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Joint ergodicity for group actions

VITALY BERGELSON† AND JOSEPH ROSENBLATT‡
The Ohio State University, Columbus, Ohio 43210, *USA*

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Abstract. Let $T_1, \ldots, T_n$ be continuous representations of a σ-compact separable locally compact amenable group G as measure-preserving transformations of a non-atomic separable probability space (X, β, m). Let (K_n) be a right Følner sequence of compact sets in G. If $T_1, \ldots, T_n$ are pairwise commuting in the sense that $T_i(g)T_j(h) = T_j(h)T_i(g)$ for $i \neq j$ and $g, h \in G$, then necessary and sufficient conditions can be given, in terms of the ergodicity of certain tensor products, for the following to hold: for all $F_1, \ldots, F_n \in L_\infty$, the sequence $A_N(x)$ where

$$A_N(x) = (1/|K_N|) \int_{K_N} F_1(T_1(g)^{-1}x)F_2(T_1(g)^{-1}T_2(g)^{-1}x) \cdots F_n(T_1(g)^{-1} \cdots T_n(g)^{-1}x)\, dg$$

converges in $L_2(X)$ to $\prod_{i=1}^n \int F_i \,\mathrm{dm}$. The necessary and sufficient conditions are that each of the following representations are ergodic: T_n, $T_{n-1} \otimes T_{n-1}T_n, \ldots, T_2 \otimes T_2T_3 \otimes \cdots \otimes T_2 \cdots T_n$, $T_1 \otimes T_1T_2 \otimes \cdots \otimes T_1 \cdots T_n$.

In order to prove this theorem, specific properties of the decomposition of $L_2(X)$ into its weakly mixing and compact subspaces with respect to a representation T_i are needed. These properties are also used to prove some generalizations of well-known facts from ergodic theory in the case where G is the integer group Z.

0. *Introduction*

In [7], Furstenberg proved a fundamental theorem on multiple recurrence of measure preserving systems. He showed that for any measure preserving system (X, β, m, T), for any $k \geq 1$, $A \in \beta$, and $m(A) > 0$; there exists $n \geq 1$ such that $m(\bigcap_{i=1}^k T^{-in}A) > 0$. In the special case where T is weakly mixing, more was proved: T is weakly mixing of all orders. That is, if T is weakly mixing, then for all $A_0, \ldots, A_k \in \beta$,

$$\lim_{N\to\infty} (1/N) \sum_{n=1}^N \left[m(A_0 \cap T^{-n}A_1 \cap \cdots \cap T^{-kn}A_k) - \prod_{i=0}^n m(A_i) \right]^2 = 0. \qquad (1)$$

Indeed, Furstenberg uses (1) as part of the proof in [**7**]. See also Furstenberg, Katznelson and Ornstein [**10**] for a discussion of this connection.

As in [**10**], (1) can be shown by proving for weakly mixing T, that all $F_1, \ldots, F_k \in L_\infty(X)$,

$$\lim_{N\to\infty} \left\| (1/N) \sum_{n=1}^N \left(\prod_{l=1}^k T^{ln}F_l \right) - \prod_{l=1}^k \int F_l \, dm \right\|_2 = 0. \qquad (2)$$

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This was generalized by Furstenberg and Katznelson in [**9**], see also Furstenberg [**8**], in the process of proving an ergodic Szemeredi theorem for commuting transformations. They show that if $S_1, \ldots, S_n$ are commuting transformations with $S_iS_j^{-1}$ weakly mixing for all $i \neq j$, then for all $F_1, \ldots, F_k \in L_\infty(X)$,

$$\lim_{N\to\infty} \left\| (1/N) \sum_{n=1}^{N} \left(\prod_{l=1}^{k} S_l^n F_l \right) - \prod_{l=1}^{k} \int F_l \, dm \right\|_2 = 0. \tag{3}$$

The hypotheses that are necessary and sufficient for (3) are given in Berend and Bergelson [**1**] where it is shown that (3) holds for commuting S_i if and only if $S_1 \otimes \cdots \otimes S_n$ is ergodic and $S_iS_j^{-1}$ is ergodic for each $i \neq j$. This latter jointly ergodic criteria is much weaker than assuming $S_1, \ldots, S_n$ are weakly mixing (for instance, let S_i be irrational rotations by rationally independent angles). See also Berend and Bergelson [**2**] where necessary and sufficient criteria for (3) are given when $T_1, \ldots, T_n$ do not commute.

In § 2, (3) is generalized by replacing each S_i by commuting actions $S_i(g)$ of a general σ-compact separable amenable locally compact group. First, in § 1, some facts about representations of a σ-compact locally compact group G as measure preserving transformations are derived which are generalizations of similar theorems for abelian groups G. These are then used in § 2 to give the joint ergodicity conditions on commuting actions S_i of G which are necessary and sufficient for the generalization of (3).

1. *Groups of measure-preserving transformations*

Assume G is a σ-compact locally compact Hausdorff group (called a *group* in the sequel). Let $T: G \to \mathcal{M}(X)$ be a homomorphism of G into the invertible measure-preserving transformations $\mathcal{M}(X)$ of a probability space (X, β, m). For $F: X \to C$, $g \in G$, $(T(g)F)(x) = F(T(g)^{-1}x)$ for all $x \in X$. A *representation T of G in $\mathcal{M}(X)$* will be any such homomorphism for which the mapping $g \to \int T(g)F_1F_2\,dm$ is continuous for all $F_1, F_2 \in L_2(X)$. Generally, $L_2(X)$ is a direct sum of two orthogonal closed subspaces, denoted here by $L_2(X)_w$ and $L_2(X)_c$. The *compact summand* $L_2(X)_c$ consists of all $F \in L_2(X)$ such that $\{T(g)F: g \in G\}$ is totally-bounded in $L_2(X)$. Let M denote the unique G-invariant mean on WAP(G), the weakly almost periodic functions on G. Then the weakly mixing part $L_2(X)_w$ consists of all $F \in L_2(X)$ such that, if $f(g) = \int \bar{F}T(g)F\,dm - |\int F\,dm|^2$ for all $g \in G$, then $M(|f|) = 0$. See Bergelson and Rosenblatt [**3**] for a proof of the above using the work of Godement, or see Krengel [**14**, p. 111], where this theorem is discussed in relationship to the work of Jacobs, Deleeuw and Glicksberg.

