J. Appl. Prob. **42,** 163–174 (2005)
Printed in Israel

ON ULTIMATE RUIN IN A DELAYED-CLAIMS RISK MODEL

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Abstract

In this paper, we consider a risk model in which each main claim induces a delayed claim called a by-claim. The time of delay for the occurrence of a by-claim is assumed to be exponentially distributed. From martingale theory, an expression for the ultimate ruin probability can be derived using the Lundberg exponent of the associated nondelayed risk model. It can be shown that the Lundberg exponent of the proposed risk model is the same as that of the nondelayed one. Brownian motion approximations for ruin probabilities are also discussed.

Keywords: Brownian motion; by-claim; Lundberg exponent; main claim; martingale; ultimate ruin probability; weak convergence

2000 Mathematics Subject Classification: Primary 60J65; 62P05
Secondary 60H05; 60K99; 90A09

1. Introduction

In reality, insurance claims may be delayed due to various reasons. Risk models with this special feature have been discussed in the literature for some years. Waters and Papatriandafylou (1985) considered a discrete-time risk model allowing for delay in claims settlements and used martingale techniques to derive upper bounds for ruin probabilities. Boogaert and Haezendonck (1989) studied the mathematical properties of a liability process with settling delay within the framework of an economics environment.

In this paper, we assume that each main claim induces another type of claim, called a by-claim. The two types of claim have different distributions of severity. The time of occurrence of a by-claim is later than that of its main claim, and the time of delay for a by-claim is random. This kind of risk modeling may be of practical use. For instance, a serious motor accident causes different kinds of claim, such as car damage, injury, and death; some can be dealt with immediately while others need a random period of time to be settled.

The paper is organized as follows. In Section 2, we give a detailed description of the model. In Section 3, we use the martingale method to derive an expression for the ultimate ruin probability, which involves a nondelayed surplus process as well as its Lundberg exponent. In Section 4, we show that the Lundberg exponent of the proposed model with delayed claims is the same as that of the associated nondelayed risk model. Finally, in Section 5, we investigate the

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weak convergence of the aggregate claims process and derive Brownian motion approximations for ruin probabilities.

2. The model

Yuen and Guo (2001) used the method of probability-generating functions to derive ruin probabilities for the compound binomial model with discrete delay time for by-claims. In this paper, we consider a similar problem in the continuous-time setting. Specifically, we study various aspects of the compound Poisson model with delayed claims.

Let the aggregate main claims process be a compound Poisson process and let $N(t)$ be the corresponding Poisson claim number process, with intensity λ. Its jump times are denoted by $\{T_i, i = 1, 2, \ldots\}$ with $T_0 = 0$. The main claim amount random variables $\{X_i, i = 1, 2, \ldots\}$ follow a common distribution F with mean μ_F and variance σ_F^2. In our model, if X_i occurs at T_i, it will generate a by-claim, denoted by Y_i, occurring at time $T_i + W_i$, where W_i is the random time of delay for Y_i. The by-claim amount random variables $\{Y_i, i = 1, 2, \ldots\}$ have a common distribution G with mean μ_G and variance σ_G^2, while the delay times $\{W_i, i = 1, 2, \ldots\}$ are exponentially distributed with mean λ_1^{-1}. It is assumed that the X_i, Y_i, T_i, and W_i are independent. All the random variables and random processes considered in this paper are defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

In this setup, the surplus process takes on the form

$$S(t) = u + ct - D(t), \tag{2.1}$$

where u is the initial surplus, c is the rate of premium,

$$D(t) = \sum_{i=1}^{N(t)} X_i + \sum_{i=1}^{\infty} Y_i\, \mathbf{1}(T_i + W_i \le t),$$

and $\mathbf{1}(\cdot)$ is the indicator function of the event $\{\cdot\}$. The associated by-claim number process is given by

$$\bar{N}(t) = \sum_{i=1}^{\infty} \mathbf{1}(T_i + W_i \le t).$$

It is obvious that the incorporation of the aggregate by-claims process complicates the analysis of surplus process (2.1).

Write the distribution of T_i as H_{T_i}. The expectation of the aggregate by-claims at time t is given by

$$\begin{aligned}
\mathrm{E}\bigg[\sum_{i=1}^{\infty} Y_i\, \mathbf{1}(T_i + W_i \le t)\bigg] &= \sum_{i=1}^{\infty} \mu_G\, \mathrm{P}(T_i + W_i \le t) \\
&= \mu_G \int_0^t \sum_{i=1}^{\infty} H_{T_i}(t-x)\lambda_1 \mathrm{e}^{-\lambda_1 x}\, \mathrm{d}x \\
&= \mu_G \int_0^t \lambda(t-x)\lambda_1 \mathrm{e}^{-\lambda_1 x}\, \mathrm{d}x \\
&= \lambda\mu_G\bigg(t - \frac{1}{\lambda_1} + \frac{1}{\lambda_1}\mathrm{e}^{-\lambda_1 t}\bigg).
\end{aligned} \tag{2.2}$$

Hence, to satisfy the positive safety loading condition, we may simply assume that $c > \lambda(\mu_F + \mu_G)$. Let $T = \inf\{t : S(t) < 0\}$ be the time of ruin for surplus process (2.1). Then, the ultimate ruin probability for (2.1) is given by $\psi(u) = \mathrm{P}(T < \infty)$.

