)}(\{1,z\})\}$ for $n \in \mathbb{N}$. Then

$$K := T_{\mathbb{C}}^{(0)}(\{1,z\}) \oplus T_{\mathbb{C}}^{(1)}(\{1,z\})$$

inherits a ring structure from $T^*(\{1,z\})$ in the obvious way. For a generator z_1 of $T_{\mathbb{C}}^{(1)}(\{1,z\})$, we have

$$K \cong \mathbb{Z}[z_1]/\langle z_1^3 - 2z_1\rangle$$

as rings.

Given non-negative integers k and l, let

$$\phi_{k,l}\colon \hat{S}_k \hat{Y} \hat{S}_l \longrightarrow \hat{S}_{k+l}$$

be the $\mathcal{G}$-embedding onto the subgroup $\langle \hat{S}_k, \hat{S}_l\rangle$ given in the Example 2. Then define multiplication in

$$T^* := \sum_{n=0}^{\infty} \oplus T^* \hat{S}_n$$

as the following composite:

$$T^*\hat{S}_k \times T^*\hat{S}_l \xrightarrow{\otimes} T^*(\hat{S}_k \hat{Y} \hat{S}_l) \xrightarrow{(\phi_{k,l})_*} T^*\hat{S}_{k+l},$$

where $(\phi_{k,l})_*$ is the map induced by $\phi_{k,l}$ and

$$T^*\hat{S}_n := T_{\mathbb{C}}^{(0)}(\hat{S}_n) \oplus T_{\mathbb{C}}^{(1)}(\hat{S}_n).$$

Then T^* is a pseudo-commutative $\mathbb{Z}/2 \times \mathbb{N}$ graded K-algebra. Let $T_{\mathbb{C}}^{(0)}(\{1,z\}_n) = \mathbb{Z}M_n$ (resp. $\mathbb{Z}M_n + \mathbb{Z}1^*M_n$) if n is odd (resp. even). This follows from Table 1 in Section 2 and Note 3 in Section 1. Hence M_{2n} is a special irreducible and M_{2n+1} is not. By using

$$z_1^2 = 1 + 1^*, \quad M_1^2 = M_2 + 1^*M_2$$

and Lemma 1.3, we have

$$z_1 M_{2n+1} = N_{2n+1} + 1^* N_{2n+1}$$

for some special irreducible $N_{2n+1} \in T_{\mathbb{C}}^{(1)}(\{1,z\}_{2n+1})$. Recall the last paragraph of Section 2. We have $\hat{S}_n \subset B'_{n-1} \subset C'_n$ and an irreducible module for B'_{n-1} will restrict to an irreducible representation of $\hat{S}_n$. Let

$$c_{2n+1} = M_{2n} \in T_{\mathbb{C}}^{(0)}\hat{S}_{2n+1}, \quad c_{2n} = N_{2n-1} \in T_{\mathbb{C}}^{(1)}\hat{S}_{2n},$$

then the K-algebra $\sum_{n=0}^{\infty} \oplus T^*\hat{S}_n$ is isomorphic to the quotient of the free algebra with generators $\{c_1, c_2, \ldots\}$ by relations

(a) $c_i c_j = (1^*)^{i+j+1} c_j c_i$;
(b) $c_n^2 = (-1)^{n+1} z_1 \left(c_{2n} + z_1 \sum_{i=1}^{n-1} (-1)^i c_i c_{2n-i}\right)$;

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

see [10, A8.14 and A8.15] for details.

In this section we shall set up a $K_{\mathbb{R}}$ algebra $T^*_{\mathbb{R}}$, in a way parallel to that we have recalled above. Its structure will be determined in Section 6, from which the real projective representations which are $\mathbb{R}$-projectively equivalent to linear representations of $\hat{S}_n$ can be described when $n \geq 4$.

First we will use information contained both in [3] and Section 2 to explicitly describe the structure of the ground ring $K_{\mathbb{R}}$.

The operation $(\mathcal{V}, \mathcal{W}) \longmapsto \mathcal{V} \otimes \mathcal{W}$ gives rise to a pairing

$$T^{(k)}_{\mathbb{R}}(\{1, z\}) \otimes T^{(l)}_{\mathbb{R}}(\{1, z\}) \longrightarrow T^{(k+l)}_{\mathbb{R}}(\{1, z\})$$

and induces a $\mathbb{Z}$-graded ring structure on the direct sum

$$T^*_{\mathbb{R}}(\{1, z\}) := \sum_{n=0}^{\infty} \oplus T^{(n)}_{\mathbb{R}}(\{1, z\}).$$

We denote this product by $(u, v) \longmapsto uv$, which is associative from [6, Proposition 2.1(ii)]. If we view equivalent categories as being the same, there are only 8 different non-equivalent ones among $T^{(n)}_{\mathbb{R}}(\{1, z\})$ for $n \in \mathbb{N}$. So

$$K_{\mathbb{R}} := \sum_{n=0}^{7} \oplus T^{(n)}_{\mathbb{R}}(\{1, z\})$$

inherits a ring structure from $T^*_{\mathbb{R}}(\{1, z\})$ in the obvious way. By table 1 and Proposition 5.5 in [3], we can assume that

$$T^{(0)}_{\mathbb{R}}(\{1, z\}) = \mathbb{Z}1 \oplus \mathbb{Z}1^*, \quad T^{(4)}_{\mathbb{R}}(\{1, z\}) = \mathbb{Z}x_4 \oplus \mathbb{Z}x_4^* \quad T^{(i)}_{\mathbb{R}}(\{1, z\}) = \mathbb{Z}x_i$$

for $1 \leq i \leq 3$ or $5 \leq i \leq 7$ and here and below $*$ is given by: $\mathcal{V}^* = \mathcal{V}1^*$. We choose the generators x_i to be the isomorphism classes of modules. By dimension counts, it follows that all these generators commute with each other. All but one of the relations below follow by the same argument:

$$\begin{gathered} x_2 = x_1^2, \quad x_3 = x_1^3, \quad x_5 = x_1 x_4, \quad x_6 = x_7^2, \quad 1 + 1^* = x_1 x_7, \\ x_4^2 = 4, \quad x_1 x_4 = x_7^3, \quad x_4 x_7 = x_1^3, \quad x_1^2 x_7 = 2x_1, \quad x_1 x_7^2 = 2x_7, \\ x_1^4 = x_4 + x_4^*, \quad x_1^6 = 4x_6, \quad x_1^7 = 8x_7. \end{gathered}$$

The equation $x_4^2 = 4$ needs more explanation. One can refer to the course of proving Theorem 6.9 in [3].

Note that we will use the above thirteen relations in finding the basic real special irreducible Clifford modules in Section 6.

The complete ring-structure of $K_{\mathbb{R}}$ is given by:

THEOREM 3.1. *$K_{\mathbb{R}}$ is a commutative $\mathbb{Z}/8$-graded ring generated by a unit* $1 \in T^{(0)}_{\mathbb{R}}(\{1, z\})$, *and by* $x_1 \in T^{(1)}_{\mathbb{R}}(\{1, z\}), x_4 \in T^{(4)}_{\mathbb{R}}(\{1, z\}), x_7 \in T^{(7)}_{\mathbb{R}}(\{1, z\})$ *with relations:*

$$\mathcal{R} := \{x_4^2 - 4, x_1 x_4 - x_7^3, x_4 x_7 - x_1^3, x_1^2 x_7 - 2x_1, x_1 x_7^2 - 2x_7\}.$$

i.e.,

$$K_{\mathbb{R}} \cong \mathbb{Z}[x_1, x_4, x_7]/\langle \mathcal{R} \rangle.$$

PROOF. Clearly there is a homomorphism from $\mathbb{Z}[x_1, x_4, x_7]/\langle\mathcal{R}\rangle$ onto $K_{\mathbb{R}}$, since we know that the relations hold and that the "missing" generators are not needed by other relations. To see this is also an isomorphism, it suffices to prove that the minimum number of generators needed to generate the abelian group $\mathbb{Z}[x_1, x_4, x_7]/\langle\mathcal{R}\rangle$ is not greater than that for free abelian group $K_{\mathbb{R}}$, namely 10. Indeed, by a straightforward check, it follows that

$$\{1, x_1x_7 - 1, x_1, x_1^2, x_1^3, x_4, x_1x_4x_7 - x_4, x_7^3, x_7^2, x_7\}$$

generates $\mathbb{Z}[x_1, x_4, x_7]/\langle\mathcal{R}\rangle$ as an abelian group, as required. ∎

LEMMA 3.1. *The operation* $\otimes$ *is bilinear over* $K_{\mathbb{R}}$, *i.e.,*

$$x(\mathcal{V}\mathcal{W}) = (x\mathcal{V})\mathcal{W} = \mathcal{V}(x\mathcal{W})$$

for any $x \in K_{\mathbb{R}}$ *and* $\mathcal{V} \in T_{\mathbb{R}}^{(n)}(G)$, $\mathcal{W} \in T_{\mathbb{R}}^{(m)}(H)$.

PROOF. By associativity of $\otimes$, it suffices to show that $1^*\mathcal{V} = \mathcal{V}1^*$ and $x_i\mathcal{V} = \mathcal{V}x_i$ for $i = 1, 4, 7$.