The representation T is *weakly mixing* if and only if $L_2(X)_c$ consists of just the constants. Of particular importance in § 2 are some of the structural aspects of $L_2(X)_c$. In case G is abelian, in particular $G = Z$ as in Halmos [**11**], the space $L_2(X)_c$ has an orthonormal basis $(F_\lambda: \lambda \in \Lambda)$ of eigenvectors. That is $(F_\lambda: \lambda \in \Lambda)$ is an orthonormal basis of $L_2(X)_c$ such that for all $g \in G$, there exists $c_\lambda(g) \in C$ such that $T(g)F_\lambda = c_\lambda(g)F_\lambda$ a.e. $[m]$. Also, T is ergodic if and only if the constant functions are the only eigenvectors with eigenvalue one. If T is ergodic and $\lambda_1, \lambda_2 \in \Lambda$ are such that $c_{\lambda_1}(g) = c_{\lambda_2}(g)$ for all $g \in G$, then $F_{\lambda_1} = F_{\lambda_2}$ a.e. $[m]$. That is, the eigenvalue homomorphisms $c_\lambda: G \to T$ are *simple*. The analogues of these results for general groups are described in Propositions 1.1 and 1.4.

1.1. PROPOSITION. *The representation T restricted to $L_2(X)_c$ decomposes as an orthogonal direct sum of finite-dimensional subrepresentations.*

There are a number of proofs of this theorem. See [3] for a discussion of the proof via Godement's decomposition of positive definite functions. Also, see Dye [6] for a proof for amenable groups that can be generalized to any locally compact group. These proofs all suggest that there should be a direct argument using only the Peter–Weyl theorem for compact groups. Indeed, in Deleeuw and Glicksberg [4, p. 72], the necessary fact about compactifications is observed which is needed in the proof of the following:

1.2. THEOREM. *Suppose G is a group of unitary transformations $\{V_g: g \in G\}$ of a Hilbert space H. Then a necessary and sufficient condition for H to decompose as a direct sum of G-invariant finite-dimensional subspaces is that for all $F \in H$, $\{V_gF: g \in G\}$ is totally bounded in the L_2-norm topology.*

Proof. Because orbits $\{V_gF: g \in G\}$ are totally bounded when F lies in a G-invariant finite-dimensional subspace of H, the condition is necessary. Conversely, suppose all orbits are precompact. Let $\mathscr{G}$ be the weak-operator closure of G in the bounded operators $B(H)$. Give $\mathscr{G}$ the weak-operator topology.

Theorem 3.2 in Deleeuw and Glicksberg [3] proves that $\mathscr{G}$ is a compact semigroup with a jointly continuous multiplication. Clearly, if $A \in \mathscr{G}$, then $A^* \in \mathscr{G}$, too. But also $A^* = A^{-1}$. That is, $\mathscr{G}$ consists of unitary transformations. Indeed, suppose $A = \lim_i V_{g_i}$ in the weak-operator topology. Then $\lim_i V_{g_i-1} = A^*$ in that topology and so

$$\begin{aligned} I &= \lim_i V_{g_i} V_{g_i-1} = (\lim_i V_{g_i})(\lim_i V_{g_i-1}) \\ &= (\lim_i V_{g_i-1})(\lim_i V_{g_i}) \\ &= AA^* = A^*A \end{aligned}$$

by the joint continuity of the multiplication. Finally, this also shows that $A \to A^{-1} = A^*$ is continuous in $\mathscr{G}$. Thus, $\mathscr{G}$ is a compact group, continuing $\{V_g: g \in G\}$ as a dense subgroup, and acts continuously as unitary transformations on H.

Now we apply the Peter–Weyl principle (as in Greenleaf and Moskowitz [11] or in Hewitt and Ross [13, p. 29]) to argue that H is a direct sum of $\mathscr{G}$-invariant, and hence G-invariant, finite-dimensional subspaces. □

Remark 1. Notice that the positive definite functions arising from the representation T on $L_2(X)_c$ (or of G on H in 1.2) are almost-periodic. Hence, the almost periodic compactification of a group as in Loomis [15] can be used to envelop the representation and prove 1.2 analogously to the above. Also, the uniqueness part of theorem 27.44 [13] applies to the decomposition and hence each finite-dimensional T-invariant T-irreducible subspace $A \subset L_2(X)_c$ has a multiplicity $\mu = \mu(H, T)$ associated with it. As in [13, p. 29], for $\sigma \in \hat{\mathscr{G}}$, let M_σ be the smallest closed subspace of $L_2(X)_c$ containing all such A with $\mathscr{G}$ equivalent to σ on A. This subspace will be used in the proof of 1.6.

Remark 2. Let $\mathcal{G}_c$ be the enveloping compact group as in the proof of 1.2 for $\{T(g): g \in G\}$ when $H = L_2(X)_c$. Let β_c be $\{A \in \beta: 1_A \in L_2(X)_c\}$. It is easy to show β_c is a σ-algebra too and the largest compact factor in β, see [**3**] and [**10**]. Also, $L_2(X)_c$ is naturally isometric to $L_2(X, \beta_c, m)$ since each $F \in L_2(X)_c$ is β_c-measurable. For any $A \in \mathcal{G}_c$, there exists $(g_i) \subset G$ such that for all $F_1, F_2 \in L_2(X)_c$, $\lim_i \langle T(g_i)F_1, F_2\rangle = \langle A(F_1), F_2\rangle$ and $\lim_i \langle T(g_i^{-1})F_1, F_2\rangle = \langle A^{-1}(F_1), F_2\rangle$. Hence, $(T(g_i): i)$ is Cauchy in the weak topology of $\mathcal{M}(X, \beta_c)$, cf. Halmos [**12**]. Thus, $\mathcal{G}_c$ as a group is identical to $\{T(g): g \in G_c\}$ where G_c is the closure of $T(G)$ in $\mathcal{M}(X, \beta_c)$ in the weak topology and T denotes the regular representation on $L_2(X)_c$. Moreover, since $\mathcal{G}_c$ is compact, this identification gives a topological isomorphism of $\mathcal{G}_c$ with the weak-operator topology into G_c with the weak topology of $\mathcal{M}(X, \beta_c)$. If the probability space (X, β, m) is not standard, this may only identify $\mathcal{G}_c$ with Boolean σ-isomorphisms in $\mathcal{M}(X, \beta_c)$. To have $\mathcal{G}_c$ realized completely as point transformations as in Mackey [**17**] requires some further separability hypotheses on G and/or (X, β, m).

1.3. *Definition.* Assume G has a countable dense subset. A *separable (measure-preserving) representation* of G in $\mathcal{M}(X)$ is a representation T of G in $\mathcal{M}(X)$ such that (X, β, m) is a non-atomic separable probability space.