3. A martingale approach for the ultimate ruin probability

In this section, we use martingale techniques to derive an expression for the ultimate ruin probability for $S(t)$. Specifically, the ruin probability can be expressed in terms of a slightly modified surplus process and its Lundberg exponent.

Define a modification of $S(t)$ by

$$S^*(t) = u + ct - D^*(t), \tag{3.1}$$

where $D^*(t) = \sum_{i=1}^{N(t)}(X_i + Y_i)$. Note that (3.1) is a compound Poisson process (a nondelayed risk model) with $X_i + Y_i$ being the claim amount for the ith claim. Let $\mathcal{F}_t = \sigma\{S(u), u \le t\}$ and $\mathcal{F}_t^* = \sigma\{S^*(u), u \le t\}$ be the natural filtrations of $S(t)$ and $S^*(t)$, respectively. The Lundberg exponent R^* for $S^*(t)$ is defined to be the positive solution of $\mathrm{E}[\mathrm{e}^{r(X+Y)}] = 1 + \lambda^{-1}cr$. From classical ruin theory, we know that

$$M^*(t) = \mathrm{e}^{-R^* S^*(t)}$$

is a martingale relative to the filtration $\mathcal{F}_t^*$. Now, consider two other filtrations given by

$$\mathcal{G}_t^* = \mathcal{F}_t^{*X} \vee \mathcal{F}_t^{*Y} \quad \text{and} \quad \mathcal{G}_t = \mathcal{G}_t^* \vee \sigma\{W_1, W_2, \dots\},$$

where $\mathcal{F}_t^{*X}$ is the natural filtration of $\sum_{i=1}^{N(t)} X_i$ and $\mathcal{F}_t^{*Y}$ is the natural filtration of $\sum_{i=1}^{N(t)} Y_i$. It is obvious that $\mathcal{G}_t^* \subset \mathcal{G}_t$ and that $M^*(t)$ is a $\mathcal{G}_t^*$ martingale. Moreover, we make the following proposition.

Proposition 3.1. *For the time of ruin T and the filtrations $\mathcal{F}_t$, $\mathcal{F}_t^*$, $\mathcal{G}_t$, $\mathcal{G}_t^*$, and $M^*(t)$, the following statements hold.*

(a) (i) $\mathcal{F}_t^* \subset \mathcal{G}_t^*$ and (ii) $\mathcal{F}_t \subset \mathcal{G}_t$.

(b) $M^*(t)$ is a martingale relative to filtration $\mathcal{G}_t$.

(c) T is a stopping time relative to $\mathcal{F}_t$ and $\mathcal{G}_t$.

Proof. Since $D^*(t) = \sum_{i=1}^{N(t)} X_i + \sum_{i=1}^{N(t)} Y_i$, part (a)(i) follows immediately. Write

$$W_i^{(k)} = \sum_{j=0}^{\infty} \frac{j+1}{k} \mathbf{1}\left(\frac{j}{k} < W_i \le \frac{j+1}{k}\right)$$

and

$$D_Y^{(k)}(t) = \sum_{i=1}^{N(t)} Y_i \mathbf{1}(T_i + W_i^{(k)} \le t).$$

Then, we have

$$\lim_{k\to\infty} W_i^{(k)} = W_i \quad \text{and} \quad \lim_{k\to\infty} D_Y^{(k)}(t) = \sum_{i=1}^{N(t)} Y_i \mathbf{1}(T_i + W_i \le t).$$

Thus, to prove part (a)(ii) it suffices to show that $D_Y^{(k)}(t)$ is $\mathcal{G}_t$ measurable. It can easily be shown that

$$\begin{aligned}\mathbf{1}(N(t)=n)D_Y^{(k)}(t)\\ &= \sum\biggl(\mathbf{1}(N(t)=n)\biggl(\prod_{i=1}^{n}\mathbf{1}\biggl(\frac{j_i}{k}<W_i\le\frac{j_i+1}{k}\biggr)\biggr)\sum_{i=1}^{n}Y_i\,\mathbf{1}\biggl(T_i\le t-\frac{j_i+1}{k}\biggr)\biggr),\end{aligned} \tag{3.2}$$

where the first summation on the right-hand side is over all $j_i = 0, 1, \dots$ for $i = 1, \dots, n$. Notice that

$$\begin{aligned}&\mathbf{1}(N(t)=n)\biggl(\prod_{i=1}^{n}\mathbf{1}\biggl(\frac{j_i}{k}<W_i\le\frac{j_i+1}{k}\biggr)\biggr)\sum_{i=1}^{n}Y_i\,\mathbf{1}\biggl(T_i\le t-\frac{j_i+1}{k}\biggr)\\ &\quad= \mathbf{1}(N(t)=n)\biggl(\prod_{i=1}^{n}\mathbf{1}\biggl(\frac{j_i}{k}<W_i\le\frac{j_i+1}{k}\biggr)\biggr)\sum_{i=1}^{n}Y_i\,\mathbf{1}\biggl(N\biggl(t-\frac{j_i+1}{k}\biggr)\ge i\biggr).\end{aligned} \tag{3.3}$$

Equalities (3.2) and (3.3) imply that $\mathbf{1}(N(t)=n)D_Y^{(k)}(t)$ is $\mathcal{G}_t$ measurable, because the product term on both sides of (3.3) is $\sigma\{W_1, W_2, \dots\}$ measurable.