Let $\mathcal{V} = (V, V', \eta_1, \dots, \eta_n) \in T_{\mathbb{R}}^{(n)}(G)$. Then $\mathcal{V}1^* = 1^*\mathcal{V}$ as we saw in Section 1. Let x_1 to be $(R^{(0)}, R^{(1)}, \xi)$ with $R^{(0)} = R^{(1)} = \mathbb{R}$ as vector space and $\xi(r_0, r_1) = (r_1, r_0)$. Then

$$\mathcal{V}x_1 = (V \otimes R^{(0)} + V' \otimes R^{(1)}, V \otimes R^{(1)} + V' \otimes R^{(0)}, 1 \otimes \xi, \eta_1 \widehat{\otimes} 1, \dots, \eta_n \widehat{\otimes} 1).$$

On the other hand, using [6, Proposition 1.2] for the second equation and $1^*x_1 = x_1 1^*$ for the third equation, we have

$$\begin{aligned} x_1\mathcal{V} &= (R^{(0)} \otimes V + R^{(1)} \otimes V', R^{(1)} \otimes V + R^{(0)} \otimes V', 1 \otimes \eta_1, \dots, 1 \otimes \eta_n, \xi \widehat{\otimes} 1) \\ &= (1^*)^n(R^{(0)} \otimes V + R^{(1)} \otimes V', R^{(1)} \otimes V + R^{(0)} \otimes V', \xi \widehat{\otimes} 1, 1 \otimes \eta_1, \dots, 1 \otimes \eta_n) \\ &= (R^{(0)} \otimes V + R^{(1)} \otimes V', R^{(1)} \otimes V + R^{(0)} \otimes V', \xi \widehat{\otimes} 1, 1 \otimes \eta_1, \dots, 1 \otimes \eta_n) \end{aligned}$$

Then

$$r \otimes v \longmapsto (-1)^{|r|\,|v|} v \otimes r$$

gives a natural isomorphism between $x_1\mathcal{V}$ and $\mathcal{V}x_1$. Similarly, we have $x_4\mathcal{V} = \mathcal{V}x_4$ and $x_7\mathcal{V} = \mathcal{V}x_7$ as required. ∎

We are now in a position to introduce the promised algebra . Given non-negative integers k and l, let

$$\phi_{k,l}\colon \hat{S}_k \hat{Y} \hat{S}_l \longrightarrow \hat{S}_{k+l}$$

be the $\mathcal{G}$-embedding onto the subgroup $\langle \hat{S}_k, \hat{S}_l\rangle$ given in Example 2 in Section 1. Then the algebra multiplication is defined to be the following composite:

$$T_{\mathbb{R}}^*\hat{S}_k \times T_{\mathbb{R}}^*\hat{S}_l \xrightarrow{\otimes} T_{\mathbb{R}}^*(\hat{S}_k\hat{Y}\hat{S}_l) \xrightarrow{(\phi_{k,l})_*} T_{\mathbb{R}}^*\hat{S}_{k+l}.$$

where $(\phi_{k,l})_*$ is the map induced by $\phi_{k,l}$. Denote the above product by $(x, y) \longmapsto xy$ again, *i.e.*, $xy = (\phi_{k,l})_*(x \otimes y)$. This makes $T_{\mathbb{R}}^* := \sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^*\hat{S}_n$ into a $K_{\mathbb{R}}$-algebra; that is, the multiplication $x\mathcal{V} \in T_{\mathbb{R}}^{(n+m)}(\{1, z\})\hat{Y}\hat{S}_k$ $(\simeq T_{\mathbb{R}}^{(n+m)}\tilde{S}_k)$ if $x \in T_{\mathbb{R}}^{(n)}(\{1, z\})$ and $\mathcal{V} \in T_{\mathbb{R}}^{(m)}\hat{S}_k$, is $K_{\mathbb{R}}$-linear by above proposition. So we have:

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

THEOREM 3.2. *$\sum_{n=0}^{\infty} \oplus T^*_{\mathbb{R}}\hat{S}_n$ is a pseudo-commutative $\mathbb{Z}/8 \times \mathbb{N}$ graded ring in that $xy = (1^*)^{ij+kl}yx$ for $x \in T^{(i)}_{\mathbb{R}}(\hat{S}_k)$ and $y \in T^{(j)}_{\mathbb{R}}(\hat{S}_l)$, and is also a $K_{\mathbb{R}}$-algebra.*

PROOF. Everything is clear except the pseudo-commutativity and the associtivity. The latter has essentially the same proof as Theorem A8.8 [10]. ■

4. **The structure of the $K_{\mathbb{C}}$-algebra $\sum_{n=0}^{\infty} \oplus T^*_{\mathbb{C}}\hat{S}_n$.** The categories $Z^{(n)}_{\mathbb{C}}(G)$, $n \in \mathbb{Z}$, are 2-fold and therefore 8-fold periodic. If we let

$$T^*G := \sum_{i=0}^{1} \oplus T^{(i)}_{\mathbb{C}}(G), \quad K := T^*(\{1, z\}), \quad T^* := \sum_{n=0}^{\infty} \oplus T^*\hat{S}_n \quad \text{and}$$

$$T^*_{\mathbb{C}}G := \sum_{i=0}^{7} \oplus T^{(i)}_{\mathbb{C}}(G), \quad K_{\mathbb{C}} := T^*_{\mathbb{C}}(\{1, z\}), \quad T^*_{\mathbb{C}} := \sum_{n=0}^{\infty} \oplus T^*_{\mathbb{C}}\hat{S}_n,$$

then T^*G (resp. $T^*_{\mathbb{C}}G$) is a K- (resp. $K_{\mathbb{C}}$-) module and T^* (resp. $T^*_{\mathbb{C}}$) is a $\mathbb{Z}/2 \times \mathbb{N}$-graded (resp. $\mathbb{Z}/8 \times \mathbb{N}$-graded) K- (resp. $K_{\mathbb{C}}$-) algebra. The two algebras are associative and pseudo-commutative in that $xy = (1^*)^{ij+kl}yx$ for $x \in T^{(i)}_{\mathbb{C}}\hat{S}_k$ and $y \in T^{(j)}_{\mathbb{C}}\hat{S}_l$. The main goal of this section is to relate the structure of the $K_{\mathbb{C}}$-algebra $T^*_{\mathbb{C}}$ to that of the K-algebra T^*. This will be convenient for the purpose of comparing the complex and real cases, and using known facts about the complex case to shorten proofs for the real case in Section 6. We shall proceed by analogy with Appendix 8 of [10].

Let $T^{(1)}_{\mathbb{C}}(\{1, z\}) = \mathbb{Z}z_1$ and $T^{(2)}_{\mathbb{C}}(\{1, z\}) = \mathbb{Z}z_2 \oplus \mathbb{Z}1^*z_2$. Then the map

$$\mathcal{V} \longmapsto z_2\mathcal{V} \quad (\text{or } \mathcal{V}z_2)$$

accomplished the category equivalence: $Z^{(n)}_{\mathbb{C}}(G) \simeq Z^{(n+2)}_{\mathbb{C}}(G)$ as in [6]. Hence

$$T^{(2i)}_{\mathbb{C}}(\{1, z\}) \cong z_2^i T^{(0)}_{\mathbb{C}}(\{1, z\}) = \mathbb{Z}z_2^i \oplus \mathbb{Z}1^*z_2^i$$

and

$$T^{(2i+1)}_{\mathbb{C}}(\{1, z\}) \cong z_2^i T^{(1)}_{\mathbb{C}}(\{1, z\}) = \mathbb{Z}z_1 z_2^i.$$

The following lemma gives the structure of $K_{\mathbb{C}}$ as we did in real number field case or 2-fold periodicity over complex numbers.

LEMMA 4.1. *$K_{\mathbb{C}}$ is a commutative ring and is generated by z_1, z_2 with relations:*

$$z_1^3 = 2z_1z_2, \quad z_2^4 = 1;$$

i.e.,

$$K_{\mathbb{C}} \cong \mathbb{Z}[z_1, z_2]/\langle z_1^3 - 2z_1z_2, z_2^4 - 1\rangle.$$

PROOF. By dimension count and 8-fold periodicity, we have

$$z_1^2 = z_2 + 1^*z_2, \quad z_1^2z_2^3 = 1 + 1^*, \quad z_1^3 = 2z_1z_2, \quad z_2^4 = 1.$$

So there is an epimorphism from $\mathbb{Z}[z_1, z_2]/\langle z_1^3 - 2z_1z_2, z_2^4 - 1\rangle$ into $K_{\mathbb{C}}$. Then, by checking that the minimum number of abelian group generators of

$\mathbb{Z}[z_1, z_2]/\langle z_1^3 - 2z_1z_2, z_2^4 - 1\rangle$ is not greater than that of free abelian group $K_{\mathbb{C}}$, the above homomorphism is an isomorphism. ∎

Note that the four relations involving 1, 1^*, z_1, and z_2 in the above proof will be frequently used in the later part of the paper.

LEMMA 4.2. *$T^*_{\mathbb{C}}G$ is a free $K_{\mathbb{C}}$-module, and any K-basis for T^*G is a $K_{\mathbb{C}}$-basis for $T^*_{\mathbb{C}}G$ (where we regard $T^*(G)$ as the subgroup of $T^*_{\mathbb{C}}(G)$ consisting of the zero-th and first component of the $\mathbb{Z}/8$-grading).*

PROOF. As we know from [10, A8.4], a K-basis of T^*G consists of special irreducibles, with exactly one chosen from each pair x, 1^*x. The set of all irreducibles is the union of the triples $\{x, 1^*x, z_1x\}$. Now, since $T^*_{\mathbb{C}}G = \sum_{i=0}^3 \oplus z_2^i T^*G$ and $1^*, z_1$ have the same meaning in both K and $K_{\mathbb{C}}$, then the number of generators of $K_{\mathbb{C}}$-module $T^*_{\mathbb{C}}G$ is not greater than that of K-module T^*G. On the other hand, $z_2 \longmapsto 1$ defines an (ungraded) group homomorphism from $T^*_{\mathbb{C}}G$ onto T^*G, and our result follows. ∎

LEMMA 4.3. *The operator $\otimes$ determines an isomorphism of $K_{\mathbb{C}}$-modules*

$$T^*_{\mathbb{C}}G \otimes_{K_{\mathbb{C}}} T^*_{\mathbb{C}}H \xrightarrow{\otimes} T^*_{\mathbb{C}}(G\hat{Y}H).$$

Moreover, $\mathcal{V} \otimes \mathcal{W}$ is a special irreducible if and only if $\mathcal{V}$ and $\mathcal{W}$ are.

PROOF. The map is an isomorphism because

$$T^*G \otimes_K T^*H \xrightarrow{\otimes} T^*(G\hat{Y}H)$$

is an isomorphism (see [10, A8.6]) and $T^*_{\mathbb{C}}G = \sum_{i=0}^3 \oplus z_2^i T^*G$.

Write $\mathcal{V} = z_2^s\mathcal{V}'$, $\mathcal{W} = z_2^t W'$ with $V'\,(W') \in T^*G\,(T^*H)$. So $\mathcal{V}\otimes\mathcal{W}$ is special irreducible

$\iff \mathcal{V}' \otimes \mathcal{W}'$ is special irreducible

$\iff \mathcal{V}'$ and $\mathcal{W}'$ are special irreducibles in T^*G and T^*H respectively by [10, A8.6]

$\iff z_2^s\mathcal{V}'$ $(z_2^t\mathcal{W}')$ is special irreducible in $T^*_{\mathbb{C}}G$ $(T^*_{\mathbb{C}}H)$.