1.4. PROPOSITION. *Suppose T is an ergodic separable representation of G in $\mathcal{M}(X)$ and let H be a finite-dimensional T-invariant T-irreducible subspace of $L_2(X)_c$. Then the multiplicity μ_H of H in $L_2(X)_c$ is at most* dim (H).

Proof. The hypothesis on G guarantees that there is a countable subgroup G_0 which is dense in G and so $\{T(g): g \in G_0\}$ is a countable dense subset of $\mathcal{G}_c$. The decomposition of $L_2(X)_c$ into finite-dimensional T-invariant T-irreducible subspaces is completely determined by $\mathcal{G}_c$, and hence, because T is continuous, by the representation T restricted to G_0. Because G_0 is countable, we may assume without affecting this decomposition, up to unitary equivalence, that (X, β, m) is a standard probability space. Hence, corollary 2 of Zimmer [**20**] applies and shows that there is a standard probability space (Y, Γ, p) and an action of G_0 on it as measure preserving transformation so that the action of G_0 on $L_2(X)_c$ is equivalent to the action on $L_2(Y, \Gamma, p)$. Now Mackey [**16**], theorem 1, applies to the G_0-space (Y, Γ, p). This shows that all μ_H are finite. Actually, an examination of the proof in [**16**] shows that the action of $\mathcal{G}_c$ on $L_2(X)_c$ is equivalent to the regular action of $\mathcal{G}_c$ on $L_2(\mathcal{G}_c/K)$ for a suitable closed subgroup K of $\mathcal{G}_c$. Hence, the Peter–Weyl Theorem for $\mathcal{G}_c$ shows that any finite-dimensional T-invariant T-irreducible subspace H of $L_2(X)_c$ has multiplicity no larger than dim (H). □

1.5. COROLLARY. *Suppose T is an ergodic separable representation of G in $\mathcal{M}(X)$. Then there is $F \in L_2(X)_c$ such that* span $\{T(g)F: g \in G\}$ *is norm dense in $L_2(X)_c$.*

Proof. Proposition 1.4 shows that the hypotheses of theorem 1.10 in Greenleaf and Moskowitz [**11**] are satisfied. This theorem gives exactly the above. □

Questions. (1) When does there also exist a cyclic vector for $L_2(X)_w$ or $L_2(X)$ for ergodic separable group actions? (2) If T is an ergodic separable action, when does there exist $A \in \beta$ such that the smallest σ-algebra containing $\{T(g)A: g \in G\}$ is β up to null sets?

Another general property of the finite-dimensional decomposition of $L_2(X)_c$ is the following.

1.6. PROPOSITION. *Let T be an ergodic separable representation of G in $\mathcal{M}(X)$. Then there is an orthogonal direct-sum decomposition of $L_2(X)_c$ into finite-dimensional T-invariant subspaces H_i such that each $H_i \subset L_\infty(X)$ and hence each H_i has an orthonormal basis formed by L_∞-functions.*

Proof. As in proof of theorem 1.2, there is a compact group $\mathcal{G}$ of unitary transformations of $L_2(X)_c$ such that $\{T(g): g \in G\}$ is dense in $\mathcal{G}$ in the weak-operator topology on $L_2(X)_c$. The proof of theorem 27.44 [13] first constructs a projection $P_\sigma: L_2(X)_c \to M_\sigma$ for each $\sigma \in \hat{\mathcal{G}}$. The definition of P_σ shows that for all $F_1, F_2 \in L_\infty(X) \cap L_2(X)_c$,

$$\left| \int P_\sigma(F_1) F_2 \, dm \right| \le d_\sigma^2 \|F_1\|_\infty \, \|F_2\|_1$$

since

$$\sup \{|X_{\bar\sigma}(g)|: g \in \mathcal{G}\} \le d_\sigma.$$

Since $L_\infty(X) \cap L_2(X)_c$, is dense in $L_1(X, \beta_c, m)$, this shows $\|P_\sigma(F_1)\|_\infty \le d_\sigma^2 \|F_1\|_\infty$. Also, P_σ maps $L_\infty(X) \cap L_2(X)_c$ onto a dense subspace of M_σ. Thus, $L_\infty(X) \cap M_\sigma$ is dense in M_σ for all $\sigma \in \hat{\mathcal{G}}$. Since M_σ is finite-dimensional by proposition 1.4, this is enough to prove this proposition. □

Remark. It may well be that this theorem is true without the assumption that T is a separable action of G in $\mathcal{M}(X)$. Indeed, the extreme case is where G is a compact group and T is the action by left multiplication on $(G, \beta_\lambda, \lambda)$ where λ is a left-invariant Haar measure on G. There $L_2(G)_c = L_2(G)$ and this does have a basis of $L_\infty(G)$ functions as above because the coefficient functions of finite-dimensional irreducible representations of G can be used in this role.

Yet another application of proposition 1.4 is this generalization of the well-known fact that if $T, S \in \mathcal{M}(X)$ and $TS = ST$, then T ergodic and S weakly mixing implies T is weakly mixing too.

1.7. PROPOSITION. *Suppose S is a weakly mixing representation of G and T is an ergodic separable representation of G in $\mathcal{M}(X)$. If S and T commute, then T is weakly mixing too.*

Proof. If T were not weakly mixing, then there would exist a finite-dimensional T-invariant subspace $H \subset L_2^0(G)_c$, the mean zero functions in $L_2(X)_c$. We may assume H is T-irreducible, For $g \in G$, $S(g)H$ is T irreducible and the representation of T on $S(g)H$ is equivalent to the representation T on H. By the property of M_σ where σ is T restricted to H, $S(g)H \subset M_\sigma$. Thus, M_σ is an S invariant subspace which is finite-dimensional by proposition 1.4. Hence, S cannot be weakly mixing.

There are other theorems that can be generalized along these lines. A particularly important one in relation to criteria used in § 2 is to determine when a tensor product

$T\otimes S$ is ergodic. As in [3], if S is weakly mixing and T is ergodic, then $T\otimes S$ is ergodic. More generally, in the abelian case, $T\otimes S$ is ergodic if T and S are ergodic and share no common eigenvalues other than 1. The following generalization of this fact holds.