For part (b), we must prove that $\mathrm{E}[M^*(t)\mid\mathcal{G}_s] = M^*(s)$ almost surely for $t \ge s$, i.e. that, for each $G\in\mathcal{G}_s$, $\int_G M^*(t)\,\mathrm{d}\,\mathrm{P} = \int_G M^*(s)\,\mathrm{d}\,\mathrm{P}$. Because of the structure of $\mathcal{G}_s$, this is equivalent to proving that $G = K_1\cap K_2$, where $K_1\in\mathcal{G}_t^*$ and $K_2\in\sigma\{W_1, W_2, \dots, W_n\}$. Since K_1 and K_2 are independent, K_2 is independent of $M^*(t)$. Therefore,

$$\int_{K_1\cap K_2} M^*(t)\,\mathrm{d}\,\mathrm{P} = \mathrm{P}(K_2)\int_{K_1} M^*(t)\,\mathrm{d}\,\mathrm{P} = \mathrm{P}(K_2)\int_{K_1} M^*(s)\,\mathrm{d}\,\mathrm{P} = \int_{K_1\cap K_2} M^*(s)\,\mathrm{d}\,\mathrm{P},$$

because $M^*(t)$ is a $\mathcal{G}_t^*$ martingale.

Part (c) follows from part (a)(ii) and the definition of T.

We now present the main result of this section, which states that $S^*(t)$ and its Lundberg exponent R^* play an important role in studying the ruin probability for $S(t)$.

Theorem 3.1. *The ultimate ruin probability for $S(t)$ can be expressed as*

$$\psi(u) = \frac{\mathrm{e}^{-R^*u}}{\mathrm{E}[\mathrm{e}^{-R^*S^*(T)}\mid T<\infty]},$$

where T is the time of ruin for $S(t)$.

Proof. Since $M^*(t)$ is a martingale and T is a stopping time, it follows from the optional stopping theorem that

$$\begin{aligned}\mathrm{e}^{-R^*u} &= M^*(0) = \mathrm{E}[M^*(t\wedge T)]\\ &= \mathrm{E}[M^*(t\wedge T)\mid T\le t]\,\mathrm{P}(T\le t) + \mathrm{E}[M^*(t\wedge T)\mid T>t]\,\mathrm{P}(T>t).\end{aligned}$$

Hence, the theorem is proved if

$$\mathrm{E}[M^*(t\wedge T)\mid T>t]\,\mathrm{P}(T>t)\to 0 \quad \text{as } t\to\infty. \tag{3.4}$$

https://doi.org/10.1239/jap/1110381378 Published online by Cambridge University Press

The following two expectations will be used to prove (3.4):

$$Z_1(R^*) = \mathrm{E}[S(t)\mathrm{e}^{R^*(D^*(t)-ct)}] \quad \text{and} \quad Z_2(R^*) = \mathrm{E}[S(t)^2\mathrm{e}^{R^*(D^*(t)-ct)}].$$

We first consider $Z_1(R^*)$. Let

$$Z_1(R^*) = u + ct - Z_{11}(R^*) - Z_{12}(R^*), \tag{3.5}$$

where

$$Z_{11}(R^*) = \mathrm{E}\bigg[\sum_{i=1}^{N(t)} X_i \mathrm{e}^{R^*(D^*(t)-ct)}\bigg],$$

$$Z_{12}(R^*) = \mathrm{E}\bigg[\sum_{i=1}^{\infty} Y_i\, \mathbf{1}(T_i + W_i \le t)\mathrm{e}^{R^*(D^*(t)-ct)}\bigg],$$

and define $h_X(r) = \mathrm{E}[\mathrm{e}^{rX}] - 1$, $h_Y(r) = \mathrm{E}[\mathrm{e}^{rY}] - 1$, and $h(r) = \mathrm{E}[\mathrm{e}^{r(X+Y)}] - 1$. Then,

$$\begin{aligned}
Z_{11}(r) &= \mathrm{e}^{-rct}\sum_{n=1}^{\infty}\bigg(\frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}\sum_{i=1}^{n}\mathrm{E}\bigg[X_i e^{rX_i}\exp\bigg\{r\sum_{j\le n, j\ne i} X_j\bigg\}\exp\bigg\{r\sum_{i=1}^{n} Y_i\bigg\}\bigg]\bigg) \\
&= \mathrm{e}^{-rct}\sum_{n=1}^{\infty}\bigg(\frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}(h_Y(r)+1)^n(h_X(r)+1)^{n-1}\sum_{i=1}^{n}\mathrm{E}[X_i\mathrm{e}^{rX_i}]\bigg).
\end{aligned}$$