∎

LEMMA 4.4. *For all positive integers m and n,*

$$\phi^*_{m,n}(c_{m+n}) = \begin{cases} z_1 c_m \otimes c_n & \text{if } m \text{ and } n \text{ are odd} \\ z_1 z_2^3 c_m \otimes c_n & \text{otherwise,} \end{cases}$$

where c_i are defined in Section 3 for $i = 1, 2, \ldots$.

PROOF. Consider the commutative diagram:

$$\begin{array}{ccccc} (g,h)Z & & \hat{S}_m\hat{Y}\hat{S}_n & \xrightarrow{\phi_{m,n}} & \hat{S}_{m+n} \\ \downarrow & & \downarrow & & \downarrow \\ g\otimes h & & C_{m-1}\otimes C_{n-1} & & C_{m+n-1} \\ & & \downarrow & & \downarrow \\ & & C_m \otimes C_n & \xrightarrow{\cong} & C_{m+n} \end{array}$$

The symbol $\otimes$ denotes the $\mathbb{Z}/2$-graded tensor product of $\mathbb{Z}/2$-graded algebras, as we stated before Theorem 2.2. If M and M' are modules for Clifford algebras C and C' respectively, then $M \otimes M'$ is a module for the algebra $C \otimes C'$ as usual. The following diagrams show the effect of restricting modules along the embeddings in the previous diagram, according to the parity of m and n.

1. Both m and n are odd.

$$\begin{array}{ccc} z_1(1+1^*)^2 c_m \otimes c_n & \xleftarrow{\phi^*_{m,n}} & (1+1^*)^2 c_{m+n} \\ \uparrow & & \uparrow \\ z_1(1+1^*)^2 M_{m-1} \otimes M_{n-1} & & z_1(1+1^*) M_{m+n-1} \\ \uparrow & & \uparrow \\ z_1 M_m \otimes M_n & \longleftarrow & z_1(1+1^*) M_{m+n}. \end{array}$$

2. m is odd and n is even.

$$\begin{array}{ccc} (1+1^*)^2 c_m \otimes c_n & \xleftarrow{\phi^*_{m,n}} & z_1(1+1^*) c_{m+n} \\ \uparrow & & \uparrow \\ (1+1^*) M_{m-1} \otimes z_1 M_{n-1} & & z_1(1+1^*) M_{m+n-1} \\ \uparrow & & \uparrow \\ z_1 M_m \otimes M_n & \longleftarrow & z_1 M_{m+n}. \end{array}$$

3. m is even and n is odd. We have a similar diagram as in case 2.

4. Both m and n are even.

$$\begin{array}{ccc} (1+1^*)^2 c_m \otimes c_n & \xleftarrow{\phi^*_{m,n}} & z_1(1+1^*) c_{m+n} \\ \uparrow & & \uparrow \\ z_1 M_{m-1} \otimes z_1 M_{n-1} & & z_1^2 M_{m+n-1} \\ \uparrow & & \uparrow \\ z_1^2 M_m \otimes M_n & \longleftarrow & z_1^2 M_{m+n}. \end{array}$$

We can get the desired conclusions by using following:

(i) All the diagrams are commutative;

(ii) All the "restricting" maps are $K_{\mathbb{C}}$-module homormophisms;

(iii) In a free $K_{\mathbb{C}}$-module, $z_1^l x = 0$ implies $z_1 x = 0$ and $z_2^l x = 0$ implies $x = 0$;

(iv) $1 + 1^* = z_1^2 z_2^3$ holds in $K_{\mathbb{C}}$; see Lemma 4.1.

■

Now define, for all $n > 0$,

(1) $$\hat{c}_n := z_2^{[\frac{n-1}{2}]} c_n$$

a special irreducible in $T_{\mathbb{C}}^{(n-1)}(\hat{S}_n)$, where [ ] is the "rounding down" (integer part) function. Then, by a straightforward calculation, we can reduce Lemma 4.4 to one equation, *i.e.*,

(2) $$\phi^*_{m,n}(\hat{c}_{m+n}) = z_1 \hat{c}_m \otimes \hat{c}_n.$$

Now let us define a $K_{\mathbb{C}}$-valued inner product on $T^*_{\mathbb{C}} G$. This will be uniquely specified by requiring the following to hold:

(i) $\langle\cdot,\cdot\rangle$ is bilinear over $K_{\mathbb{C}}$;
(ii) $\langle \mathcal{V}, \mathcal{V}\rangle = z_2^i$ if $\mathcal{V}$ is a special irreducible in $T_{\mathbb{C}}^{(i)}(G)$;
(iii) $\langle \mathcal{V}, \mathcal{W}\rangle = 0$ if $\mathcal{V}, \mathcal{W}$ are special irreducibles with $\mathcal{V} \neq (1^*)^t z_2^s \mathcal{W}$ for any $t, s \geq 0$.

By the definition, it is not hard to see that

$$\langle \hat{c}_n, \hat{c}_n\rangle = z_2^{n-1}, \tag{3}$$

and b (or $-b$) is a special irreducible if and only if $\langle b, b\rangle = z_2^s$ for some $s \geq 0$.

If we denote by $(\cdot,\cdot)$ the K-valued inner product defined on T^*G [10, Appendix 8], then

$$\langle \mathcal{V}, \mathcal{W}\rangle = z_2^{i+j+k}(\mathcal{V}', \mathcal{W}')$$

where $\mathcal{V} = z_2^i\mathcal{V}'$, $\mathcal{W} = z_2^j\mathcal{W}'$ and $V' \in T^{(k)}(G)$, $W' \in T^{(l)}(G)$ with $k, l = 0$ or 1. Therefore the Frobenius reciprocity property holds for $\langle\cdot,\cdot\rangle$, since it holds for $(\cdot,\cdot)$, *i.e.*, if $\phi\colon G \longrightarrow G'$ is an injective $\mathcal{G}$-map, then

$$\langle \phi_*\mathcal{V}, \mathcal{W}\rangle = \langle \mathcal{V}, \phi^*\mathcal{W}\rangle$$

for $\mathcal{V} \in T_{\mathbb{C}}^*G$, $\mathcal{W} \in T_{\mathbb{C}}^*G'$.

For each $x \in T_{\mathbb{C}}^{(i)}\hat{S}_k$, define an operator $x^\perp$ on $T_{\mathbb{C}}^*$, which reduces the $\mathbb{N}$-grading by k and increases the $\mathbb{Z}/8$-grading by i, as follows:

$$\langle x^\perp(y), u\rangle = \langle y, xu\rangle.$$

Equivalently, if $\{a_\lambda\}$ is an orthogonal $K_{\mathbb{C}}$-basis of special irreducibles for $T_{\mathbb{C}}^*$, we have

$$x^\perp(y) = \sum_\lambda z_2^{s(\lambda)}\langle y, xa_\lambda\rangle a_\lambda$$

where

$$z_2^{s(\lambda)}\langle a_\lambda, a_\lambda\rangle = 1,$$

i.e., if $a_\lambda \in T_{\mathbb{C}}^{(j)}$, then

$$s(\lambda) \equiv 8 - j \pmod 8.$$

Thus, $x^\perp$ is a homomorphism of $K_{\mathbb{C}}$-modules. We have

$$(xy)^\perp = y^\perp x^\perp = (1^*)^{ij+kl}x^\perp y^\perp$$

for $x \in T_{\mathbb{C}}^{(i)}\hat{S}_k$ and $y \in T_{\mathbb{C}}^{(j)}\hat{S}_l$. This follows from pseudo-commutativity.

Now please allow us to use $\langle\ ,\ \rangle$ also to denote the derived inner product on $T_{\mathbb{C}}^*G \otimes_{K_{\mathbb{C}}} T_{\mathbb{C}}^*G$, *i.e.*, a bilinear map which satisfies

$$\langle x \otimes u, y \otimes v\rangle := \langle x, y\rangle\langle u, v\rangle.$$

LEMMA 4.5. *For $i < n$,*

$$\hat{c}_i^\perp(\hat{c}_n) = z_1 z_2^{i-1}\hat{c}_{n-i};$$

whereas

$$\hat{c}_n^\perp(\hat{c}_n) = z_2^{n-1}.$$

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

PROOF. Let u be any element of $T^*_{\mathbb{C}}\hat{S}_{n-i}$. Then using the reciprocity property, equations (2) and (3) if $i < n$, and $z_2^4 = 1$, we get

$$\begin{aligned}\langle \hat{c}_i^{\perp}(\hat{c}_n), u\rangle &= \langle \hat{c}_n, \hat{c}_i u\rangle \\ &= \langle \hat{c}_n, (\phi_{i,n-i})_*(\hat{c}_i \otimes u)\rangle \\ &= \langle \phi^*_{i,n-i}(\hat{c}_n), \hat{c}_i \otimes u\rangle \\ &= \langle z_1 \hat{c}_i \otimes \hat{c}_{n-i}, \hat{c}_i \otimes u\rangle \\ &= z_1\langle \hat{c}_i, \hat{c}_i\rangle\langle \hat{c}_{n-i}, u\rangle \\ &= \langle z_1 z_2^{i-1}\hat{c}_{n-i}, u\rangle\end{aligned}$$

as required. To prove the second equation, by the same argument as above and $u \in K_{\mathbb{C}}$ if $i = n$, we have

$$\langle \hat{c}_n^{\perp}(\hat{c}_n), u\rangle = u\langle \hat{c}_n, \hat{c}_n\rangle = \langle z_2^{n-1}, u\rangle$$

as required. ∎

LEMMA 4.6. *The Clifford modules* $\{\hat{c}_1, \hat{c}_2, \ldots\}$ *satisfy the following relations:*
(a) $\hat{c}_i\hat{c}_j = (1^*)^{i+j+1}\hat{c}_j\hat{c}_i$;
(b) $\hat{c}_n^2 = (-1)^{n+1}z_1z_2^3\big(\hat{c}_{2n} + z_1\sum_{i=1}^{n-1}(-1)^i\hat{c}_i\hat{c}_{2n-i}\big)$.