1.8. PROPOSITION. *If T and S are ergodic representations, then $T\otimes S$ is ergodic if and only if T and S have disjoint spectra: there are no finite-dimensional subspaces H_1, $H_2\subset L_2^0(X)$ with H_1 T-invariant and H_2 S-invariant, such that $T|_{H_1}$ is equivalent to $S|_{H_2}$.*

Proof. Suppose $T|_{H_1}$ is equivalent to $S|_{H_2}$. Then the contragradient representation of $T|_{H_1}$ is realized as the complex conjugate $S|_{\bar H_2}$. Since the trivial representation is a subrepresentation of $T|_{H_1}\otimes S|_{\bar H_2}$, the trivial representation is a subrepresentation of $T\otimes S$.

On the other hand, suppose T and S do have disjoint spectra. Let M denote the unique G-invariant mean on WAP(G). Suppose $0\neq \mathcal{F}\in L_2^0(X\times X)$ is $T\otimes S$-invariant. Then

$$\left\langle M(g), \int_{X\times X} (T(g)\otimes S(g)\mathcal{F})\bar F\, dm\times dm\right\rangle = \|\mathcal{F}\|_2^2\neq 0.$$

But for all $\varepsilon>0$, there exist $c_i\in C$ and F_i, $G_i\in L_2(X)$, $i=1,\dots,n$ such that for each i, $\int F_i\, dm$ or $\int G_i\, dm=0$, and $\|\mathcal{F}-\sum_{i=1}^n c_iF_i\otimes G_i\|_2<\varepsilon$. If we show

$$\left\langle M(g), \int_{X\times X} (T(g)\otimes S(g)F_i\otimes G_i)\overline{F_j\otimes G_i}\, dm\times dm\right\rangle = 0$$

for all $i, j=1,\dots,n$, letting $\varepsilon\to 0$ this would show $\|\mathcal{F}\|_2^2=0$, a contradiction which would prove $T\otimes S$ is ergodic.

Suppose then F_1, F_2, G_1, $G_2\in L_2(X)$ and for $i=1,2$, $\int F_i\, dm$ or $\int G_i\, dm$ is 0. Let $L_2^T(X)_c\oplus L_2^T(X)_w$ and $L_2^S(X)_c\oplus L_2^S(X)_w$ be the orthogonal decompositions for S and T discussed earlier, and write $F_i=F_i^c+F_i^w$, $G_i=G_i^c+G_i^w$ for $i=1,2$ according to this decomposition. Since T is weakly mixing on $L_2^T(X)_w$ and S is weakly mixing on $L_2^S(X)_w$, we have

$$\left\langle M(g), \int_{X\times X} (T(g)\otimes S(g)F_1\otimes G_1)\overline{F_2\otimes G_2}\, dm\times dm\right\rangle$$
$$= \left\langle M(g), \int_{X\times X} (T(g)\otimes S(g)F_1^c\otimes G_2^c)\overline{F_2^c\otimes G_2^c}\, dm\times dm\right\rangle.$$

See [3], § 1, for further discussion. Hence, we may assume at the outset that $F_i\in L_2^T(X)_c$ and $G_i\in L_2^S(X)_c$ for $i=1,2$.

But now what we want to show is that $T_c\otimes S_c$ is ergodic where T_c is T restricted to the mean zero functions in $L_2^T(X)_c$ and S_c is S restricted to the mean zero functions in $L_2^S(X)_c$. For representations of a locally compact group that are direct sums of finite-dimensional representations, the trivial representation is a subrepresentation of $T_c\otimes S_c$ if and only if for some subrepresentation of T_c' of T_c, the contragradient representation $\check{T}_c'$ is a subrepresentation of S_c (up to unitary equivalence). Because the representations here are obtained from non-singular group actions on a measure space, $\check{T}_c'$ is just given by the complex conjugate $\bar{T}_c'$ of T_c'.

So if some S'_c is equivalent to $\check{T}'_c$, then $\bar{S}'_c$ is equivalent to T'_c. But $\bar{S}'_c$ is a subrepresentation of S_c, and so the disjointness of the spectra of S and T prevents this. Hence $T_c \otimes S_c$ is ergodic and so $T \otimes S$ is ergodic. □

In the above, if $T|_{H_1}$ is equivalent to $S|_{H_2}$ with H_1 having an orthonormal basis $F_1, \ldots, F_n \in L_2^0(X)$, and $V: H_1 \to H_2$ is a unitary transformation such that $VT(g) = S(g)V$ for all $g \in G$, then

$$\phi(\xi_1, \xi_2) = \sum_{i=1}^{n} F_i(\xi_i)\overline{VF_i}(\xi_2)$$

for $\xi_1, \xi_2 \in X$ is a non-zero mean zero $T \otimes S$-invariant function. This explicit proof of the first part of the above is observed in Moore [17]; it shows how the trivial representation is a subrepresentation of $T|_{H_1} \otimes S|_{\bar{H}_2}$.

The same argument as above can be used to prove a local version of this theorem when $T \otimes S$ is not ergodic. That is, let $F_1 \in L_2^T(X)_c$ and $F_2 \in L_2^S(X)_c$, both mean zero. Then

$$\left\langle M(g), \int_X T(g)F_1\overline{F_1}\,dm \int S(g)F_2\overline{F_2}\,dm \right\rangle = 0$$

if and only if the positive definite functions $f_i(g) = \int_X T(g)F_i\overline{F_i}\,dm$ give rise to representations of G with disjoint spectra.

2. *Averaging theorems*

The idea used in proving the major convergence theorem here is to use an abstract version of the van der Corput inequality, cf. [**1**]. To do this for amenable groups requires an approximate tiling lemma for averages in G over Følner sequences. Let $|\cdot|$ be a fixed right invariant Haar measure on G. For simplicity, let $d\lambda_G(g)$ be denoted dg.

In this section integrals of the form $\int_{K_m} T(g)F\,dg$ are needed. The explanation of the meaning of $\int_{K_m} T(g)F\,dg$ when G is not discrete and K_m is compact is that it represents the usual Pettis integral of $g \to T(g)F \in L_2(X)$. Here $p: g \to T(g)F$ is continuous since it is weakly continuous and so its range is separable by the σ-compactness of G. So the weak continuity of p shows that p is λ_G-measurable by the Pettis measurability theorem, see Dunford and Schwartz [**5**, III.6.11]. Thus, this integral can also be taken to be the Bochner integral. The following propositions are well-known.