Since $\mathrm{E}[X_i e^{rX_i}] = h_X'(r)$, $h(r)+1 = (h_X(r)+1)(h_Y(r)+1)$, and $\lambda h(R^*) - cR^* = 0$, where a prime denotes differentiation, $Z_{11}(R^*)$ becomes

$$Z_{11}(R^*) = (h_Y(R^*)+1)h_X'(R^*)\lambda t. \tag{3.6}$$

We also have

$$\begin{aligned}
Z_{12}(r) &= \mathrm{e}^{-rct}\sum_{n=1}^{\infty}\mathrm{E}[Y_n\,\mathbf{1}(T_n + W_n \le t)\mathrm{e}^{rD^*(t)}] \\
&= \mathrm{e}^{-rct}\sum_{n=1}^{\infty}\int_0^{\infty}\mathrm{E}[Y_n\,\mathbf{1}(T_n + W_n \le t)\mathrm{e}^{rD^*(t)} \mid W_n = s]\,\mathrm{P}(W_n = s)\,\mathrm{d}s \\
&= \mathrm{e}^{-rct}\sum_{n=1}^{\infty}\int_0^{t}\lambda_1\mathrm{e}^{-\lambda_1 s}\,\mathrm{E}[Y_n\,\mathbf{1}(T_n \le t-s)\mathrm{e}^{rD^*(t)}]\,\mathrm{d}s \\
&= \mathrm{e}^{-rct}\int_0^{t}\lambda_1\mathrm{e}^{-\lambda_1 s}\,\mathrm{E}\bigg[\mathrm{e}^{rD^*(t)}\sum_{n=1}^{\infty}Y_n\,\mathbf{1}(T_n \le t-s)\bigg]\,\mathrm{d}s \\
&= \mathrm{e}^{-rct}\int_0^{t}\lambda_1\mathrm{e}^{-\lambda_1 s}\,\mathrm{E}\bigg[\mathrm{e}^{rD^*(t)}\sum_{n=1}^{N(t-s)}Y_n\bigg]\,\mathrm{d}s \\
&= \mathrm{e}^{-rct}\int_0^{t}\lambda_1\mathrm{e}^{-\lambda_1 s}\,\mathrm{E}\bigg[\mathrm{e}^{rD^*(t-s)}\exp\{r(D^*(t)-D^*(t-s))\}\sum_{n=1}^{N(t-s)}Y_n\bigg]\,\mathrm{d}s \\
&= \int_0^{t}\lambda_1\mathrm{e}^{-\lambda_1 s}\,\mathrm{E}\bigg[\exp\{r(D^*(t-s)-c(t-s))\}\sum_{n=1}^{N(t-s)}Y_n\bigg]\,\mathrm{E}[e^{r(D^*(s)-cs)}]\,\mathrm{d}s.
\end{aligned}$$

https://doi.org/10.1239/jap/1110381378 Published online by Cambridge University Press

From (3.6) and the fact that $\mathrm{E}[\mathrm{e}^{R^*(D^*(s)-cs)}]=1$ for all s, $Z_{12}(R^*)$ can be written as

$$\begin{aligned}
Z_{12}(R^*) &= \int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 s}(h_X(R^*)+1)h'_Y(R^*)\lambda(t-s)\,\mathrm{d}s \\
&= (h_X(R^*)+1)h'_Y(R^*)\bigg(\lambda t - \frac{\lambda}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 t})\bigg).
\end{aligned}$$

Then, substituting the derived forms of $Z_{11}(R^*)$ and $Z_{12}(R^*)$ into (3.5), we obtain

$$Z_1(R^*) = u + ct - \lambda t h'(R^*) + \frac{\lambda}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 t})(h_X(R^*)+1)h'_Y(R^*). \tag{3.7}$$

We next look for an upper bound for $Z_2(R^*)$. First, write

$$Z_2(r) = (u+ct)^2 - 2(u+ct)Z_{21}(r) + Z_{22}(r) + Z_{23}(r) + 2Z_{24}(r), \tag{3.8}$$

where

$$\begin{aligned}
Z_{21}(r) &= \mathrm{E}\bigg[\bigg(\sum_{i=1}^{N(t)} X_i + \sum_{i=1}^{\infty} Y_i\,\mathbf{1}(T_i+W_i\le t)\bigg)\mathrm{e}^{r(D^*(t)-ct)}\bigg], \\
Z_{22}(r) &= \mathrm{E}\bigg[\bigg(\sum_{i=1}^{N(t)} X_i\bigg)^2 \mathrm{e}^{r(D^*(t)-ct)}\bigg], \\
Z_{23}(r) &= \mathrm{E}\bigg[\bigg(\sum_{i=1}^{\infty} Y_i\,\mathbf{1}(T_i+W_i\le t)\bigg)^2 \mathrm{e}^{r(D^*(t)-ct)}\bigg], \\
Z_{24}(r) &= \mathrm{E}\bigg[\bigg(\sum_{i=1}^{N(t)} X_i\bigg)\bigg(\sum_{i=1}^{\infty} Y_i\,\mathbf{1}(T_i+W_i\le t)\bigg)\mathrm{e}^{r(D^*(t)-ct)}\bigg].
\end{aligned}$$

Note that $Z_{21}(R^*) = Z_{11}(R^*) + Z_{12}(R^*)$, i.e.

$$Z_{21}(R^*) = \lambda t h'(R^*) - \frac{\lambda}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 t})h'_Y(R^*)(h_X(R^*)+1).$$

To evaluate $Z_{22}(r)$, we need

$$\begin{aligned}
H(r) &= \mathrm{E}\bigg[\bigg(\sum_{i=1}^{n} X_i\bigg)^2 \exp\bigg\{r\sum_{i=1}^{n} X_i\bigg\}\bigg] \\
&= [(h_X(r)+1)^n]'' \\
&= n(h_X(r)+1)^{n-2}((n-1)(h'_X(r))^2 + (h_X(r)+1)h''_X(r)),
\end{aligned}$$

which implies that

$$\begin{aligned}
Z_{22}(r) &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}\,\mathrm{E}\bigg[\bigg(\sum_{i=1}^{n} X_i\bigg)^2 \mathrm{e}^{r(D^*(t)-ct)}\bigg] \\
&= \mathrm{e}^{-rct}\sum_{n=1}^{\infty} \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}(h_Y(r)+1)^n H(r).
\end{aligned}$$