PROOF. (a) follows from the pseudo-commutativity. By a case-by-case calculation, we can get the following six equalities:

1. $1 \le s < m$, $\langle \hat{c}_s\hat{c}_{m-s}, \hat{c}_m\rangle = z_1z_2^{m+2}$;
2. $\langle \hat{c}_n^2, \hat{c}_{2n}\rangle = z_1z_2^{2n+2}$;
3. $\langle \hat{c}_n^2, \hat{c}_n^2\rangle = (2n-1)z_1^2z_2^{2n+1}$;
4. $1 \le s < \frac{m}{2}$, $\langle \hat{c}_s\hat{c}_{m-s}, \hat{c}_s\hat{c}_{m-s}\rangle = z_2^{m+2} + (2s-1)z_1^2z_2^{m+1}$;
5. $1 \le s < t < \frac{m}{2}$, $\langle \hat{c}_s\hat{c}_{m-s}, \hat{c}_t\hat{c}_{m-t}\rangle = 2sz_1^2z_2^{m+1}$;
6. $1 \le s < n$, $\langle \hat{c}_n^2, \hat{c}_s\hat{c}_{2n-s}\rangle = 2sz_1^2z_2^{2n+1}$.

Then by using these equalities, we can prove

$$\Big\langle \hat{c}_n^2 + (-1)^n z_1z_2^3\Big[\hat{c}_{2n} + z_1\sum_{i=1}^{n-1}(-1)^i\hat{c}_i\hat{c}_{2n-i}\Big], \hat{c}_n^2 + (-1)^n z_1z_2^3\Big[\hat{c}_{2n} + z_1\sum_{i=1}^{n-1}(-1)^i\hat{c}_i\hat{c}_{2n-i}\Big]\Big\rangle = 0.$$

This proves (b). ∎

Here and below we denote $\mathcal{D}$, as in [10], to be the set of all strict partitions of all non-negative integers. For $\lambda = (\lambda_1 > \cdots > \lambda_r > 0) \in \mathcal{D}$, we define $|\lambda| := \lambda_1 + \cdots + \lambda_r$ and $l(\lambda) := r$. Let $\mathcal{D}(n)$ be the subset of $\mathcal{D}$ consisting of elements with $|\lambda| = n$. Now we come to our main result of this section: the structure of the $K_{\mathbb{C}}$-algebra $\sum_{n=0}^{\infty}\oplus T^*_{\mathbb{C}}\hat{S}_n$ as follows, which is $\mathbb{Z}/8$-graded version of that of the K^*-algebra $\sum_{n=0}^{\infty}\oplus T^*\hat{S}_n$.

THEOREM 4.1. *1. The sets*

$$\{c_\lambda := c_{\lambda_1}\cdots c_{\lambda_r} \mid \lambda = (\lambda_1 > \cdots > \lambda_r > 0) \in \mathcal{D}\}$$

and

$$\{\hat{c}_\lambda := \hat{c}_{\lambda_1} \cdots \hat{c}_{\lambda_r} \mid \lambda = (\lambda_1 > \cdots > \lambda_r > 0) \in \mathcal{D}\}$$

are two $K_{\mathbb{C}}$*-bases for* $\sum_{n=0}^{\infty} \oplus T_{\mathbb{C}}^* \hat{S}_n$*;*

2. The algebra $\sum_{n=0}^{\infty} \oplus T_{\mathbb{C}}^* \hat{S}_n$ *is isomorphic to the quotient of the free algebra with generators* $\{\hat{c}_1, \hat{c}_2, \ldots\}$ *by relations in Lemma 4.6.*

PROOF. Since $\{c_\lambda\}$ is a basis for $\sum_{n=0}^{\infty} \oplus T_{\mathbb{C}}^* \hat{S}_n$, 1. follows immediately from Lemma 4.2 and the fact that the transition matrix from $\{c_\lambda\}$ to $\{\hat{c}_\lambda\}$ is an invertible diagonal matrix. By 1., the quotient algebra in 2. maps canonically onto $\sum_{n=0}^{\infty} \oplus T^* \hat{S}_n$. But its defining relations imply that $\{\hat{c}_\lambda \mid \lambda \in \mathcal{D}\}$ is a set of $K_{\mathbb{C}}$-module generators for that quotient algebra, so the map is an isomorphism. ∎

5. $Z_{\mathbb{R}}^{(n)}(G), Z_{\mathbb{H}}^{(n)}(G)$ **and** $Z_{\mathbb{C}}^{(n)}(G)$**.** We shall relate $Z_{\mathbb{R}}^{(n)}(G)$ and $Z_{\mathbb{H}}^{(n)}(G)$ to $Z_{\mathbb{C}}^{(n)}(G)$ in this section, using "structure maps" j in a manner analogous to the classic case (see [1, 3.2, 3.3]) when G is a finite group. For the classic background, we refer to [1].

Let $\mathcal{V} = (V^{(0)}, V^{(1)}, \eta_1, \ldots, \eta_n) \in Z_{\mathbb{C}}^{(n)}(G)$. A real (or quaternionic) structure on $\mathcal{V}$ is a conjugate linear G-map $j\colon V^{(0)} + V^{(1)} \longrightarrow V^{(0)} + V^{(1)}$ with

$$jV^{(i)} = V^{(i)},\ j^2 = \mathrm{id}\ \ (\text{or} - \mathrm{id})\ \text{and}\ j\eta_k = \eta_k j, \quad \text{for } i = 0, 1;\ 1 \leq k \leq n.$$

In both cases j is called a *structure map*. An object in $Z_{\mathbb{C}}^{(n)}(G)$ is said to be of *real* (or *quaternionic*) type if it admits a real (or quaternionic) structure. Let $Z_{\mathbb{C}}^{(n)}(G,+)$ (or $Z_{\mathbb{C}}^{(n)}(G,-)$) be the category of all sequences $(V, V', \eta_1, \ldots, \eta_n, j)$ with $(V, V', \eta_1, \ldots, \eta_n) \in Z_{\mathbb{C}}^{(n)}(G)$ is of real (or quaternionic) type with structure map j. A morphism in $Z_{\mathbb{C}}^{(n)}(G, \pm)$, say,

$$(V, V', \eta_1, \ldots, \eta_n, j_{\mathcal{V}}) \xrightarrow{\varphi} (W, W', \zeta_1, \ldots, \zeta_n, j_{\mathcal{W}})$$

is a morphism in $Z_{\mathbb{C}}^{(n)}(G)$ such that $\varphi j_{\mathcal{V}} = j_{\mathcal{W}} \varphi$.

THEOREM 5.1. *We have*

$$Z_{\mathbb{R}}^{(n)}(G) \simeq Z_{\mathbb{C}}^{(n)}(G,+) \quad \textit{and} \quad Z_{\mathbb{H}}^{(n)}(G) \simeq Z_{\mathbb{C}}^{(n)}(G,-)$$

for $n \in \mathbb{Z}$.

PROOF. We shall prove the case of real representations and omit the case of quaternionic ones. Define a functor

$$\Delta\colon Z_{\mathbb{R}}^{(n)}(G) \longrightarrow Z_{\mathbb{C}}^{(n)}(G,+)$$

by

$$(V, V', \eta_1, \ldots, \eta_n) \longmapsto (\mathbb{C} \otimes V, \mathbb{C} \otimes V', 1 \otimes \eta_1, \ldots, 1 \otimes \eta_n, j)$$

where

$$j(z \otimes v) = \bar{z} \otimes v \quad \text{and} \quad g(z \otimes v) = z \otimes gv.$$

It is easy to check that we have defined an object of $Z_{\mathbb{C}}^{(n)}(G,+)$. If φ is a morphism in $Z_{\mathbb{R}}^{(n)}(G)$, then let Δ send φ to $1\otimes\varphi$, which is a morphism in $Z_{\mathbb{C}}^{(n)}(G,+)$. By a routine check, we have defined a functor Δ from $Z_{\mathbb{R}}^{(n)}(G)$ to $Z_{\mathbb{C}}^{(n)}(G,+)$.

Conversely , define

$$\Theta\colon Z_{\mathbb{C}}^{(n)}(G,+)\longrightarrow Z_{\mathbb{R}}^{(n)}(G)$$

by

$$(W^{(0)},W^{(1)},\xi_1,\dots,\xi_n,j_{\mathcal{W}})\longmapsto(W_+^{(0)},W_+^{(1)},\zeta_1,\dots,\zeta_n)$$

where

$$W_+^{(i)}:=\{1-\text{eigenspace of } j_{\mathcal{W}}|_{W^{(i)}}\}\quad\text{for } i=0,1$$

and

$$\zeta_k:=\eta_k|_{W_+^{(0)}+W_+^{(1)}}\quad\text{for } 1\le k\le n.$$

Since $j_{\mathcal{W}}|_{W^{(i)}}$ commutes with $\xi_j(j=1,\dots,n)$ so ζ_k maps $W_+^{(0)}$ to $W_+^{(1)}$ and $W_+^{(1)}$ to $W_+^{(0)}$, it is easy to see that $(W_+^{(0)},W_+^{(1)},\zeta_1,\dots,\zeta_n)$ is a well-defined object of $Z_{\mathbb{R}}^{(n)}(G)$. Also if ψ is a morphism in $Z_{\mathbb{C}}^{(n)}(G,+)$, define $\Theta\psi$ to be its restriction to $W_+^{(0)}+W_+^{(1)}$, then $\Theta\psi$ determines a morphism in $Z_{\mathbb{R}}^{(n)}(G)$. The construction supplies a functor Θ.

It is fairly easy to prove the compositions $\Delta\Theta$ and $\Theta\Delta$ are naturally equivalent to identity functors. ■

Suppose now that $\mathcal{V}$ and $\mathcal{W}$ admits structure maps $j_{\mathcal{V}},j_{\mathcal{W}}$ such that

$$j_{\mathcal{V}}^2=\epsilon_{\mathcal{V}}=\pm1,\quad j_{\mathcal{W}}^2=\epsilon_{\mathcal{W}}=\pm1.$$

Then $\mathcal{V}\otimes\mathcal{W}$ admits a structure map $j=j_{\mathcal{V}}\otimes j_{\mathcal{W}}$ such that $j^2=\epsilon_{\mathcal{V}}\epsilon_{\mathcal{W}}=\pm1$. Therefore we can separate three cases.

CASE 1. The product of two real representations is real.

CASE 2. The product of one real representation and one quaternionic representation is quaternionic.