2.1. PROPOSITION. *Let G be an amenable group and let (K_m) be a right Følner sequence of compact sets. Let $A: G \to H$ be a function where H is a Hilbert space. Assume $A(G)$ is bounded. Let*

$$S_1(m) = (1/|K_m|)\int_{K_m} A(g)\,dg$$

and

$$S_2(m, h) = (1/|K_m|)\int_{K_m} (1/|K_h|)\int_{K_h} A(gz)\,dz\,dg.$$

Then, for all h,

$$\lim_{m\to\infty} \|S_1(m)-S_2(m,h)\|=0.$$

In the sequel (K_m) will be a Følner sequence chosen as above.

2.2. PROPOSITION. *If $T:G\to\mathcal{M}(X)$ is an ergodic representation of G, and*

$$A_m(F)=(1/|K_m|)\int_{K_m} T(g)F\,dg, \qquad \text{then for all } F\in L_p(X),\ 1\le p<\infty,$$

$$\lim_{m\to\infty}\left\|A_mF-\int F\,dm\right\|_p=0.$$

Now suppose $T_1,\ldots,T_n$ are commuting representations of G in $\mathcal{M}(X)$ and consider products $S_j(g)=T_1(g)\ldots T_j(g)$, $j=1,\ldots,n$. For $F_1,\ldots,F_n\in L_\infty(X)$, the product $S_1(g)F_1S_2(g)F_2\ldots S_n(g)F_n$ is a well-defined element of $L_\infty(X)$. It will be important that S_j is a representation of G, too, so it is necessary to assume that the T_j are commuting in the sense that $T_i(g)T_j(h)=T_j(h)T_i(g)$ for $i\ne j$ and $g,h\in G$. It might seem that the theorems to follow could be phrased and proved for products $S_1F_1\ldots S_nF_n$ without knowing each S_i is itself a product. However, we will need to use the fact that $S_j^{-1}S_k$, $j<k$, is a representation of G; and generally, if S_1, S_2 are representations of G such that $S_1^{-1}S_2$ is a representation of G, then $S_2=S_1T$ for some representation T of G commuting with S_1. For these reasons, we generally assume S_i are formed as above from commuting representations T_i. Either notation is appropriate in the case of abelian groups G.

2.3. *Definition.* We say $(T_1,\ldots,T_n)$ is *mutually ergodic* if $T_1\otimes T_1T_2\otimes\cdots\otimes T_1\ldots T_n$ is ergodic. We say $(T_1,\ldots,T_n)$ is *fully mutually ergodic* if $(T_j,\ldots,T_n)$ is mutually ergodic for all $j=1,\ldots,n$. This type of joint ergodicity of $T_1,\ldots,T_n$ should be contrasted with the one in [1].

2.4. THEOREM. *Let $T_1,\ldots,T_n$ be commuting fully mutually ergodic representations of an amenable group G in $\mathcal{M}(X)$. Let $S_i=T_i\cdots T_i$ for $i=1,\ldots,n$. Let (K_m) be a right Følner sequence in G and let $F_1,\ldots,F_n\in L_\infty(X)$. Then*

$$\lim_{m\to\infty}\left\|1/|K_m|\int_{K_m} S_1(g)F_1\cdots S_n(g)F_n\,dg-\int F_1\,dm\cdots\int F_n\,dm\right\|_{L_2(X)}=0. \quad (5)$$

Proof. The proof is by induction on n. If $n=1$, we are assuming T_1 is ergodic, and proposition 2.2 proves the theorem. Assume the theorem has been proved for a fully mutually ergodic system $(T_1,\ldots,T_{n-1})$. Assume $T_1,\ldots,T_n$ is fully mutually ergodic. Without loss of generality, $\int F_1\,dm=0$. Indeed, suppose the theorem is proved in case $\int F_1\,dm=0$. Then let $E(F_1)=\int F_1\,dm$. We have

$$\begin{aligned}&S_1(g)(F_1-E(F_1))S_2(g)F_2\cdots S_n(g)F_n\\ &\quad=S_1(g)F_1S_2(g)F_2\cdots S_n(g)F_n-E(F_1)(S_2(g)F_2\cdots S_n(g)F_n). \qquad (6)\end{aligned}$$

Since $(T_1,\ldots,T_n)$ is mutually ergodic, $(T_1T_2,T_3,\ldots,T_n)$ is mutually ergodic. By induction,

$$(1/|K_m|)\int_{K_m} S_2(g)F_n\cdots S_n(g)F_n\,dg\to\prod_{i=2}^{m}\int F_i\,dm$$

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as $m \to \infty$ in $L_2(X)$-norm. Thus,

$$\lim_{m\to\infty} (1/|K_m|) \int_{K_m} E(F_1)S_2(g)F_2 \cdots S_n(g)F_n\, dg = \prod_{i=1}^{m} \int F_i\, dm$$

in $L_2(X)$-norm, too. Hence, (6) shows the theorem follows from the case $E(F_1) = 0$.

We assume $\int F_1\, dm = 0$ and show

$$\lim_{m\to\infty} (1/|K_m|) \int_{K_m} S_1(g)F_1S_2(g)F_2 \cdots S_n(g)F_n\, dg = 0$$

in $L_2(X)$-norm. If we let

$$A(g) = S_1(g)F_1S_2(g)F_2 \cdots S_n(g)F_n,$$
$$H = L_2(X)$$

in proposition 2.1, we see that it suffices to show that for all $\varepsilon > 0$, there is an $h \geq 1$ such that for some $M \geq 1$ if $m \geq M$, then

$$\mathcal{N} = \left\|(1/|K_m|) \int_{K_m} (1/|K_h|) \int_{K_h} S_1(gz)F_1 \cdots S_n(gz)F_n\, dz\, dg\right\|_2^2 < \varepsilon.$$

Now

$$\mathcal{N} \leq (1/|K_m|) \int_{K_m} \left\|(1/|K_h|) \int_{K_h} S_1(gz)F_1 \cdots S_n(gz)F_n\, dz\right\|_2^2 dg.$$

Indeed, if $A: G \to L_2(X)$ is bounded and weakly measurable, then for $K \subset G$, K compact

$$\left\| \int_K A(g)\, dg \right\|_2^2 \leq \left(\int_K \|A(g)\|_2\, dg \right)^2$$
$$\leqq \left[\left(\int_K \|A(g)\|_2^2\, dg \right)^{1/2} \left(\int_K 1\, dg \right)^{1/2} \right]^2 = |K| \int_K \|A(g)\|_2^2\, dg.$$