After some algebraic manipulation, we obtain

$$\begin{aligned}Z_{22}(R^*) &= \exp\{t(\lambda h(R^*) - cR^*)\}\lambda t(h_Y(R^*)+1)(\lambda t(h_Y(R^*)+1)(h'_X(R^*))^2 + h''_X(R^*))\\ &= \lambda t(h_Y(R^*)+1)(\lambda t(h_Y(R^*)+1)(h'_X(R^*))^2 + h''_X(R^*)).\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}Z_{23}(R^*) &\le \mathrm{E}\bigg[\bigg(\sum_{i=1}^{N(t)} Y_i\bigg)^2 \mathrm{e}^{R^*(D^*(t)-ct)}\bigg]\\ &= \lambda t(h_X(R^*)+1)(\lambda t(h_X(R^*)+1)(h'_Y(R^*))^2 + h''_Y(R^*)).\end{aligned}$$

For the last term on the right-hand side of (3.8), we obtain

$$\begin{aligned}Z_{24}(R^*) &\le \mathrm{E}\bigg[\bigg(\sum_{i=1}^{N(t)} X_i\bigg)\bigg(\sum_{i=1}^{N(t)} Y_i\bigg)\mathrm{e}^{R^*(D^*(t)-ct)}\bigg]\\ &= \mathrm{e}^{-R^*ct-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\,\mathrm{E}\bigg[\sum_{i=1}^{n} X_i \exp\bigg\{R^*\sum_{i=1}^{n} X_i\bigg\}\sum_{i=1}^{n} Y_i \exp\bigg\{R^*\sum_{i=1}^{n} Y_i\bigg\}\bigg]\\ &= \mathrm{e}^{-R^*ct-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}[(h_X(r)+1)^n]'_{r=R^*}[(h_Y(r)+1)^n]'_{r=R^*}\\ &= \mathrm{e}^{-R^*ct-\lambda t}h'_X(R^*)h'_Y(R^*)\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{(n-1)!}n(h(R^*)+1)^{n-1}\\ &= \mathrm{e}^{-R^*ct-\lambda t}h'_X(R^*)h'_Y(R^*)\lambda t\mathrm{e}^{\lambda t(h(R^*)+1)}(1+\lambda t(h(R^*)+1))\\ &= h'_X(R^*)h'_Y(R^*)\lambda t(1+\lambda t(h(R^*)+1)).\end{aligned}$$

Substituting the derived forms of $Z_{21}(R^*)$, $Z_{22}(R^*)$, $Z_{23}(R^*)$, and $Z_{24}(R^*)$ into (3.8) gives

$$\begin{aligned}Z_2(R^*) \le{}& (u+ct)^2 - 2(u+ct)\bigg(\lambda t h'(R^*) - \frac{\lambda}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 t})h'_Y(R^*)(h_X(R^*)+1)\bigg)\\ &+ \lambda t(h_Y(R^*)+1)(\lambda t(h_Y(R^*)+1)(h'_X(R^*))^2 + h''_X(R^*))\\ &+ \lambda t(h_X(R^*)+1)(\lambda t(h_X(R^*)+1)(h'_Y(R^*))^2 + h''_Y(R^*))\\ &+ 2h'_X(R^*)h'_Y(R^*)\lambda t(1+\lambda t(h(R^*)+1)). \qquad (3.9)\end{aligned}$$

The proof of (3.4) is as follows. Note that

$$\mathrm{E}[M^*(t\wedge T) \mid T > t]\,\mathrm{P}(T > t) = \int_{T>t} M^*(t)\,\mathrm{d}\,\mathrm{P}.$$

It is clear that if $T > t$ then $S(t) \geq 0$. Since $c - \lambda h'(R^*) < 0$, it is easily seen from (3.7) that $Z_1(R^*) < 0$ for large t. Therefore,

$$\begin{aligned}
\int_{T>t} M^*(t)\,\mathrm{d}\,\mathrm{P} &\leq \int_{S(t)\geq 0} M^*(t)\,\mathrm{d}\,\mathrm{P} \\
&\leq \int_{S(t)\geq Z_1(R^*)/2} M^*(t)\,\mathrm{d}\,\mathrm{P} \\
&\leq \int_{|S(t)-Z_1(R^*)|\geq -Z_1(R^*)/2} M^*(t)\,\mathrm{d}\,\mathrm{P} \\
&\leq \int_{|S(t)-Z_1(R^*)|\geq -Z_1(R^*)/2} \frac{4\mathrm{e}^{-R^*S^*(t)}(S(t)-Z_1(R^*)^2)}{Z_1(R^*)^2}\,\mathrm{d}\,\mathrm{P} \\
&\leq \frac{4\mathrm{e}^{-R^*u}(Z_2(R^*)-Z_1(R^*)^2)}{Z_1(R^*)^2}. \qquad (3.10)
\end{aligned}$$