CASE 3. The product of two quaternionic representations is real.

Now let us consider relationships between real representations and quaternionic representations. We have following category equivalence which is a special case of a result in [7].

THEOREM 5.2.

$$Z_{\mathbb{R}}^{(n)}(G)\simeq Z_{\mathbb{H}}^{(n+4)}(G)$$

for all $n\in\mathbb{Z}$.

PROOF. Let x_4 and z_2 be as given in Sections 3 and 4. Using Table 1 and a dimension count, one can see that cx_4 is either $2z_2^2$ or $2(1^*z_2^2)$, say $2z_2^2$. Therefore z_2^2 is of quaternionic type.

Define a functor

$$\Phi\colon Z_{\mathbb{C}}^{(n)}(G,+) \longrightarrow Z_{\mathbb{C}}^{(n+4)}(G,-)$$

by

$$\mathcal{V} \longmapsto z_2^2\mathcal{V}.$$

By the Case 2 above, $z_2^2\mathcal{V}$ is a well-defined element in $Z_{\mathbb{C}}^{(n+4)}(G,-)$. If φ is a morphism in $Z_{\mathbb{C}}^{(n)}(G,+)$, then let Φ send φ to $1\otimes\varphi$, which is a morphism in $Z_{\mathbb{C}}^{(n+4)}(G,-)$. By an elementary check, Φ is a functor. We can define the "same" functor to be an inverse functor of Φ by using $z_2^4 = 1$. So the desired result follows from Theorem 5.1. ■

We will also consider relations (or functors) between different types of graded representations coming from restriction, extension and conjugation.

DEFINITION 5.1. 1. If $\mathcal{V} = (V, V', \eta_1, \ldots, \eta_n) \in Z_{\mathbb{R}}^{(n)}(G)$, define $c\mathcal{V}$ to be $(\mathbb{C}\otimes V, \mathbb{C}\otimes V', 1\otimes\eta_1, \ldots, 1\otimes\eta_n)$, then obviously $c\mathcal{V}$ is an object of $Z_{\mathbb{C}}^{(n)}(G)$.

2. Similarly, if $\mathcal{W} = (W, W', \xi_1, \ldots, \xi_n) \in Z_{\mathbb{C}}^{(n)}(G)$, define $q\mathcal{W}$ to be $(\mathbb{H}\otimes W, \mathbb{H}\otimes W', 1\otimes\xi_1, \ldots, 1\otimes\xi_n)$, an object of $Z_{\mathbb{H}}^{(n)}(G)$.

3. $\mathcal{W} = (W, W', \zeta_1, \ldots, \zeta_n) \in Z_{\mathbb{H}}^{(n)}(G)$, let $c'\mathcal{W}$ have the same underlying set as W, W' and the same operation from G and $\zeta_1, \ldots, \zeta_n$, but regard it as an object of $Z_{\mathbb{C}}^{(n)}(G)$.

4. Similarly, if $\mathcal{W} \in Z_{\mathbb{C}}^{(n)}(G)$, define $r\mathcal{W} \in Z_{\mathbb{R}}^{(n)}(G)$.

5. Let $\mathcal{W} = (W, W', \xi_1, \ldots, \xi_n) \in Z_{\mathbb{C}}^{(n)}(G)$, define $t\mathcal{W}$ to have the same underlying set as W, W' and the same operations from G and $\xi_1, \ldots, \xi_n$, but we make $\mathbb{C}$ act in a new way: $z * w := \bar{z}w$, where $w \in t\mathcal{W}$, and $\bar{z}$ is acting on $W + W'$ as given in $\mathcal{W}$.

It is straightforward to see $\{r, c, q, c', t\}$ are natural and commute with direct sum. Also, given a morphism, say φ, we can construct morphisms $r\varphi, c\varphi, q\varphi, c'\varphi$ and $t\varphi$ in the obvious way. The next theorem expresses the relations among these functors. Since all these constructions don't touch the gradation, the classical proof can be transplanted here.

THEOREM 5.3. *We have*

$$\begin{gathered} rc = 2, \quad cr = 1 + t, \\ qc' = 2, \quad c'q = 1 + t, \\ tc = c, \quad rt = r, \\ tc' = c', \quad qt = q, \\ t^2 = 1, \end{gathered}$$

where $rc = 2$ is understood $rc\mathcal{V} \cong \mathcal{V}\oplus\mathcal{V}$, and so on.

We now come to the main goal of this section: To relate the irreducible graded real and quaternionic representations to the irreducible graded complex representations. Let G be a finite group. The analogue for $M(\mathbb{F}[G])$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, of the following theorem is very well known. Therefore, Note 3 in Section 1, which reduces it from Z to M, eliminates the need for any further proof.

THEOREM 5.4. *There are sets of irreducibles* $\{\mathcal{U}_m\} \subset Z_{\mathbb{R}}^{(n)}(G)$, $\{\mathcal{V}_n\} \subset Z_{\mathbb{C}}^{(n)}(G)$ *and* $\{\mathcal{W}_p\} \subset Z_{\mathbb{H}}^{(n)}(G)$ *which satisfy the following conditions:*

1. *The inequivalent irreducible representations of* $Z_{\mathbb{R}}^{(n)}(G)$ *are precisely the* $\mathcal{U}_m$, $r\mathcal{V}_n$ *and* $rc'\mathcal{W}_p$.
2. *The inequivalent irreducible representations of* $Z_{\mathbb{C}}^{(n)}(G)$ *are precisely the* $c\mathcal{U}_m$, $\mathcal{V}_n$, $t\mathcal{V}_n$ *and* $c'\mathcal{W}_p$.
3. *The inequivalent irreducible representations of* $Z_{\mathbb{H}}^{(n)}(G)$ *are precisely the* $qc\mathcal{U}_m$, $q\mathcal{V}_n$ *and* $\mathcal{W}_p$.

The following result will be used in the proof of Theorem 6.1 in next section. Its proof will be clear from the course of proofs of Theorems 5.1 and 5.2, and Definitions 5.1.

LEMMA 5.1. *We have following commutative diagram*

$$\begin{array}{ccc} T_{\mathbb{R}}^{(k-4)}(G) & \xrightarrow{\theta} & T_{\mathbb{H}}^{(k)}(G) \\ \downarrow & & \downarrow \\ T_{\mathbb{C}}^{(k-4)}(G) & \xrightarrow{z_2^2} & T_{\mathbb{C}}^{(k)}(G) \end{array}$$

where θ *is an isomorphism induced from the corresponding category equivalence in Theorem 5.2.*

6. **The structure of the** $K_{\mathbb{R}}$**-algebra** $\sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^{*}\hat{S}_n$**.** The central result of this paper is to give the structure of the $K_{\mathbb{R}}$-algebra $\sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^{*}\hat{S}_n$. Now, let us consider relations between the ground rings $K_{\mathbb{R}}$ and $K_{\mathbb{C}}$ under the maps c and r defined in Section 5:

$$\begin{array}{ccc} K_{\mathbb{R}} = \sum_{n=0}^{7} \oplus T_{\mathbb{R}}^{(n)}(\{1,z\}) & \overset{c}{\rightleftharpoons} & K_{\mathbb{C}} = \sum_{n=0}^{7} \oplus T_{\mathbb{C}}^{(n)}(\{1,z\}) \\ \| & & \| \\ \mathbb{Z}[x_1, x_4, x_7]/\langle \mathcal{R} \rangle & \overset{c}{\rightleftharpoons} & \mathbb{Z}[z_1, z_2]/\langle \mathcal{C} \rangle \end{array}$$

where

$$\mathcal{R} = \{x_4^2 - 4, x_1x_4 - x_7^3, x_4x_7 - x_1^3, x_1^2x_7 - 2x_1, x_1x_7^2 - 2x_7\},$$

and

$$\mathcal{C} = \{z_1^3 - 2z_1z_2, z_2^4 - 1\};$$

recall that $cr = 1 + t$, $rc = 2$. We have following result. Its proof is straightforward from dimension counts.

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

LEMMA 6.1.

$$
\begin{array}{cc}
c1 = 1 & r1 = 2 \\
c1^* = 1^* & r1^* = 2(1^*) \\
cx_1 = z_1 & rz_1 = 2x_1 \\
cx_1^2 = z_1^2 & rz_1^2 = 2x_1^2 \\
cx_1^2 = z_2 + 1^* z_2 & rz_2 = r(1^* z_2) = x_1^2 \\
c(x_4 x_7) = 2z_1 z_2 & r(z_1 z_2) = x_4 x_7 \\
cx_4 = 2z_2^2 & rz_2^2 = x_4, \quad r(1^* z_2^2) = 1^* x_4 \\
c(x_4 + 1^* x_4) = 2z_1^2 z_2 & r(z_1^2 z_2) = x_4 + 1^* x_4 \\
c(x_1 x_4) = 2z_1 z_2^2 & r(z_1 z_2^2) = x_1 x_4 \\
cx_7 = z_1 z_2^3 & r(z_1 z_2^3) = 2x_7 \\
c(x_1 x_7) = z_1^2 z_2^3 & r(z_1^2 z_2^3) = 2x_1 x_7 \\
cx_7^2 = z_1^2 z_2^2 & r(z_1^2 z_2^2) = 2x_7^2 \\
cx_7^2 = z_2^3 + 1^* z_2^3 & rz_2^3 = r(1^* z_2^3) = x_7^2.
\end{array}
$$

Note that the choice of x_4 is 1^*-unique, *i.e.*, either cx_4 or $c(1^*x_4)$ is $2z_2^2$. Therefore both z_2^2 and $1^*z_2^2$ are quaternionic.

Let recall that for every group G and each $l \geq 0$, G_l is the group defined in Note 3 after Definition 1.1 in Section 1. We shall use this notation several times in the rest of this section.

The following lemma gives the image under c, t and $*$ of the real and complex Clifford modules in Table 1 in Section 2.