But we have,

$$\left\|(1/|K_h|) \int_{K_h} S_1(gz)F_1 \cdots S_n(gz)F_n\, dz\right\|_2^2$$
$$= (1/|K_h|^2) \int_{K_h} \int_{K_h} \left(\int_X S_1(gs)F_1 \cdots S_n(gs)F_nS_1(gt)\bar{F}_1 \cdots S_n(gt)\bar{F}_n\, dm \right) ds\, dt$$
$$= (1/|K_h|^2) \int_{K_h} \int_{K_h} \left(\int_X T_1(s)F_1(T_1(s)T_2(gs)F_2) \cdots (T_1(s)T_2(gs) \cdots T_n(gs)F_n) \right.$$
$$\left. \times T_1(t)\bar{F}_1(T_1(t)T_2(gt)\bar{F}_2) \cdots (T_1(t)T_2(gt) \cdots T_n(gt)\bar{F}_n)\, dm \right) ds\, dt$$
$$= (1/|K_h|^2) \int_{K_h} \int_{K_h} \left(\int_X (T_1(s)F_1)T_2(g)[T_1(s)T_2(s)F_2] \cdots T_2(g)T_3(g) \cdots \right.$$
$$\times (T_n(g)[T_1(s)T_2(s) \cdots T_n(s)F_n](T_1(t)\bar{F}_1)T_2(g)[T_1(t)T_2(t)\bar{F}_2] \cdots$$
$$\left. \times T_2(g)T_3(g) \cdots T_n(g)[T_1(t) \cdots T_n(t)\bar{F}_n]\, dm \right) ds\, dt$$
$$= (1/|K_h|^2) \int_{K_h} \int_{K_h} \left(\int_X [S_1(s)F_1S_1(t)\bar{F}_1](T_2(g)[S_2(s)F_2S_2(t)\bar{F}_2]) \cdots \right.$$
$$\left. \times (T_2(g) \cdots T_n(g)[S_n(s)F_nS_n(t)\bar{F}_n]\, dm \right) ds\, dt.$$

Hence, the induction hypothesis applied to the fully mutually ergodic system $(T_2, \ldots, T_n)$ shows that for all $h \geq 1$,

$$\lim_{m\to\infty} (1/|K_m|) \int_{K_m} \left\| \left(1/|K_h| \int_{K_h} S_1(gz)F_1 \cdots S_n(gz)F_n \, dz\right\|_2^2 dg$$

$$= (1/|K_h|^2) \int_{K_h} \int_{K_h} \int_X S_1(s)F_1 S_1(t)\bar{F}_1 \, dm \int_X S_2(s)F_2 \cdot$$

$$\times S_2(t)\bar{F}_2 \, dm \cdots \int_X S_n(s)F_n S_n(t)\bar{F}_n \, dm \Big) \, ds \, dt$$

$$= \left\| 1/|K_h| \int_{K_h} S_1(g)\otimes\cdots\otimes S_n(g)(F_1\otimes\cdots\otimes F_n)\right\|_{L_2(X^n)}^2.$$

But $S_1\otimes\cdots\otimes S_n$ is ergodic and so for any $\varepsilon > 0$, there is an $h > 1$ such that this last norm is no larger than ε because

$$\int_X \cdots \int_X F_1\otimes\cdots\otimes F_n \, dm\cdots dm = \prod_{l=1}^n \int_X F_l \, dm = 0.$$

For this h, there is an $M \geq 1$ such that for all $m \geq M$, $\mathcal{N} \leq 2\varepsilon$. □

2.5. *Examples.* (a) Suppose G and X are separable as in definition 1.3. Then assume $T_1, \ldots, T_n$ are commuting representations of G such that T_1 is weakly mixing and each T_i is ergodic. By proposition 1.7, each T_i is weakly mixing. Thus, if also each S_i is ergodic (and hence weakly mixing by the same argument), then $S_1\otimes\cdots\otimes S_n$ is weakly mixing. It follows that $(T_1, \ldots, T_n)$ is fully mutually ergodic. Indeed, without the separability assumption, if each T_i and S_i is weakly mixing, then $(T_1, \ldots, T_n)$ is fully mutually ergodic. This situation is the generalization of a weakly mixing system of commuting transformations as in [**9**]. But as in [**1**] where the exact hypotheses for (3) were given, the hypotheses for theorem 2.4 above are less than assuming that all T_i and S_i are weakly mixing. In fact, theorem 2.6 will show that the hypotheses of theorem 2.4 are essentially necessary, thus generalizing the work in [**1**].

(b) Let $H = \bigoplus_{i=1}^n G$, with the product topology. Any representation $T: H \to L$ on a Hilbert space L gives commuting representations $T_i: G \to L$ defined by $T_i(g) = T(e_i(g))$ where $e_i: G \to H$ is the ith-coordinate injection. If T is strongly mixing, then so is each T_i. The proof of theorem 2.5 in Bergelson and Rosenblatt [**3**] shows, for amenable groups G among others, that in the weak topology, the representations T such that each T_i and S_i is weakly mixing from a residual subset of all the representations of H on L. So, at the unitary level, there are many representations T of H such that $(T_1, \ldots, T_n)$ is fully mutually ergodic as representations of G as unitary operators on L. Using the Gaussian measure space construction, see Neveu [**18**] or Schmidt [**19**], each such T and system $(T_1, \ldots, T_n)$ gives a representation $T: H \to \mathcal{M}(X)$ such that $(T_1, \ldots, T_n)$ is fully mutually ergodic as point transformations. Hence, for any group G, there are examples of systems $(T_1, \ldots, T_n)$ where the hypotheses theorem 2.4 hold. If G is discrete, it is easier to give an example. Let $X = \prod_H [0, 1]$ and let H act on X by permutation of the coordinate indices.

Then in the product probability measure given by Lebesgue measure in [0, 1], each T_i and S_i is strongly mixing.

(c) If G is abelian, then the existence of eigenvalues and orthonormal bases of eigenvectors for a representation T in $\mathcal{M}(X)$ restricted to $L_2(X)_c$ allows a simplification of the hypotheses of theorem 2.4. It is not too hard to show that if $T_1, \ldots, T_n$ are ergodic and $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ is ergodic, then $T_2 \otimes T_2 T_3 \otimes \cdots \otimes T_2 \ldots T_n$ is ergodic. So, if $T_1, \ldots, T_n$ and $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ are ergodic, then $(T_1, \ldots, T_n)$ is fully mutually ergodic. Thus, by proposition 1.8, when G is abelian, $(T_1, \ldots, T_n)$ is fully mutually ergodic if and only if just the constants have the eigenvalue 1 for T_i and if Λ_i denotes the set of eigenvalues of S_i, then for $\lambda_i \in \Lambda_i$, $1 = 1, \ldots, n$, $\lambda_1 \lambda_2 \ldots \lambda_n = 1$ if and only if $\lambda_i = 1$ for all i. Of course, this includes the case where $\Lambda_i = \varnothing$ for all i, i.e. each T_i is weakly mixing. Is there a simplification of the hypotheses of theorem 2.5 along these lines for general groups?