Making use of (3.7) and (3.9), we find that

$$\begin{aligned}
Z_2(R^*) - Z_1(R^*)^2 &\leq 2\lambda t h'(R^*)\frac{\lambda}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 t})h_Y'(R^*)(h_X(R^*)+1) \\
&\quad + \lambda t h''(R) - \left(\frac{\lambda}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 t})h_Y'(R^*)(h_X(R^*)+1)\right)^2.
\end{aligned}$$

Hence, the numerator on the right-hand side of (3.10) is of order t and $Z_1(R^*)^2$ is of order t^2. We conclude that

$$\mathrm{E}[M^*(t\wedge T) \mid T > t]\,\mathrm{P}(T > t) = \int_{T>t} M^*(t)\,\mathrm{d}\,\mathrm{P} \to 0$$

as $t \to \infty$.

4. Lundberg exponent

In this section, we show that R^* is also the Lundberg exponent for model (2.1). Brémaud (2000) also discussed such an issue for the Poisson shot noise delayed-claims model.

Since the ruin probability for model (2.1) is smaller than that for the nondelayed model, we have $\psi(u) \leq \mathrm{e}^{-R^*u}$. To verify that R^* is the Lundberg exponent, we must calculate $\mathrm{E}[\mathrm{e}^{rD(t)}]$. For $N(t) = k$, it is well known that the random vector $(T_1, T_2, \dots, T_k)$ has the same distribution as the order statistics of k independent and identically distributed uniform $[0, t]$ random variables. Furthermore, $N(t)$, X_i, Y_i, and W_i are independent. Thus, we have

$$\begin{aligned}
\mathrm{E}[\mathrm{e}^{rD(t)}] &= \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^k}{k!}\frac{k!}{t^k}\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{k-1}}^t \mathrm{d}s_k \prod_{i=1}^k \mathrm{E}[\exp\{r(X_i + Y_i\,\mathbf{1}(s_i + W_i \leq t))\}] \\
&= \mathrm{e}^{-\lambda t}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(\int_0^t \mathrm{E}[\exp\{r(X + Y\,\mathbf{1}(W \leq s))\}]\,\mathrm{d}s\right)^k \\
&= \exp\left\{-\lambda t\left(1 - t^{-1}\int_0^t \mathrm{E}[\exp\{r(X + Y\,\mathbf{1}(W \leq s))\}]\,\mathrm{d}s\right)\right\},
\end{aligned}$$

where (X, Y, W) is the generic random vector of (X_i, Y_i, W_i).

Now let $g(r) = \lim_{t\to\infty} t^{-1} \ln \mathrm{E}[\mathrm{e}^{r(D(t)-ct)}]$. It can easily be shown that

$$g(r) = \lambda\,\mathrm{E}[\mathrm{e}^{r(X+Y)} - 1] - cr$$
$$= \lambda r \int_0^\infty \mathrm{e}^{rx}(1 - F * G(x))\,\mathrm{d}x - cr, \tag{4.1}$$

where $F * G$ represents the convolution of F and G. On the other hand, the results of Duffield and O'Connell (1995) give

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi(u) = -R,$$

where $R = \sup\{r : g(r) \le 0\}$. It follows from (4.1) that R is exactly the Lundberg exponent of the nondelayed risk model (3.1). Therefore, $R = R^*$.

5. Approximations for ruin probabilities

This section is devoted to deriving Brownian motion approximations for ruin probabilities for model (2.1). This kind of approximation for the compound Poisson model can be found in Grandell (1977), (1978) and Iglehart (1969). Although the by-claim number process $\bar{N}(t)$ is neither Poisson nor renewal, we are still able to extend the classical results to the proposed model.

Define $\kappa^2 = \sigma_F^2 + \sigma_G^2 + (\mu_F + \mu_G)^2$. The mean and variance of $D^*(t)$ are $\lambda t(\mu_F + \mu_G)$ and $\lambda t\kappa^2$, respectively. The following theorem states that, asymptotically, $D(t)$ has the same mean and standard deviation, and that the delay time distribution does not come into play in the limit.

Theorem 5.1. *As $t \to \infty$,*

$$U(t) = \frac{D(t) - \lambda t(\mu_F + \mu_G)}{(\lambda t\kappa^2)^{1/2}} \xrightarrow{\mathrm{D}} N(0, 1),$$

where '$\xrightarrow{\mathrm{D}}$' stands for convergence in distribution and $N(0, 1)$ is a standard normal random variable. As a result, $(\lambda t)^{-1/2}(\bar{N}(t) - \lambda t) \xrightarrow{\mathrm{D}} N(0, 1)$.

Proof. It is well known that

$$\frac{D^*(t) - \lambda t(\mu_F + \mu_G)}{(\lambda t\kappa^2)^{1/2}} \xrightarrow{\mathrm{D}} N(0, 1).$$

From (2.2), we find that

$$\mathrm{E}\bigg[\frac{D^*(t) - \lambda t(\mu_F + \mu_G)}{(\lambda t\kappa^2)^{1/2}} - U(t)\bigg] = \mathrm{E}\bigg[\frac{\sum_{i=1}^{N(t)} Y_i - \sum_{i=1}^{\infty} Y_i\,\mathbf{1}(T_i + S_i \le t)}{(\lambda t\kappa^2)^{1/2}}\bigg]$$
$$= \frac{\lambda\mu_G(1 - \mathrm{e}^{-\lambda_1 t})}{\lambda_1(\lambda t\kappa^2)^{1/2}}$$
$$\to 0$$

as $t \to \infty$. Hence,

$$\frac{D^*(t) - \lambda t(\mu_F + \mu_G)}{(\lambda t\kappa^2)^{1/2}} - U(t) \to 0$$

in probability. This implies the weak convergence of $U(t)$. If we let $X_i = 0$ and $Y_i = 1$, the convergence of the by-claim number process is simply a special case of the convergence of $U(t)$.