LEMMA 6.2. *We have*

n	$T_{\mathbb{R}}^{(0)}(\{1,z\}_n)$	a'_n	$T_{\mathbb{C}}^{(0)}(\{1,z\}_n)$	$a'^{\mathbb{C}}_n$	$c, t, *$
0	$\mathbb{Z}m_0 \oplus \mathbb{Z}m_0^*$	1	$\mathbb{Z}M_0 \oplus \mathbb{Z}M_0^*$	1	$cm_0 = M_0, cm_0^* = M_0^*$, $tM_0 = M_0, tM_0^* = M_0^*$
1	$\mathbb{Z}m_1$	2	$\mathbb{Z}M_1$	2	$cm_1 = M_1$, $tM_1 = M_1$
2	$\mathbb{Z}m_2$	4	$\mathbb{Z}M_2 \oplus \mathbb{Z}M_2^*$	2	$cm_2 = M_2 + M_2^*$, $tM_2 = M_2^*$
3	$\mathbb{Z}m_3$	8	$\mathbb{Z}M_3$	4	$cm_3 = 2M_3$, $tM_3 = M_3$
4	$\mathbb{Z}m_4 \oplus \mathbb{Z}m_4^*$	8	$\mathbb{Z}M_4 \oplus \mathbb{Z}M_4^*$	4	$cm_4 = 2M_4, cm_4^* = 2M_4^*$, $tM_4 = M_4, tM_4^* = M_4^*$
5	$\mathbb{Z}m_5$	16	$\mathbb{Z}M_5$	8	$cm_5 = 2M_5$, $tM_5 = M_5$
6	$\mathbb{Z}m_6$	16	$\mathbb{Z}M_6 \oplus \mathbb{Z}M_6^*$	8	$cm_6 = M_6 + M_6^*$, $tM_6 = M_6^*$
7	$\mathbb{Z}m_7$	16	M_7	16	$cm_7 = M_7$, $tM_7 = M_7$

where a'_n *(resp.* $a'^{\mathbb{C}}_n$*) is the* $\mathbb{R}$ *(resp.* $\mathbb{C}$*)-dimension of an irreducible object of* $T_{\mathbb{R}}^{(0)}(\{1,z\}_n)$ *(resp.* $T_{\mathbb{C}}^{(0)}(\{1,z\}_n)$*).*

PROOF. ($n = 0$) $cm_0 = M_0$ or M_0^* by a dimension count, say, $cm_0 = M_0$. Then $cm_0^* = M_0^*$. This follows because c is a monomorphism. Hence $tM_0 = tcm_0 = M_0$ and $tM_0^* = tcm_0^* = M_0^*$, since $tc = c$.

($n = 1$ or 7) $cm_1 = M_1$ by a dimension count. Hence $tM_1 = tcm_1 = M_1$. Similarly for $n = 7$.

($n = 2$) If $tM_2 = M_2$ then $tM_2^* = M_2^*$. Now $rM_2 = m_2 = rM_2^*$ implies that $crM_2 = crM_2^*$, *i.e.*, $M_2 + tM_2 = M_2^* + tM_2^*$ and therefore $M_2 = M_2^*$, a contradiction. Hence $tM_2 = M_2^*$, $tM_2^* = M_2$ and $cm_2 = crM_2 = M_2 + tM_2 = M_2 + M_2^*$ as required.

($n = 3$ or 5) $tM_3 = M_3$ and $rM_3 = m_3$ by a dimension count. Hence $cm_3 = crM_3 = M_3 + tM_3 = 2M_3$. Similarly for $n = 5$.

($n = 4$) $rM_4 = m_4$ or m_4^* by a dimension count, say $rM_4 = m_4$. If $tM_4 = M_4^*$, then $cm_4 = crM_4 = M_4 + tM_4 = M_4 + M_4^*$ and $cm_4^* = (cm_4)^* = cm_4$, which implies $m_4 = m_4^*$, a contradiction. Hence $tM_4 = M_4$ and $tM_4^* = M_4^*$. Therefore $cm_4 = crM_4 = 2M_4$ and $cm_4^* = 2M_4^*$.

($n = 6$) $rM_6 = m_6 = rM_6^*$ by a dimension count. If $tM_6 = M_6$, then $tM_6^* = M_6^*$ and $cm_6 = crM_6 = crM_6^*$, *i.e.*, $M_6 = M_6^*$, a contradiction. Hence $tM_6 = M_6^*$ and $cm_6 = M_6 + M_6^*$. ■

LEMMA 6.3. *Let m_l and M_l be as given in Lemma 6.2. Let N_l be given by, for l odd,*

$$z_1 M_l = N_l + 1^* N_l$$

as in Section 3. For $l \geq 0$, there exists a special irreducible $n_l \in T_{\mathbb{R}}^{(l)}(\{1, z\}_l)$ with

$$cn_l = \begin{cases} z_2^{\frac{l-1}{2}} N_l & \text{if } l \text{ is odd,} \\ z_2^{\frac{l}{2}} M_l & \text{otherwise.} \end{cases}$$

PROOF. Lemmas 1.3 and 6.2 will play key roles in the following steps. We shall use them several times without specifically referring to them.

($l = 0$) Define n_0 to be m_0, then $cn_0 = M_0$ as required.

($l = 1$)

$$x_1 m_1 = n_1 + 1^* n_1 \quad \text{and} \quad z_1 M_1 = N_1 + 1^* N_1.$$

Then $c(x_1 m_1) = z_1 cm_1 = z_1 M_1$ and $N_1 + 1^* N_1 = cn_1 + 1^* n_1$. Hence $cn_1 = N_1$ or $1^* N_1$, say, $cn_1 = N_1$, so n_1 is irreducible and special since N_1 is.

($l = 2$) From

$$x_1^2 m_2 = x_1^2 m_1^2 = 2(n_2 + n_2^*) \quad \text{and} \quad c(x_1^2 m_2) = z_1^2 (M_2 + M_2^*) = 2(z_2 M_2 + z_2 M_2^*)$$

we have $cn_2 = z_2 M_2$ or $z_2 M_2^*$, say, $cn_2 = z_2 M_2$, so n_2 is special irreducible since M_2 is. Similarly for $l = 3$.

($l = 4$) From

$$x_1^4 = x_4 + 1^* x_4, \quad m_1^4 = m_4 + 1^* m_4, \quad \text{and} \quad x_1^4 m_1^4 = 8(n_4 + 1^* n_4)$$

we have $x_4m_4 = 4n_4$. Then $c(x_4m_4) = 4z_2^2M_4$ implies

$$cn_4 = z_2^2M_4$$

as desired.

For $l = 5, 6, 7$, we use the following equalities. Their proofs are similar to the above argument.

$$x_1x_4m_5 = 4(n_5 + 1^*n_5), \quad z_1M_5 = N_5 + 1^*N_5,$$
$$x_7^2m_6 = 2(n_6 + 1^*n_6), \quad x_7m_7 = n_7 + 1^*n_7, \quad z_1M_7 = N_7 + 1^*N_7.$$

Our result follows from 8-fold periodicity of the real Clifford modules. ∎

Recall that $\hat{S}_l \subset B'_{l-1} \subset C'_l$ and therefore an irreducible module of B'_{l-1} will restrict to an irreducible module of $\hat{S}_l$, as stated in the last paragraph in Section 2. The fact that the category of B'_{l-1}-modules is equivalent to the category of $\{1, z\}_{l-1}$-modules in Note 3 in Section 1 allows us to define the real basic Clifford modules, for each $l > 0$, as follows:

$$\hat{d}_l = n_{l-1} \in T_{\mathbb{R}}^{(l-1)}\hat{S}_l.$$

Let us recall the definition of c_l and $\hat{c}_l$:

$$c_l = \begin{cases} M_{l-1} \in T_{\mathbb{C}}^{(0)}\hat{S}_l & \text{if } l \equiv 1 \pmod 2, \\ N_{l-1} \in T_{\mathbb{C}}^{(1)}\hat{S}_l & \text{if } l \equiv 0 \pmod 2 \end{cases}$$

and

$$\hat{c}_l = z_2^{[\frac{l-1}{2}]}c_l.$$

Therefore

$$\hat{c}_l = \begin{cases} z_2^{\frac{l-1}{2}}M_{l-1} & \text{if } l \equiv 1 \pmod 2, \\ z_2^{\frac{l-2}{2}}N_{l-1} & \text{otherwise.} \end{cases}$$

By Lemma 6.3, we have

$$c\hat{d}_l = cn_{l-1} = \hat{c}_l, \quad \text{for all } l > 0.$$

Now we are coming to the main goal of this paper: to determine the structure of the $K_{\mathbb{R}}$-algebra $\sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^*\hat{S}_n$ from that of the $K_{\mathbb{C}}$-algebra $\sum_{n=0}^{\infty} \oplus T_{\mathbb{C}}^*\hat{S}_n$ using the maps c and r. The latter is known by Theorem 4.1.

THEOREM 6.1. *Let $\{\hat{d}_1, \hat{d}_2, \ldots\}$ be the real basic Clifford modules defined as above.*
1. The set

$$\{\hat{d}_\lambda := \hat{d}_{\lambda_1} \cdots \hat{d}_{\lambda_r} \mid \lambda = (\lambda_1 > \cdots \lambda_r > 0) \in \mathcal{D}\}$$

is a $K_{\mathbb{R}}$-basis for $\sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^*\hat{S}_n$;

2. The algebra $\sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^*\hat{S}_n$ is isomorphic to the quotient of the free algebra with the generators $\{\hat{d}_1, \hat{d}_2, \ldots\}$ by relations:
(a) $\hat{d}_i\hat{d}_j = (1^*)^{i+j+1}\hat{d}_j\hat{d}_i$;
(b) $\hat{d}_n^2 = (-1)^{n+1}x_7(\hat{d}_{2n} + x_1\sum_{i=1}^{n-1}(-1)^i\hat{d}_i\hat{d}_{2n-i})$.

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

PROOF. Applying c to the both sides of (a) and (b) yields the relations involving the complex Clifford modules $\hat{c}_n, n > 0$ in Lemma 4.6. Hence the relations (a) and (b) hold since c is a monomorphism.

We argue that 1. implies 2. as follows. By 1., the quotient algebra maps canonically onto $\sum_{n=0}^{\infty} \oplus T_{\mathbb{R}}^{*}\hat{S}_n$. But its defining relations (a) and (b) obvoiously imply that $\{\hat{d}_\lambda \mid \lambda \in \mathcal{D}\}$ is a set of $K_{\mathbb{R}}$-module generators for that quotient algebra, so the map is an isomorphism.