One of the interesting aspects of the hypotheses of theorem 2.4 is that they are necessary given $T_1, \ldots, T_n$ commute.

2.6. THEOREM. *Let $T_1, \ldots, T_n$ be commuting, separable representations of* an amenable *group G in $\mathcal{M}(X)$ and fix a right Følner sequence (K_m). Suppose that for all $F_0, \ldots, F_n \in L_\infty(X)$;*

$$\lim_{m\to\infty} \int_X \left(1/|K_m| \int_{K_m} F_0 S_1(g) F_1 \ldots S_n(g) F_n \, dg \right) dm = \prod_{i=0}^{n} \int_X F_i \, dm. \qquad (7)$$

Then $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ is ergodic. Also, $(T_1, \ldots, T_n)$ *is fully mutually ergodic.*

Proof. Only the ergodicity of $S_1 \otimes \cdots \otimes S_n$ is needed to get the rest. Indeed, let $F_1 = 1$. Then

$$\int_X \left(1/|K_m| \int_{K_m} S_1(g) F_1 \cdots S_n(g) F_n \, dg \right) dm$$

$$= (1/|K_m|) \int_{K_m} \left(\int_X S_2(g) F_2 \cdots S_n(g) F_n \, dm \right) dx$$

$$= (1/|K_m|) \int_{K_m} \left(\int_X (T_2(g) F_2) \cdots (T_2(g) \ldots T_n(g) F_n \, dm \right) dg$$

$$= \int_X \left(1/|K_m| \int_{K_m} \mathcal{S}_1(g) F_2 \ldots \mathcal{S}_{n-1}(g) F_n \, dg \right) dm,$$

where $\mathcal{S}_i = T_2, \ldots, T_{i+1}$, $i = 1, \ldots, n-1$. Hence, (7) holds for $(\mathcal{S}_1, \ldots, \mathcal{S}_{n-1})$. So, by induction, we would get $(T_1, \ldots, T_n)$ being fully mutually ergodic. Note also that (7) entails the ergodicity of each T_i and S_i. For example, fix $i = 1, \ldots, n$ and let $F_j = 1$ for $j \in \{i-1, i\}$. Then (7) implies

$$\lim_{m\to\infty} (1/|K_m|) \int_X \left(\int_{K_m} F_{i-1} T_i(g) F_i(g) \, dm \right) = \int F_{i-1} \, dm \int F_i \, dm.$$

Hence, if $F \in L_\infty(X)$ is T_i-invariant, then

$$\int_X |F_i|^2 \, dm = \left| \int_X F_i \, dm \right|^2$$

by taking $\overline{F_{i-1}} = F_i = F$. So, if $A \in \beta$ is T_i-invariant, then $m(A) = m(A)^2$ and $m(A)$ is 0 or 1.

Let $L_2(X) = L_c^i \oplus L_w^i$ where L_c^i is the compact summand for S_i and L_w^i is the weakly mixing summand for S_i. We identify $L_2(X^n)$ with $\otimes_{i=1}^n L_2(X)$ in the usual fashion. On an invariant subspace of the form $L_2(X) \otimes \cdots \otimes L_w^i \otimes \cdots \otimes L_2(X)$, $S_1 \otimes \cdots \otimes S_n$ is ergodic because S_i is weakly mixing on L_i^w. So, if $S_1 \otimes \cdots \otimes S_n$ has an invariant function, it must be in $L_c^1 \otimes \cdots \otimes L_c^n$. But the existence of an S_i-invariant finite-dimensional orthogonal direct sum decomposition of L_c^i with an L_∞ basis on the summands is guaranteed by proposition 1.6. Hence, if $S_1 \otimes \cdots \otimes S_n$ is not ergodic, then there exist S_i-invariant finite-dimensional subspaces H_i of $L_c^i \cap L_\infty(X)$, with some $H_i \subset L_2^0(X)$, such that some $0 \neq F \in H_1 \otimes \cdots \otimes H_n \subset L_2^0(X)$ is invariant under $S_1 \otimes \cdots \otimes S_n$. Let $(f_k^i\colon k = 1, \ldots, \dim(H_i))$ be orthonormal bases in $L_\infty(X)$ for each H_i. Then F has unique expansion as a finite sum

$$F(\xi_1, \ldots, \xi_n) = \sum_{(m_1, \ldots, m_n)} a(m_1, \ldots, m_n) f_{m_1}^1(\xi_1) \cdots f_{m_n}^n(\xi_n)$$

for a.e. $(\xi_1, \ldots, \xi_n) \in X^n$.

For each $g \in G$, $S_i(g)|_{H_i}$ has a matrix expansion $S_i(g) = [\alpha_{mn}^i]$ in the basis $(f_m^i : m)$ with α_{mn}^i depending on g. Since $(S_1 \otimes \cdots \otimes S_n)F = F$, for all $g \in G$,

$$F = \sum_{(k_1, \ldots, k_n)} a(k_1, \ldots, k_n) \Big(\sum_{l_1} \alpha_{l_1 k_1}^1 f_{l_1}^1 \Big) \cdots \Big(\sum_{l_n} \alpha_{l_n k_n}^n f_{l_n}^n \Big)$$

$$= \sum_{(k_1, \ldots, k_n)} \sum_{(l_1, \ldots, l_n)} a(k_1, \ldots, k_n) \alpha_{l_1 k_1}^1 \cdots \alpha_{l_n k_n}^n f_{l_1}^1 \cdots f_{l_n}^n.$$

Hence, for all $(l_1, \ldots, l_n)$,

$$a(l_1, \ldots, l_n) = \sum_{(k_1, \ldots, k_n)} a(k_1, \ldots, k_n) \alpha_{l_1 k_1}^1 \cdots \alpha_{l_n k_n}^n.$$