Now define

$$U_n(t) = \frac{D(nt) - \lambda nt(\mu_F + \mu_G)}{(\lambda n\kappa^2)^{1/2}}.$$

To obtain the desired Brownian motion approximations, we must establish the weak convergence of $U_n(t)$. The symbol '$\xrightarrow{\mathrm{W}}$' represents weak convergence for stochastic processes.

Theorem 5.2. *For constant t, $U_n(t) \xrightarrow{\mathrm{W}} B(t)$ as $n \to \infty$, where $B(t)$ is a standard Brownian motion.*

Proof. To prove the theorem, we need to prove (i) the convergence of the finite-dimensional distributions of $U_n(t)$ and (ii) the uniform tightness of $U_n(t)$, i.e. that, for every $\varepsilon > 0$,

$$\lim_{c\to 0} \limsup_{n\to\infty} \Delta^{\mathrm{P}}_{J_1}(c, U_n(t), \varepsilon) = 0, \tag{5.1}$$

where

$$\Delta^{\mathrm{P}}_{J_1}(c, U_n(t), \varepsilon) = \sup_{t_1<t<t_2} \min(\mathrm{P}(|U_n(t) - U_n(t_1)| > \varepsilon), \mathrm{P}(|U_n(t_2) - U_n(t)| > \varepsilon)),$$

with $t_1 \geq t - c$ and $t_2 \leq t + c$. Note that, according to Skorokhod (1957), (i) and (ii) imply that the distribution of $l(U_n(t))$ converges to the distribution of $l(B(t))$ for any J_1-continuous functional l.

By the definition of $U_n(t)$, we have

$$\begin{aligned} |U_n(t_1) - U_n(t_2)| &\leq \left| \frac{\sum_{i=N(nt_1)+1}^{N(nt_2)} (X_i + Y_i) - \lambda n(t_2 - t_1)(\mu_F + \mu_G)}{(\lambda n\kappa^2)^{1/2}} \right| \\ &\quad + \left| \frac{\sum_{i=1}^{\infty} Y_i(\mathbf{1}(nt_1 < T_i \leq nt_2) - \mathbf{1}(nt_1 < T_i + S_i \leq nt_2))}{(\lambda n\kappa^2)^{1/2}} \right| \\ &=: L_1 + L_2, \end{aligned}$$

with $t_2 > t_1$. By Chebyshev's inequality,

$$\mathrm{P}(L_1 \geq \varepsilon) \leq \frac{t_2 - t_1}{\varepsilon^2}. \tag{5.2}$$

Furthermore,

$$\begin{aligned} L_2 &\leq \left| \frac{\sum_{i=1}^{\infty} Y_i(\mathbf{1}(nt_1 < T_i \leq nt_2) - \mathbf{1}(nt_1 < T_i \leq nt_2, nt_1 < T_i + S_i \leq nt_2))}{(\lambda n\kappa^2)^{1/2}} \right| \\ &\quad + \frac{\sum_{i=1}^{\infty} Y_i \, \mathbf{1}(T_i \leq nt_1, nt_1 < T_i + S_i \leq nt_2)}{(\lambda n\kappa^2)^{1/2}} \\ &= \frac{\sum_{i=1}^{\infty} Y_i \, \mathbf{1}(nt_1 < T_i \leq nt_2, T_i + S_i > nt_2)}{(\lambda n\kappa^2)^{1/2}} \\ &\quad + \frac{\sum_{i=1}^{\infty} Y_i \, \mathbf{1}(T_i \leq nt_1, nt_1 < T_i + S_i \leq nt_2)}{(\lambda n\kappa^2)^{1/2}} \end{aligned}$$

https://doi.org/10.1239/jap/1110381378 Published online by Cambridge University Press

and, hence,

$$\begin{aligned}\mathrm{E}[L_2] &\le \frac{\mu_G \sum_{i=1}^{\infty} \mathrm{P}(nt_1 < T_i \le nt_2, T_i + S_i > nt_2)}{(\lambda n \kappa^2)^{1/2}} \\ &\quad + \frac{\mu_G \sum_{i=1}^{\infty} \mathrm{P}(T_i \le nt_1, nt_1 < T_i + S_i \le nt_2)}{(\lambda n \kappa^2)^{1/2}} \\ &\le \frac{\mu_G \sum_{i=1}^{\infty} (\mathrm{P}(T_i \le nt_2 < T_i + S_i) + \mathrm{P}(T_i \le nt_1 < T_i + S_i))}{(\lambda n \kappa^2)^{1/2}}. \end{aligned} \tag{5.3}$$