Let $M_{\mathbb{R}}$ be the $K_{\mathbb{R}}$-module generated by $\{\hat{d}_\lambda : \lambda \in \mathcal{D}\}$. By applying c, it is immediate to see that $\{\hat{d}_\lambda\}$ is a $K_{\mathbb{R}}$-basis for $M_{\mathbb{R}}$. To complete our proof, it suffices to prove 1. by showing that

$$T_{\mathbb{R}}^{(k)}\hat{S}_n = M_{\mathbb{R}}^{n,k} \tag{4}$$

for $n \geq 0$ and $0 \leq k \leq 7$, where

$$M_{\mathbb{R}}^{n,k} := \operatorname{span}_{\mathbb{Z}}\{x\hat{d}_\lambda : \lambda \in \mathcal{D}(n), n - l(\lambda) + |x| \equiv k \pmod 8\}, \tag{5}$$

where $|x|$ is the grading of x in $K_{\mathbb{R}}$.

We first claim that

$$2T_{\mathbb{R}}^{(k)}\hat{S}_n \subset M_{\mathbb{R}}^{n,k}. \tag{6}$$

Fix any $d \in T_{\mathbb{R}}^{(k)}\hat{S}_n$. By Theorem 4.1, we have

$$c(d) = \sum \big(n_{i,j,k,\lambda}(1^*)^i z_1^j z_2^k\big)\hat{c}_\lambda \tag{7}$$

where $n_{i,j,k,\lambda} \in \mathbb{Z}$ and $0 \leq i \leq 1, 0 \leq j \leq 2, 0 \leq k \leq 3$. Now

$$c1 = 1, \quad c1^* = 1^*, \quad cx_1^2 = z_1^2 = z_2 + 1^*z_2,$$
$$cx_1 = z_1, \quad cx_7 = z_1z_2^3, \quad cx_7^2 = z_1^2z_2^2 = z_2^3 + 1^*z_2^3,$$

and

$$c\hat{d}_\lambda = \hat{c}_\lambda.$$

So the equality (7) becomes, by moving the terms of real type to left hand side,

$$c(d + d') = \sum(m_1z_2 + m_2z_2^2 + m_3 1^*z_2^2 + m_4z_2^3 + m_5z_1z_2 + m_6z_1z_2^2)\hat{c}_\lambda,$$

for some $d' \in M_{\mathbb{R}}^{n,k}$ and $m_i \in \mathbb{Z}$. By comparing the coefficients of $\hat{c}_\lambda$ in $tc(d+d') = c(d+d')$ and using $tz_2 = 1^*z_2$, we have

$$m_1 = 0 = m_4.$$

Since

$$c(x_4x_7) = 2z_1z_2, \quad cx_4 = 2z_2^2, \quad c(x_1x_4) = 2z_1z_2^2, \quad c(x_4 + 1^*x_4) = 2z_1^2z_2,$$

we have

$$c(2d + 2d') = c(d'')$$

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

for some $d'' \in M_{\mathbb{R}}^{n,k}$. Therefore

$$2d = -2d' + d'' \in M_{\mathbb{R}}^{n,k}$$

as required.

Let

$$a := \frac{1}{2}\#\{\text{complex irreducibles in } T_{\mathbb{C}}^{(k)}\hat{S}_n\} + \#\{\text{quarternionic irreducibles in } T_{\mathbb{C}}^{(k)}\hat{S}_n\},$$
$$b := \frac{1}{2}\#\{\text{complex irreducibles in } T_{\mathbb{C}}^{(k)}\hat{S}_n\} + \#\{\text{real irreducibles in } T_{\mathbb{C}}^{(k)}\hat{S}_n\},$$
$$a' := \dim_{\mathbb{Z}/2}(A' \otimes \mathbb{Z}/2), \quad \text{where} \quad A' := T_{\mathbb{C}}^{(k)}\hat{S}_n / c(T_{\mathbb{R}}^{(k)}\hat{S}_n),$$
$$b' := \dim_{\mathbb{Z}/2}(B' \otimes \mathbb{Z}/2), \quad \text{where} \quad B' := T_{\mathbb{C}}^{(k)}\hat{S}_n / c'(T_{\mathbb{H}}^{(k)}\hat{S}_n),$$
$$a'' := \dim_{\mathbb{Z}/2}(A'' \otimes \mathbb{Z}/2), \quad \text{where} \quad A'' := T_{\mathbb{C}}^{(k)}\hat{S}_n / c(M_{\mathbb{R}}^{n,k}),$$
$$b'' := \dim_{\mathbb{Z}/2}(B'' \otimes \mathbb{Z}/2), \quad \text{where} \quad B'' := T_{\mathbb{C}}^{(k)}\hat{S}_n / c'\theta(M_{\mathbb{R}}^{n,k-4}),$$

where

$$\theta\colon T_{\mathbb{R}}^{(k-4)}\hat{S}_n \longrightarrow T_{\mathbb{H}}^{(k)}\hat{S}_n$$

is an isomorphism with $c'\theta = z_2^2 c$ as defined in Lemma 5.1. Let

$$\begin{aligned} a''' := \#\big\{\lambda \in \mathcal{D}(n) : n - l(\lambda) + \{2, 3, 5, \text{ or } 6\} \equiv k \pmod 8\big\} \\ + 2\#\{\lambda \in \mathcal{D}(n) : n - l(\lambda) + 4 \equiv k \pmod 8\}, \end{aligned}$$

and

$$\begin{aligned} b''' := \#\big\{\lambda \in \mathcal{D}(n) : n - l(\lambda) + \{1, 2, 6 \text{ or } 7\} \equiv k \pmod 8\big\} \\ + 2\#\{\lambda \in \mathcal{D}(n) : n - l(\lambda) \equiv k \pmod 8\}. \end{aligned}$$

By Theorem 5.4, we have $a = a'$ and $b = b'$. By Lemma 6.1, $a''' \geq a''$. By Lemma 6.1 and $B'' = T_{\mathbb{C}}^{(k)}\hat{S}_n / z_2^2 c(M_{\mathbb{R}}^{n,k-4})$, $b''' \geq b''$. Since

$$M_{\mathbb{R}}^{n,k} \subset T_{\mathbb{R}}^{(k)}\hat{S}_n, \quad c'\theta(M_{\mathbb{R}}^{n,k-4}) \subset c'(T_{\mathbb{H}}^{(k)}\hat{S}_n),$$

we have

$$a'' \geq a' \quad \text{and} \quad b'' \geq b'.$$

Therefore

$$a''' \geq a'' \geq a' = a \quad \text{and} \quad b''' \geq b'' \geq b' = b.$$

Now by counting the number of the complex irreducibles, we get

$$\begin{aligned} a''' + b''' &= \#\{\lambda : n - l(\lambda) + 1 \equiv k \pmod 2\} + 2\#\{\lambda : n - l(\lambda) \equiv k \pmod 2\} \\ &= a + b. \end{aligned}$$

https://doi.org/10.4153/CJM-1994-029-9 Published online by Cambridge University Press

This implies that all the inequalities above become equalities. Therefore there exists an isomorphism between

$$A' \otimes \mathbb{Z}/2 \quad \text{and} \quad A'' \otimes \mathbb{Z}/2.$$

Let

$$A := c(T^{(k)}_{\mathbb{R}}\hat{S}_n)/c(M^{n,k}_{\mathbb{R}}).$$

By Theorem 5.4 and equation (6), we can decompose both A' and A as follows

$$A' = \mathbb{Z}^m \oplus (\mathbb{Z}/2)^p, \quad A = (\mathbb{Z}/2)^q,$$

since A and A' are finitely generated abelian groups. Now

$$A' \cong A''/A$$

implies

$$A'' = \mathbb{Z}^m \oplus (\mathbb{Z}/2)^{p+q}.$$

On the other hand,

$$A' \otimes \mathbb{Z}/2 \cong A'' \otimes \mathbb{Z}/2$$

forces that $q = 0$ and consequently (4) holds. This completes our proof. ∎

It is a result of Schur [19] that there are exactly two nonisomorphic complex representation groups of S_n when $n \geq 4$ and $n \neq 6$, namely $\hat{S}_n$ and $\tilde{S}_n$ (defined below), corresponding to distinct elements of $H^2(S_n, \mathbb{R}^\times)$. $\hat{S}_6$ is (up to isomorphism) the only complex representation group of S_6. We have dealt with the group $\hat{S}_n$ so far. We will define $\tilde{S}_n$ as follows.

EXAMPLE 3. Let n be a positive integer. Define $\tilde{S}_n$ in the same way as $\hat{S}_n$ in Section 1, except that t_j^2 and $(t_jt_{j+1})^3$ are z, not 1. So $(\tilde{S}_n, z, \sigma)$ is another object in $\mathcal{G}$, where σ is defined as it was for $\hat{S}_n$. The map $\tilde{\phi}_{k,l}$ defined in [10, Chapter 3] is an embedding of $\tilde{S}_k \hat{Y} \tilde{S}_l$ into $\tilde{S}_{k+l}$. Its image may be identified with the subgroup of $\tilde{S}_{k+l}$ generated by the union of the $\tilde{S}$-double covers of the symmetric groups on $\{1, 2, \ldots, k\}$ and $\{k+1, k+2, \ldots, k+l\}$.

Define

$$\tilde{T}^*_{\mathbb{R}}G := \sum_{i=0}^{7} \oplus T^{(-i)}_{\mathbb{R}}(G), \quad K^-_{\mathbb{R}} := \tilde{T}^*_{\mathbb{R}}(\{1, z\}) \quad \text{and} \quad \tilde{T}^*_{\mathbb{R}} := \sum_{n=0}^{\infty} \oplus \tilde{T}^*_{\mathbb{R}}\tilde{S}_n.$$

Then we can prove in a way parallel to that as above that $\tilde{T}^*_{\mathbb{R}}$ is a pseudo-commutative $K^-_{\mathbb{R}}$-algebra with a product defined as for $T^*_{\mathbb{R}}$ and satisfying $xy = (1^*)^{ij+kl}yx$ for $x \in T^{(i)}_{\mathbb{R}}(\tilde{S}_k)$ and $y \in T^{(j)}_{\mathbb{R}}(\tilde{S}_l)$.

