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HOW TO OBTAIN AN ASYMPTOTIC EXPANSION OF A SEQUENCE FROM AN ANALYTIC IDENTITY SATISFIED BY ITS GENERATING FUNCTION

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Abstract

Let f_n be a sequence of nonnegative integers and let $f(x) := \sum_{n\geq 0} f_n x^n$ be its generating function. Assume $f(x)$ has the following properties: it has radius of convergence r, $0 < r < 1$, with its only singularity on the circle of convergence at $x = r$ and $f(r) = s$; $y = f(x)$ satisfies an analytic identity $F(x, y) = 0$ near (r, s); for some $k \geq 2$ $F_{0,j} = 0$, $0 \leq j < k$, $F_{0,k} \neq 0$ where $F_{i,j}$ is the value at (r, s) of the i^{th} partial derivative with respect to x and the j^{th} partial derivative with respect to y of F. These assumptions form the basis of what we call the typical and general cases. In both cases we show how to obtain an asymptotic expansion of f_n. We apply our technique to produce several terms in the asymptotic expansion of combinatorial sequences for which previously only the first term was known.

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1. Introduction

Pólya [8] developed a technique to use an analytic identity satisfied by the generating function of a sequence (arising in combinatorics) to asymptotically determine the terms in the sequence. This technique was further developed and studied by many authors including [1, 3, 5, 6]. It involves the following two situations:

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Situation 1

Let f_n be a sequence of nonnegative integers arising in combinatorics. Let $f(x) := \sum_{n\geq 0} f_n x^n$ be its generating function, viewed as a function of a complex variable. Let r be its radius of convergence.

ASSUMPTION 1. The radius of convergence r satisfies $0 < r < 1$ and $x = r$ is the only singularity of f on its circle of convergence.

ASSUMPTION 2. The function $f(r)$ converges, say to s.

ASSUMPTION 3. There is a function $F(x, y)$, which is analytic in a neighborhood of (r, s), such that x close to and less than or equal to r implies $F(x, f(x)) = 0$.

NOTATION. If $G(x, y)$ is a function, then $G_{i,j}$ is the value at (r, s) of the i^{th} partial derivatives with respect to x of the j^{th} partial derivative with respect to y of G.

ASSUMPTION 4. For some $k \geq 2$, $F_{0,j} = 0$ for $0 \leq j < k$ and $F_{0,k} \neq 0$.

In most cases in the literature $k = 2$ in Assumption 4 and furthermore

ASSUMPTION 5. That $F_{1,0} \neq 0$.

DEFINITION. The *typical case* consists of Assumptions 1 through 5 with $k = 2$ in Assumption 4. The *general case* consists of Assumptions 1 through 4 and Assumptions 6 through 9 (presented in Section 2).

REMARK. Assumptions 6 through 9 hold in the typical case.

Situation 2

Let f_n be a sequence of nonnegative integers arising in combinatorics. Let $f(x)$ be its generating function. Assume there is a sequence g_n with generating function $g(x)$ such that the general case holds for g_n and $f(x)$ differs from a polynomial in x and $g(x)$ by a function with a radius of convergence greater than that of $g(x)$.

In this paper we are interested in obtaining in both situations not only an asymptotic formula for f_n (as in the previously cited works) but also an asymptotic expansion of f_n. In Section 2 we show how to obtain such an expansion

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in the general case (of Situation 1) giving special emphasis to the typical case. We also observe how the work used to obtain asymptotic expansions in Situation 1 can also often be used to obtain asymptotic expansions in Situation 2. In Section 3 we apply these results to obtain the first few terms in the asymptotic expansion of combinatorial sequences for which previously only the first term was known. In particular we study several sequences that arise in [7], and in counting various types of trees studied in [5] and [3]. We correct errors in many of the constants obtained in [3].

2. Abstract discussion of the two situations

2.1. Situation 1 To obtain the asymptotic expansion we show: (1) how to convert the analytic identity into a 'polynomial' identity; (2) how to use the 'polynomial' identity to obtain a fractional power series expansion of $f(x)$ about r; (3) how to use the fractional power series expansion to obtain an asymptotic expansion of f_n in terms of $\binom{a+kb}{n}$'s, where k is a nonnegative integer and a and b are rational numbers; and (4) how to convert this asymptotic expansion into an asymptotic expansion in terms of powers of n.