Note that

$$\begin{aligned}&\mathrm{P}(T_i \le nt_2 < T_i + S_i) + \mathrm{P}(T_i \le nt_1 < T_i + S_i) \\ &\quad = \int_0^{\infty} \lambda_1 \mathrm{e}^{-\lambda_1 s} (\mathrm{P}(T_i \le nt_2 < T_i + s) + \mathrm{P}(T_i \le nt_1 < T_i + s))\,\mathrm{d}s \\ &\quad = \int_0^{nt_2} \lambda_1 \mathrm{e}^{-\lambda_1 s}\, \mathrm{P}(T_i \le nt_2 < T_i + s)\,\mathrm{d}s + \int_{nt_2}^{\infty} \lambda_1 \mathrm{e}^{-\lambda_1 s}\, \mathrm{P}(T_i \le nt_2)\,\mathrm{d}s \\ &\qquad + \int_0^{nt_1} \lambda_1 \mathrm{e}^{-\lambda_1 s}\, \mathrm{P}(T_i \le nt_1 < T_i + s)\,\mathrm{d}s + \int_{nt_1}^{\infty} \lambda_1 \mathrm{e}^{-\lambda_1 s}\, \mathrm{P}(T_i \le nt_1)\,\mathrm{d}s, \end{aligned} \tag{5.4}$$

and that

$$\begin{aligned}\mathrm{P}(T_i \le nt_k < T_i + s) &= \int_{nt_k - s}^{nt_k} \frac{\lambda^{i+1} x^i}{i!} \mathrm{e}^{-\lambda x}\,\mathrm{d}x \\ &= \int_0^{s} \frac{\lambda^{i+1} (nt_k - x)^i}{i!} \mathrm{e}^{-\lambda (nt_k - x)}\,\mathrm{d}x \end{aligned}$$

for $k = 1, 2$. Let

$$\begin{aligned}\eta_n(t_k) &= \int_0^{nt_k} \lambda_1 \mathrm{e}^{-\lambda_1 s} \int_0^{s} \lambda \sum_{i=1}^{\infty} \frac{\lambda^i (nt_k - x)^i}{i!} \mathrm{e}^{-\lambda(nt_k - x)}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_{nt_k}^{\infty} \lambda_1 \mathrm{e}^{-\lambda_1 s} \sum_{i=1}^{\infty} \mathrm{P}(T_i \le nt_k)\,\mathrm{d}s \\ &= \int_0^{nt_k} \lambda_1 \mathrm{e}^{-\lambda_1 s} \int_0^{s} \lambda (1 - \mathrm{e}^{-\lambda(nt_k - x)})\,\mathrm{d}x\,\mathrm{d}s + \lambda n t_k \mathrm{e}^{-\lambda_1 n t_k} \\ &= \int_0^{nt_k} \lambda_1 \mathrm{e}^{-\lambda_1 s} (\lambda s + \mathrm{e}^{-\lambda n t_k} - \mathrm{e}^{-\lambda(nt_k - s)})\,\mathrm{d}s + \lambda n t_k \mathrm{e}^{-\lambda_1 n t_k} \\ &= \lambda\bigg(\frac{1}{\lambda_1} + \bigg(\frac{\lambda_1}{\lambda(\lambda_1 - \lambda)} - \frac{1}{\lambda_1}\bigg)\mathrm{e}^{-\lambda_1 n t_k} - \frac{1}{\lambda_1 - \lambda}\mathrm{e}^{-\lambda n t_k} - \frac{1}{\lambda}\mathrm{e}^{-(\lambda + \lambda_1) n t_k}\bigg). \end{aligned} \tag{5.5}$$

From (5.3), (5.4), and (5.5), we obtain

$$\mathrm{E}[L_2] \le \frac{\mu_G \sum_{k=1}^{2} \eta_n(t_k)}{(\lambda n \kappa^2)^{1/2}}$$

and, hence,

$$\mathrm{P}(L_2 \ge \varepsilon) \le \frac{\mu_G \sum_{k=1}^{2} \eta_n(t_k)}{\varepsilon(\lambda n \kappa^2)^{1/2}}. \tag{5.6}$$

It follows from (5.2) and (5.6) that (5.1) holds.

Now define

$$V_n(t) = \rho_n \lambda(\mu_F + \mu_G) n^{1/2} t - (\lambda \kappa^2)^{1/2} U_n(t).$$

Based on the weak convergence of $U_n(t)$, we obtain our final result.

Theorem 5.3. *Suppose that $n^{1/2}\rho_n$ tends to a positive constant γ as n tends to ∞. Then $V_n(t) \xrightarrow{\mathrm{W}} V(t)$, where*

$$V(t) = \gamma \lambda(\mu_F + \mu_G) t - (\lambda \kappa^2)^{1/2} B(t).$$

Furthermore, for any positive constants u and δ,

$$\lim_{n\to\infty} \psi(n^{-1/2}\delta, n^{1/2}u) = \mathrm{e}^{-2\delta u}, \tag{5.7}$$

where $\psi(n^{-1/2}\delta, n^{1/2}u)$ is the ultimate ruin probability for model (2.1) with initial surplus $n^{1/2}u$ and security loading $n^{-1/2}\delta$.

Proof. The weak convergence of $V_n(t)$ is simply due to Theorem 5.1 and the limit theorem of Skorokhod (1957). Result (5.7) follows from Grandell (1978).

Acknowledgements

The first author was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (project no. HKU 7202/99H) and the second author was supported by the National Natural Science Foundation of China (grant no. 10271062).

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