But then for a.e. $\xi \in X$, and all $g \in G$,

$$\sum_{k_1, \ldots, k_n)} a(k_1, \ldots, k_n) S_1(g) f_{k_1}^1(\xi) \ldots S_n(g) f_{k_n}^n(\xi)$$

$$= \sum_{(k_1, \ldots, k_n)} a(k_1, \ldots, k_n) \Big(\sum_{l_1} a_{l_1 k_1}^1 f_{l_1}^1(\xi) \Big) \cdots \Big(\sum_{l_n} \alpha_{l_n k_n}^n f_{l_n}^n(\xi) \Big)$$

$$= \sum_{(k_1, \ldots, k_n)} \sum_{(l_1, \ldots, l_n)} a(k_1, \ldots, k_n) \alpha_{l_1 k_1}^1 \cdots \alpha_{l_n k_n}^n f_{l_1}^1(\xi) \ldots f_{l_n}^n(\xi)$$

$$= \sum_{(l_1, \ldots, l_n)} a(l_1, \ldots, l_n) f_{l_n}^1(\xi) \cdots f_{l_n}^n(\xi).$$

Fix $F_0 \in L_2(X)$. We have

$$\int_X F_0(\xi) \Big(\sum_{(l_1, \ldots, l_n)} a(l_1, \ldots, l_n) f_{l_1}^1(\xi) \cdots f_{l_n}^n(\xi) \Big) \, dm(\xi)$$

$$= (1/|K_m|) \int_{K_m} \int F_0(\xi) \cdot \sum_{(l_1, \ldots, l_n)} a(l_1, \ldots, l_n) S_1(g) f_{l_1}^1(\xi) \cdots S_n(g) f_{l_n}^n(\xi) \, dm(\xi) \, dg$$

$$= \sum_{(l_1, \ldots, l_n)} a(l_1, \ldots, l_n) \Big(\int (1/|K_m|) \Big) \int_{K_m} F_0(\xi) S_1(g) f_{l_1}^1(\xi) \cdots S_n(g) f_{l_n}^n(\xi) \, dg \, dm(\xi)$$

By (7), letting $m\to\infty$, this shows for all $F_0\in L_\infty(X)$,

$$\int F_0\Big(\sum_{(l_1,\ldots,l_n)} a(l_1,\ldots,l_n)f^1_{l_1}\cdots f^n_{l_n}\Big)\,dm=0$$

because for some i, all f^i_m are mean zero. That is,

$$\sum_{(l_1,\ldots,l_n)} a(l_1,\ldots,l_n)f^1_{l_1}(\xi)\cdots f^n_{l_n}(\xi)=0$$

a.e. ξ $[m]$.

Now, if

$$g_1,\ldots,g_{n-1}\in G,\ T_2(g_1)\otimes T_3(g_2)\otimes\cdots\otimes T_n(g_{n-1})\otimes I$$

commutes with $S_1\otimes\cdots\otimes S_n$. So, $(T_2(g_1)\otimes\cdots\otimes T_n(g_{n-1})\otimes I)F$ is $S_1\otimes\cdots\otimes S_n$ invariant, too. Moreover, $T_{i+1}(g_i)H_i$, $i=1,\ldots,n-1$, is S_i-invariant with orthonormal basis $(T_{i+1}(g_i)f^i_m\colon m)$. So, the same argument as the one above shows that for all $g_1,\ldots,g_{n-1}\in G$,

$$0=\sum_{(l_1,\ldots,l_n)} a(l_1,\ldots,l_n)T_2(g_1)f^1_{l_1}(\xi)\cdots T_n(g_{n-1})f^{n-1}_{l_{n-1}}(\xi)f^n_{l_n}(\xi) \tag{8}$$

a.e. ξ $[m]$. But each T_i is ergodic, and so

$$\lim_{m\to\infty} 1/|K_m|\int_{K_m} T_i(g)F\,dx=\int F\,dm$$

in $L_2(X)$-norm. Hence, using (8),

$$\begin{aligned}0&=\lim_{m\to\infty}(1/|K_m|^{n-1})\int_{K_m}\cdots\int_{K_m}\\&\quad\times\int\sum_{(l_1,\ldots,l_n)} a(l_1,\ldots,l_n)T_2(g_1)f^1_{l_1}(\xi)\cdots T_n(g_{n-1})f^{n-1}_{l_{n-1}}(\xi)f^n_{l_n}(\xi)\\&\quad\times\sum_{(k_1,\ldots,k_n)} a(\overline{l_1,\ldots,l_n})T_2(g_1)\overline{f^1_{l_1}(\xi)}\cdots T_n(g_{n-1})\overline{f^{n-1}_{l_{n-1}}}(\xi)\overline{f^n_{l_n}}(\xi)\,dm(\xi)\,dg_1\cdots dg_{n-1}\\&=\sum_{(l_1,\ldots,l_n)}\sum_{(k_1,\ldots,k_n)} a(l_1,\ldots,l_n)\overline{a(k_1,\ldots,k_n)}E(f^1_{l_1}\overline{f^1_{k_1}})\cdots E(f^n_{l_n}\overline{f^n_{k_n}}).\end{aligned}$$

By the orthonormality of (f^j_m), this gives

$$0=\sum_{(l_1,\ldots,l_n)}|a(l_1,\ldots,l_n)|^2.$$

So all $a(l_1,\ldots,l_n)=0$ and $F=0$, a contradiction. □

Remark. Theorems 2.4 and 2.6 show that the seemingly weaker joint ergodicity of (7) is equivalent to (5) for commuting separable actions $T_1,\ldots,T_n$. Also, the proof could have been shorter if we knew that the bases $(f^j_m\colon m)$ could have been chosen to be *generic*, i.e. not only are $(f^1_{m_1}(\xi_1)\cdots f^n_{m_n}(\xi_n)\colon(m_1,\ldots,m_n))$ orthonormal in $L_2(X^n)$, but $(f^1_{m_1}(\xi)\cdots f^n_{m_n}(\xi)\colon(m_1,\ldots,m_n))$ are linearly independent. If G is abelian, this could be arranged, but it is not clear if it is always possible. See **[8]** for other uses of genericity.

There is another possible definition of joint ergodicity that one might use as a generalization of (3). For instance, we might assume for all $F_0,\ldots,F_n\in L_\infty(X)$ that

$$\lim_{m\to\infty}(1/|K_m|)\int_{K_m}\left|\int_X F_0S_1(g)F_1\ldots S_n(g)F_n-\prod_{i=0}^n\int_X F_i\,dm\right|^2dg=0. \tag{9}$$

This property is related to (5) and (7) as in [**10**, p. 534]. However, taking all $F_i = 1$, $i \geq 1$, except for F_j, shows that property (9) forces S_j to be weakly mixing and so $S_1 \otimes \cdots \otimes S_n$ is ergodic, too. Thus, if one wishes to get a joint ergodicity result that does not entail all representations being weakly mixing, then (5) or (7) are better forms to study.

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