Step 1 By the Weierstrass Preparation Theorem (see, for example, [4]) Assumption 4 implies that in a neighborhood of (r, s)

$$F(x, y) = A(x, y) \cdot P(x, y),$$

where $A(x, y)$ is analytic, $A(r, s) \neq 0$, and

$$P(x, y) = (y - s)^k + \sum_{j=0}^{k-1} p_j(x)(y - s)^j$$

where the $p_i(x)$ are analytic and vanish at $x = r$. It may not be possible to determine the polynomial P. Instead we can determine the partial derivatives $P_{i,j}$. These and the partial derivatives $A_{i,j}$ are recursively obtained from the $F_{i,j}$'s as follows:

By definition $P_{0,j} = 0$ for $j \neq k$ and $P_{0,k} = k!$.

By the product rule

$$F_{i,j} = \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} A_{p,q} P_{i-p,j-q}.$$

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So for $j < k$

$$F_{i,j} = \sum_{p=1}^{i-1} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} A_{p,q} P_{i-p,j-q} + \sum_{q=1}^{j} \binom{j}{q} A_{0,q} P_{i,j-q} + A_{0,0} P_{i,j}$$

and hence

$$P_{i,j} = \left[F_{i,j} - \sum_{p=1}^{i-1} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} A_{p,q} P_{i-p,j-q} - \sum_{q=1}^{j} \binom{j}{q} A_{0,q} P_{i,j-q} \right] / A_{0,0}.$$

(1)

And for $j \geq k$,

$$F_{i,j} = \sum_{p=0}^{i-1} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} A_{p,q} P_{i-p,j-q} + \binom{j}{k} k! A_{i,j-k}.$$

Hence, (replacing $j-k$ by j)

$$(2) \quad A_{i,j} = \left[F_{i,j+k} = \sum_{p=0}^{i-1} \sum_{q=0}^{j+k} \binom{i}{p} \binom{j+k}{q} A_{p,q} P_{i-p,j+k-q} \right] \frac{j!}{(j+k)!}.$$

By induction on $j + ik$ Formulae (1) and (2) express the $A_{i,j}$'s and the $P_{i,j}$'s in terms of $F_{p,q}$'s for $p \leq i$ and $q \leq j + ik$.

For each j from 0 to $k-1$, $p_j(x)$ can be written as $\sum_{i \geq i_j} p_{i,j}(1 - x/r)^i$, where i_j is the order of the zero of $p_j(x)$ at $x = r$ (or $+\infty$ if $p_j(x)$ is identically zero).

An easy computation shows

$$(3) \qquad p_{i,j} = \frac{P_{i,j}(-r)^i}{i!j!} \qquad \text{for} \quad 0 \leq j \leq k-1 \quad \text{and} \quad i \geq i_j.$$

Also (as observed in [6]) for $0 \leq j \leq k-1$, i_j is the least i such that $F_{i,j} \neq 0$.

Step 2 By a classical result (for example, see Walker [10, Chapter 4, Section 3]), the 'polynomial' identity $P(x, f(x)) = 0$ for x close to and less than or equal to r implies that $f(x)$ can be expressed as a fractional power series $\sum_{i=1}^{\infty} a_i(1 - x/r)^{s_i}$ or $\sum_{i=1}^{m} a_i(1 - x/r)^{s_i}$, where s_i is an increasing sequence of rational numbers, and $a_i \neq 0$. As $f(r)$ converges, $s_1 \geq 0$.

We now observe how under the weak technical Assumptions 6, 7, 8, and 9 the algorithm in Walker can be used to express the a_i's and the s_i's in terms of the $p_{i,j}$'s of Step 1.

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ASSUMPTION 6. That i_0/k is not an integer.

ASSUMPTION 7. That $i_j \geq i_0(1 - j/k)$ for $1 \leq j \leq k - 1$.

In the typical case these assumptions hold as $k = 2$ and $i_0 = 1$. As observed in [6] Assumptions 6 and 7 imply $s_1 = i_0/k$ and $g(a_1) = 0$, where

$$g(a) = a^k + \sum \left\{p_{i_j,j}a^j : 1 \leq j \leq k-1 \text{ and } i_j = s_1(k-j)\right\} + p_{i_0,0}$$

ASSUMPTION 8. That $g(a) = 0$ has only one solution for which $a/\Gamma(-s_1)$ is a positive real (where Γ is the classical gamma function).