P. Hoffman [7] shows that the $K^-_{\mathbb{R}}$-algebra $\tilde{T}^*_{\mathbb{R}}$ for $\tilde{S}_n$ is (up to isomorphism) the $\mathbb{Z}/8$-negation of the $K_{\mathbb{R}}$-algebra $T^*_{\mathbb{R}}$ for $\hat{S}_n$, *i.e.* change all the $\mathbb{Z}/8$-gradings to their negatives and leave everything else alone. This allows us to get the explicit structure of the $K^-_{\mathbb{R}}$-algebra $\tilde{T}^*_{\mathbb{R}}$ from that of the $K_{\mathbb{R}}$-algebra $T^*_{\mathbb{R}}$.

7. **Real projective representations of S_n and A_n.** Given a finite group G, a real representation group for G is a group $\mathbb{R}(G)$ satisfying the following three conditions:

(a) $\mathbb{R}(G)$ has a central subgroup A of order $H^2(G, \mathbb{R}^\times)$;

(b) there is an isomorphism $\tau: G \longrightarrow \mathbb{R}(G)/A$;

(c) every projective representation P of G over $\mathbb{R}$ lifts to a linear representation Q of $\mathbb{R}(G)$ in the sense that the map given by

$$P_1(g) = Q\big(r(g)\big),$$

where $r(g)$ is a representative from the coset $\tau(g)$, is $\mathbb{R}$-projectively equivalent to P.

In this case we shall say P is $\mathbb{R}$-projectively equivalent to linear representation Q. By a construction of a real representation group of S_n, P. Hoffman and J. Humphreys [9] proved first part of the following theorem. By a similar construction, one can prove that $\tilde{A}_n := [\tilde{S}_n, \tilde{S}_n]$ is also a real representation group of A_n when $n \geq 4$ and $n \neq 6, 7$, so we have the second part of the following theorem.

THEOREM 7.1. *(1) Every real projective representation of S_n, for $n \geq 4$, is $\mathbb{R}$-projectively equivalent either to a linear representation of S_n or to a linear representation of $\tilde{S}_n$, $\hat{S}_n$ and $\bar{S}_n$ with z acting as -1, where $\bar{S}_n$ defined below.*

(2) Every real projective representation of A_n, for $n \geq 4$ and $n \neq 6, 7$, is $\mathbb{R}$-projectively equivalent either to a linear representation of A_n or to a linear representation of $\tilde{A}_n$.

EXAMPLE 4. Let n be a positive integer. Define $\bar{S}_n$ in the same way as $\hat{S}_n$ in Section 1, except that t_j^2 and $(t_j t_{j+1})^3$ are z, not 1 and $t_j t_k = t_k t_j$, not $t_j t_k = z t_k t_j$. So $(\bar{S}_n, z, \sigma)$ is a third family of objects in $\mathcal{G}$, where σ is defined as before.

THEOREM 7.2. *Let (G, z, σ) be any object in $\mathcal{G}$ with G finite. Then we have category equivalences:*

1. *$M(\mathbb{R}[G]) \simeq Z_{\mathbb{R}}^{(1)}(G)$;*
2. *$M(\mathbb{R}[\ker \sigma]) \simeq Z_{\mathbb{R}}^{(0)}(G)$, if σ is non-zero;*
3. *$M(\mathbb{R}[\bar{S}_n]) \simeq Z_{\mathbb{R}}^{(-1)}(S_n \times \{1, z\})$, i.e., $M(\mathbb{R}[\bar{S}_n])$ is equivalent to the category of the simultaneously graded S_n-modules and graded C_1-modules.*

Note that Theorems 7.2, 6.1 and the last part of Section 6 tell us, when $n \geq 4$ for $\tilde{S}_n$ and $\hat{S}_n$, and $n \geq 4, n \neq 6, 7$ for $\tilde{A}_n$, that all the real negative representations of $\tilde{S}_n$, $\hat{S}_n$ and $\tilde{A}_n$ are $\mathbb{Z}$-linear combinations of the product of induced representations of Clifford modules. It also explains in a general way the relationship in Schur's result [20] that the number of the irreducibles in $M(\mathbb{R}[\hat{S}_n])$, $M(\mathbb{R}[\tilde{S}_n])$ and $M(\mathbb{R}[\tilde{A}_n])$ are as follows, where $|\lambda|$ and $l(\lambda)$ are defined before Theorem 4.1 in Section 4:

$$2\#\{\lambda \in \mathcal{D}(n) : |\lambda| - l(\lambda) \equiv 1 \pmod 4\} + \#\{\lambda \in \mathcal{D}(n) : |\lambda| - l(\lambda) \not\equiv 1 \pmod 4\},$$

$$2\#\{\lambda \in \mathcal{D}(n) : |\lambda| - l(\lambda) \equiv -1 \pmod 4\} + \#\{\lambda \in \mathcal{D}(n) : |\lambda| - l(\lambda) \not\equiv -1 \pmod 4\}$$

and

$$2\#\{\lambda \in \mathcal{D}(n) : |\lambda| - l(\lambda) \equiv 0 \pmod 4\} + \#\{\lambda \in \mathcal{D}(n) : |\lambda| - l(\lambda) \not\equiv 0 \pmod 4\}$$

respectively. The real negative representations of $\bar{S}_n$ can be obtained from graded S_n- and C_1-modules.

PROOF. That $M(\mathbb{R}[G]) \simeq \mathcal{Z}_{\mathbb{R}}^{(1)}(G)$ and $\mathcal{Z}_{\mathbb{R}}^{(0)}(G) \simeq M(\mathbb{R}[\ker\sigma])$ are simple cases of 2.1, 2.2, 2.3 in [5]. It remains to prove that $\mathcal{Z}_{\mathbb{R}}^{(-1)}(S_n \times \{1, z\})$ and $\mathcal{Z}_{\mathbb{R}}^{(1)}(\bar{S}_n)$ are equivalent for all n. Define a functor

$$\mathcal{Z}_{\mathbb{R}}^{(1)}(\bar{S}_n) \xrightarrow{\Lambda} \mathcal{Z}_{\mathbb{R}}^{(-1)}(S_n \times \{1, z\})$$

by

$$(V, V', \cdot, \eta) \longmapsto (V, V', *, \zeta)$$

where

$$t_k * (v, v') := (-t_k \cdot v', t_k \cdot v) \quad \zeta(v, v') := (-\eta v', \eta v)$$

Then it is easy to check that we have defined an object of $\mathcal{Z}_{\mathbb{R}}^{(-1)}(S_n \times \{1, z\})$. For a morphism ϕ in $\mathcal{Z}_{\mathbb{R}}^{(1)}(\bar{S}_n)$, define $\Lambda\phi$ to be the function $(v, v') \longmapsto (\phi v, \phi v')$, a morphism in $\mathcal{Z}_{\mathbb{R}}^{(-1)}(S_n \times \{1, z\})$. So we have defined a functor. Symmetrically, one can define a functor, say Γ, from $\mathcal{Z}_{\mathbb{R}}^{(-1)}(S_n \times \{1, z\})$ to $\mathcal{Z}_{\mathbb{R}}^{(1)}(\bar{S}_n)$. It is easy to see that identity functions are natural isomorphisms from $\mathcal{V}$ to $\Gamma\Lambda\mathcal{V}$ and $\mathcal{W}$ to $\Lambda\Gamma\mathcal{W}$. ∎

Finally I would like to note that there are several things remaining to be done in this direction. Examples are to find relationships between the two algebras in this paper and the ring of the symmetric functions; and the connection with the combinatorics of shifted tableaux.

REFERENCES

1. J. F. Adams, *Lectures on Lie Groups*, W. A. Benjamin, New York, 1969.
2. K. Asano, *Über die Darstellungen einer endlichen Gruppe durch reelle Kollineationen*, Proc. Imp. Acad. Japan, 1933.
3. M. F. Atiyah, R. Bott and A. Shapiro, *Clifford Modules*, Topology **3**, (Supplement 1) (1964), 3–38.
4. P. N. Hoffman, *τ-Rings and Wreath Product Representations*, Lecture Notes in Mathematics **746**, Springer-Verlag, Berlin, 1979.
5. ———, *Projective and Multigraded Representations of Monomial and Multisigned Groups.I.Graded Representations of a twisted Product*, Canad. J. Math. **45**(1993), 295–339.
6. ———, *A Projective Analogue of Schur's Tensor Power Construction*, Comm. Algebra **21**(1993), 2211–2249.
7. ———, *Generalizing Clifford Module Periodicity to Graded Representations*, (1992), preprint.
8. P. N. Hoffman and J. F. Humphreys, *Hopf Algebras and Projective Representations of $G \wr S_n$ and $G \wr A_n$*, Canad. J. Math. **38**(1986), 1380–1458.
9. ———, *Real Projective Representations of Finite Groups*, Math. Proc. Cambridge Philos. Soc. **107**(1990), 27–32.
10. ———, *Projective Representations of the Symmetric Groups*, Oxford University Press, Oxford, 1992.
11. J. Q. Huang, *Ph.D. Thesis*, University of Waterloo, 1993, 1–128.
12. G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics **682**, Springer-Verlag, Berlin, 1978.
13. T. Józefiak, *Characters of Projective Representations of Symmetric Groups*, Exposition. Math. **7**(1989), 193–247.
14. M. Karoubi, *K-Theory : An Introduction*, Springer-Verlag, Berlin, Heidleberg, New York, 1977.
15. P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Graduate Text in Math. **4**, Springer-Verlag, New York, 1971.

16. M. K. Makhool and A. O. Morris, *Real Projective Representations of Clifford Algebras, and Symmetric Groups*, J. London Math. Soc. (2) **43**(1991), 412–420.
17. M. L. Nazarov, *An Orthogonal Basis of Irreducible Projective Representations of the Symmetric Groups*, Functional Anal. Appl. **22**(1988), 77–78.
18. ———, *Young's Orthogonal Form of Irreducible Projective Representations of the Symmetric Groups*, J. London Math. Soc. (2) **42**(1991), 437–451.
19. I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **139**(1911), 155–250.
20. ———, *Über die reellen Kollineationsgruppen die der symmetrischen oder der alterniernden Gruppe isomorph sind*, J. Reine Angew. Math. **158**(1927), 63–79.
21. J. P. Serre, *Linear Representations of Finite Groups*, Graduate Text in Math. **42**, Springer-Verlag, New York, 1977.
22. J. R. Stembridge, *Shifted Tableaux and the Projective Representations of Symmetric Groups*, Adv. in Math. **74**(1989), 87–134.

The Fields Institute for Research in Mathematical Sciences
185 Columbia Street West
Waterloo, Ontario
N2L 5Z5