We choose a_1 to be this solution. Assumption 8 is used in Step 3.

ASSUMPTION 9. That a_1 is not a multiple root of $g(a) = 0$.

In the typical case as observed in Bender [1] $g(a) = a^2 + p_{1,0}$ and $p_{1,0} \neq 0$ and, hence, Assumptions 8 and 9 hold.

Assumption 9 tells us that the r of Walker [10, p. 100] equals 1 and hence a_i and s_i for $i \geq 2$ can be determined by the technique suggested in Walker [10, Section 3.3]. For simplicity we illustrate this in the typical case.

Say $P(x, y) = (y-s)^2 + p(x)(y-s) + q(x)$, where

$$p(x) = \sum_{i\geq 1} p_i(1 - x/r)^i \quad \text{and} \quad q(x) = \sum_{i \geq 1} q_i(1 - x/r)^i.$$

The fractional power series of $f(x) - s$ has the form

$$\sum_{i\geq 0} b_i(1 - x/r)^{i+1/2} + \sum_{i\geq 1} c_i(1 - x/r)^i.$$

Equating coefficients of the various powers of $(1 - x/r)^i$ in $P(x, f(x))$, we obtain:

$$(4)\quad 2\sum\left\{b_i c_j : i+j = m \text{ and } j \geq 1\right\} + \sum\left\{b_i p_j : i+j = m \text{ and } j \geq 1\right\} = 0$$

for $m \geq 1$ (using the half integer powers);

$$(5)\qquad (b_0)^2 + q_1 = 0$$

(using the lowest integer power); and

$$(6)\quad \sum\left\{b_i b_j : i+j = m-1\right\} + \sum\left\{c_i c_j : i+j = m,\ i \geq 1 \text{ and } j \geq 1\right\} + \sum\left\{p_i c_j : i+j = m,\ i \geq 1 \text{ and } j \geq 1\right\} + q_m = 0$$

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for $m \geq 1$ (using the remaining integer powers).

By induction and (4), $c_i = -p_i/2$. So (6) gives

$$\sum\{b_i b_j : i+j = m-1\} - 1/4 \sum\{p_i p_j : i+j = m,\ i \geq 1 \text{ and } j \geq 1\} + q_m = 0$$

and, hence, (replacing $m-1$ by m)

$$(7)\ b_m = \Big[-\sum\{b_i b_j : i+j = m,\ i \geq 1 \text{ and } j \geq 1\} + 1/4 \sum\{p_i p_j : i+j = m+1,\ i \geq 1 \text{ and } j \geq 1\} - q_{m+1}\Big]/2b_0.$$

By (5), $b_0 = -\sqrt{-q_1}$. So using (7) by induction b_m may be expressed in terms of the p_i's and the q_i's for $i \leq m+1$.

REMARK. Even if one does not make Assumptions 6 through 9, it may still be possible to use the techniques in Walker to express the coefficients of the fractional power series of $f(x)$ in terms of the $p_{i,j}$'s. (In [6] we showed how to obtain the first noninteger power term and its coefficient.) The only difficulty is that one may obtain several possible fractional power series. In this case one needs some procedure to determine which of these is the correct one.

Step 3 Next we use the following special case of a theorem of Darboux (see [9], for example).

THEOREM. *Let f_n be a sequence of nonnegative integers with generating function $f(x)$ satisfying Assumptions* 1 *and* 2. *If for x near r, $f(x)$ is expressible as the fractional power series $\sum_{i\geq 0} a_i(1-x/r)^{(a+ib)}$, where a and b are rational numbers and $b > 0$, then f_n has the asymptotic expansion*

$$\sum_{i\geq 0} a_i \binom{a+ib}{n} (-1)^n\, r^{-n}.$$

Using this theorem we can obtain an asymptotic expansion of f_n in terms of the numbers $\binom{a+ib}{n}$.

In the typical case we use this theorem for $a = 0$ and $b = 1/2$. In this case it may be stated as follows:

THEOREM. *Let f_n be a sequence of nonnegative integers with generating function $f(x)$ satisfying Assumptions* 1 *and* 2. *If for x near r, $f(x)$ is expressible as the fractional power series $\sum_{i\geq 0} b_i(1-x/r)^{i+1/2} + \sum_{i\geq 0} c_i(1-x/r)^i$;*

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then f_n has the asymptotic expansion

$$\sum_{i\geq 0} b_i \binom{i+1/2}{n} (-1)^n r^{-n}.$$

Step 4 To obtain an asymptotic expansion of f_n in terms of powers of n it remains to obtain such asymptotic expansions of $\binom{a+ib}{n}$. This may be done by using the following asymptotic expansion form of Stirling's Formula:

$$\ln\Gamma(z) = (z-1/2)\ln(z) - z + 1/2\ln(2\pi) + \sum_{t=1}^{\infty} \frac{(-1)^{t-1}B_t}{2t(t-1)z^{2t-1}},$$

where B_t is the t^{th} Bernouilli number.

Specifically we note

$$\binom{\alpha}{n} = \frac{\Gamma(n-\alpha)}{\Gamma(n+1)} \cdot \frac{1}{\Gamma(-\alpha)} \cdot (-1)^n$$

and apply Stirling's Formula to $\Gamma(n-\alpha)$ and $\Gamma(n+1)$. For the typical case (obtained with the assistance of MACSYMA):

$$\binom{1/2}{n} = -(1 + \frac{3}{8n} + \frac{25}{128n^2} + \frac{105}{1024n^3} + \frac{1659}{32768n^4} + \dots)\, \frac{(-1)^n}{n^{3/2}} \frac{1}{2\sqrt{\pi}}$$

$$\binom{3/2}{n} = -(1 + \frac{15}{8n} + \frac{385}{128n^2} + \frac{4725}{1024n^3} + \dots)\, \frac{(-1)^n}{n^{5/2}} \frac{3}{4\sqrt{\pi}}$$

$$\binom{5/2}{n} = -(1 + \frac{35}{8n} + \frac{1785}{128n^2} + \dots)\, \frac{(-1)^n}{n^{7/2}} \frac{15}{8\sqrt{\pi}}$$

$$\binom{7/2}{n} = -(1 + \frac{63}{8n} + \dots)\, \frac{(-1)^n}{n^{9/2}} \frac{105}{16\sqrt{\pi}}$$

$$\binom{9/2}{n} = -(1 + \dots)\, \frac{(-1)^n}{n^{11/2}} \frac{945}{32\sqrt{\pi}}$$

Conclusions in the typical case

In the typical case the four steps imply that f_n has the asymptotic expansion

$$(1/(2\sqrt{\pi} r^n n^{3/2}))\ (\sum_{i\geq 0} A_i/n^i), \quad \text{where } A_0 = \sqrt{(2F_{1,0}r/F_{0,2})} \text{ (as in [1])}$$

and

$$A_1 = \Big[\big\{9(F_{0,2})^3 F_{2,0} - 18(F_{0,2})^2 F_{1,0}F_{1,2} - 9(F_{0,2})^2(F_{1,1})^2 + 18F_{0,2}F_{0,3}F_{1,0}F_{1,1} + 3F_{0,2}F_{0,4}(F_{1,0})^2 - 5(F_{0,2})^3(F_{1,0})^2\big\} r^2 + 9(F_{0,2})^3 F_{1,0} r\Big] / \big[12(F_{0,2})^4 A_0\big]$$

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Each A_i may be expressed in terms of partial derivatives of F. These expressions for A_i grow rapidly in size as i increases. They are not worthy of display and should only be computed using computer algebra. Nonetheless the results of such computations can be efficiently used with further computer algebra assistance to determine several terms of the asymptotic expansion of particular sequences of combinatorial interest. This is done throughout Section 3.

2.2. Situation 2 Say $f(x) = Q(x, g(x)) + h(x)$, where $f(x)$ is the generating function of a sequence f_n, Q is a polynomial in both its variables, $g(x)$ is the generating function of a general case sequence g_n, and $h(x)$ has a radius of convergence greater than r, the radius of convergence of $g(x)$.

By Steps 1 and 2 for g_n, $g(x)$ has a fractional power series about $x = r$. In many cases plugging this fractional power series in for $g(x)$ in $Q(x, g(x))$ gives a fractional power series which has some noninteger powers of $(1 - x/r)$. As $h(x)$ is analytic at $x = r$, applying Steps 3 and 4 to this new fractional power series gives an asymptotic expansion of f_n.

3. Some sequences of combinatorial interest

All sequences considered in this section are ones in the typical case for which the first order term in the asymptotic expansion is already known.

3.1. Several sequences from [6] In [6] we introduced several sequences including $\sharp_n$, u_n, and m_n in a study of the expected complexity of an analytic tableaux algorithm for the satisfiability problem of propositional calculus. That paper shows that the generating functions satisfy the identities:

$$\begin{aligned}
\sharp(x) &= 1 + 2x(\sharp(x))^2; \\
u(x) &= 1 + 2x\sharp(x)u(x) + x(u(x))^2; \qquad \text{and} \\
m(x) &= 1 + 2x\sharp(x)m(x) + 2x(u(x))^2 + x(m(x))^2.
\end{aligned}$$

It obtains the first order terms of an asymptotic expansion of each of these sequences.

Also we observed that $\sharp_n = 2^n c_n$, where c_n is the n^{th} Catalan number. As is well known, $c_n = (1/(2n+1))\binom{2n+1}{n}$ and, hence, asymptotic expansions of c_n and $\sharp_n$ can be determined by using the asymptotic expansion version of Stirling's Formula. Alternately the same asymptotic expansions can be obtained using Steps 2 through 4. Similarly asymptotic expansions of u_n and m_n can be obtained using these steps. With computer algebra we obtained.

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THEOREM 1. (a) $\sharp_n$ *has the asymptotic expansion*

$$\frac{8^n n^{-3/2}}{\sqrt{\pi}}\left(1-\frac{9}{8n}+\frac{145}{128n^2}-\frac{1155}{1024n^3}+\frac{36939}{32768n^4}+\dots\right);$$

(b) u_n *has the asymptotic expansion*

$$9^n n^{-3/2}\Big(1.465807536-\frac{0.5496778259}{n}+\frac{11.27984705}{n^2} \\ +\frac{182.6175746}{n^3}+\frac{4914.966960}{n^4}+\dots\Big);$$

(c) m_n *has the asymptotic expansion*

$$\mu^{-n} n^{-3/2}\Big(2.020870239+\frac{0.1180427831}{n}+\frac{16.74728853}{n^2} \\ +\frac{194.4880010}{n^3}+\frac{3562.577082}{n^4}+\dots\Big),$$

where $\mu=(-13+44\sqrt{2})/529$.

It is interesting to see in each case how the higher order terms affect the accuracy of estimates.

n	Precise value of $\sharp_n$	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order	Fifth order
10	17199104	19156887	17001737	17218749	17197141	17199301

For $\sharp_n$ at $n = 100$ each of the first five orders of approximation increases the accuracy by about two decimal places. (Intuitively we should expect this as the coefficients obtained in the expansion are all roughly the same).

n	Precise value of u_n	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order	Fifth order
10	135733168	161622583	155561737	167999099	188134843	242328155

Intuitively the accuracy is so poor because the coefficients increase so rapidly. On the other hand by $n = 70$, the fifth order approximation is accurate to 4 places and is better than any of the first through fourth order approximations.

The numerical evidence for the accuracy of the approximations of m_n is quite similar to that for u_n.

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3.2. Trees Pólya [8] and Otter [5] studied the number of trees. (See also [1]).

Let b_n be the number of planted unlabelled binary trees with n terminal nodes; let t_n be the number of unlabelled trees with n vertices; and let r_n be the number of rooted unlabelled trees with n vertices.

The above authors derived the following generating function identities:

$$b(x) = x + 1/2((b(x))^2 + b(x^2));$$

$$r(x) = x \exp\left(\sum_{n\geq 1} r(x^n)/n\right); \qquad \text{and}$$

$$t(x) = r(x) - 1/2((r(x))^2 - r(x^2)).$$

They also determined the radii of convergence of the generating functions and the first order terms of asymptotic expansions. Using Steps 2 through 4 for b_n, Steps 1 through 4 for r_n, and the procedure in Section 2.2 for t_n, we obtained with computer algebra:

THEOREM 2. (a) *b_n has the asymptotic expansion*

$$\beta^n n^{-3/2}(0.3187766259 + \frac{0.2038317427}{n} + \frac{0.3682702316}{n^2} + \frac{1.476819367}{n^3} + \ldots),$$

where $\beta = 0.4026975037$.

(b) *r_n has the asymptotic expansion*

$$\rho^{-n} n^{-3/2}(0.4399240126 + \frac{0.04416990184}{n} + \frac{0.2216928060}{n^2} + \frac{0.8676554908}{n^3} + \ldots),$$

and t_n has the asymptotic expansion

$$\rho^{-n} n^{-5/2}(0.5349496061 + \frac{0.4853877311}{n} + \frac{2.379745574}{n^2} + \ldots),$$

where $\rho = 0.3383218569$.

Here is some numerical evidence of the improvement in accuracy from the use of higher order terms.

n	Precise value of b_n	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order
20	293547	283376	292436	293254	293418

n	Precise value of r_n	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order
18	1721159	1708154	1717682	1720338	1720916

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n	Precise value of t_n	Approximations (rounded to nearest integer) First order	Second order	Third order
18	123867	115396	121213	122797

3.3. Special Types of Trees In [3] the authors study: U_n, the number of nonisomorphic identity rooted trees on n points; u_n, the number of free identity trees on n points; H_n, the number of nonisomorphic planted homeomorphically irreducible trees on n points; h_n, the number of free homeomorphically irreducible trees of order n; C_n, the number of nonisomorphic planted blocky trees on n points; and c_n, the number of free blocky trees of order n. They use the generating function identities for $U(x)$, $u(x)$, $H(x)$, $h(x)$, $C(x)$, and $c(x)$ (formulae (1), (2), (7), (8), (12), and (13) of [3]) to determine the radii of convergence of the generating functions and the first order terms in asymptotic expansions. ((1), (2), (7), and (8) are originally from [2].) Using Steps 1 through 4 for U_n, H_n, and C_n, and the procedure in Section 2.2 for u_n, h_n, and c_n, we obtained with computer algebra:

THEOREM 3. (a) *U_n has the asymptotic expansion*

$$\mu^{-n}n^{3/2}(0.3625364234 - \frac{0.1044426616}{n} - \frac{0.2410458974}{n^2} - \frac{1.272504538}{n^3} + \dots),$$

and u_n has the asymptotic expansion

$$\mu^{-n}n^{-5/2}(0.2993882875 - \frac{0.3488387859}{n} - \frac{1.335147424}{n^2} + \dots),$$

where $\mu = 0.3972130969$.

(b) *H_n has the asymptotic expansion*

$$\theta^{-n}n^{-3/2}(0.4213018529 + \frac{0.1320316948}{n} + \frac{0.6099539181}{n^2} + \frac{2.983030203}{n^3} + \dots),$$

and h_n has the asymptotic expansion

$$\theta^{-n}n^{-5/2}(0.6844472720 + \frac{1.863425043}{n} + \frac{11.23522366}{n^2} + \dots),$$

where $\theta = 0.4567332096$.

(c) *C_n has the asymptotic expansion*

$$\gamma^{-n}n^{-3/2}(0.3687229874 - \frac{0.02892358469}{n} + \frac{0.05998637958}{n^2} + \frac{0.2476468907}{n^3} + \dots),$$

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and c_n has the asymptotic expansion

$$\gamma^{-n}n^{-5/2}(0.3149782093 - \frac{0.033901824744}{n} + \frac{0.3502928548}{n^2} + \ldots)$$

where $\gamma = 0.2225111687$.

REMARK. The above theorem shows that [3] incorrectly computed μ by about 1/2%, and incorrectly computed the coefficients of all first order terms.

Here is some numerical evidence of the improvement in accuracy from the use of higher order terms.

n	Precise value of U_n	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order
20	416848	423970	417863	417159	416973

n	Precise value of u_n	Approximations (rounded to nearest integer) First order	Second order	Third order
20	16104	17506	16486	16291

n	Precise value of H_n	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order
20	30802	30185	30658	30768	30794

n	Precise value of h_n	Approximations (rounded to nearest integer) First order	Second order	Third order
20	2988	2452	2786	2886

n	Precise value of C_n	Approximations (rounded to nearest integer) First order	Second order	Third order	Fourth order
10	38982	39191	38883	38947	38973

n	Precise value of c_n	Approximations (rounded to nearest integer) First order	Second order	Third order
10	3424	3348	3312	3349

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With the exception of c_n we see substantial improvement as the order of approximation is increased. For c_n the following holds. By $n = 20$ the first order approximation is larger than the precise value; by $n = 30$ the second order approximation is a slight improvement on the first order approximation; by $n = 60$ the second order approximation is a significant improvement on the first order approximation; and in all these cases the third order approximation is significantly better than the second order approximation.

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