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# EXTENSIONS OF A BRANDT SEMIGROUP BY ANOTHER

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**1. Introduction and summary.** One possible first step in considering the structure of a class $\mathscr{C}$ of semigroups is to study the ideal extensions (here simply called "extensions") of simple or 0-simple semigroups in $\mathscr{C}$ by another if the latter are of known structure. Extensions of a semigroup by another were first studied by Clifford (see [**1**, 4.4 and 4.5]). In his constructions, an extension of a semigroup $S$ by a semigroup $T$ with zero is given by a function (satisfying certain conditions) from $T^* = T\backslash 0$ into the translational hull of $S$.

We use certain results (refining those of Clifford) established in [**2**] and a description of the translational hull of a Brandt semigroup given in [**9**] (see also [**8**]), to construct all extensions $V$ of a Brandt semigroup $S$ having a finite number of idempotents by any Brandt semigroup $T$ (cf. [**10**]). Multiplication in $V$ is determined (except in the trivial case of an orthogonal sum) by three independent parameters: two are functions between sets (one of which sometimes satisfies a simple condition) and the third one is a monomial group representation. The problem is thus completely solved as far as the theory of semigroups is concerned. This is our Theorem 2, whereas Theorem 1 gives the $\mathscr{D}$-structure of the translational hull of $S$. We then give the Schützenberger-Preston representation of all extensions $V$ which represents $V$ isomorphically by matrices over a group with zero.

In addition, we state a generalization of Theorem 2 giving all extensions of $S$ by an orthogonal sum of Brandt semigroups, and derive a number of consequences of Theorem 2 concerning the existence of particular kinds of extensions and covering of idempotents in $V$.

In [**11**] Warne characterized all extensions of a Brandt semigroup $S$ by any semigroup $T$ with zero using functions satisfying certain conditions; we are dealing with a more special situation which makes it possible to solve the extension problem completely (i.e., modulo groups). To the best of our knowledge, the expressions we have obtained for multiplication are the most explicit in the case of an extension not necessarily determined by a partial homomorphism (cf. [**1**, 4.5]).

*Terminology and notation.* In general we follow the terminology and notation of Clifford and Preston [**1**]. In particular, $T^* = T\backslash 0$ if $T$ is a semigroup with zero, $A/B$ is the Rees quotient of $A$ by $B$ if $A$ is a semigroup and $B$ is an ideal of $A$, $|I|$ is the cardinality of the set $I$. $\mathscr{I}_I$ denotes the semigroup of all one-to-one

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partial transformations on $I$ written as operators on the right (the symmetric inverse semigroup on $I$), $0$ denotes the empty transformation, $\mathbf{d}\beta$ and $\mathbf{r}\beta$ for $\beta \in \mathscr{I}_I$ denote the domain and the range of $\beta$, respectively, and rank $\beta = |\mathbf{r}\beta|$. For $k > 1$ an integer, $\mathfrak{S}_k$ is the symmetric group on $k$ letters. If $G$ is a group, $G$ wr $\mathfrak{S}_k$ is the wreath product of $G$ and $\mathfrak{S}_k$. Its elements are pairs $(f, a)$ with $f: X \to G$, $a \in \mathfrak{S}_k$, and multiplication $(f, a)(g, b) = (f \cdot {}^a g, ab)$; here for $x \in X$, $a \in \mathfrak{S}_k$, $x(f \cdot g) = (xf)(xg)$, $x^a g = (xa)g$, where $|X| = k$ and $\mathfrak{S}_k$ acts on $X$ (see [**7**]). $G$ wr $\mathfrak{S}_k$ can be interpreted as the monomial group over $G$, i.e., as all $k \times k$ matrices over $G$ with exactly one non-zero element in each row and column under the usual multiplication of matrices (see [**3**] and bibliography listed there). For a semigroup $S$, $\Omega(S)$ denotes its translational hull; we write the left (right) translations as operators on the left (right), so that the product in $\Omega(S)$ is given by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$; $\Pi(S)$ denotes the inner part of $\Omega(S)$ (i.e., all pairs $\pi_a = (\lambda_a, \rho_a)$ with $\lambda_a x = ax$, $x\rho_a = xa$); recall that $\Pi(S)$ is an ideal of $\Omega(S)$ and that $\Pi(S) \cong S$ when $S$ is weakly reductive. $P(S)$ is the semigroup of right translations on $S$, $\tilde{P}(S)$ is the image of $\Omega(S)$ under the projection $(\lambda, \rho) \to \rho$, $\Delta(S) = \{\rho_a | \, a \in S\}$ (see [**2**; **8**; **9**]). If $\varphi$ is a homomorphism of a semigroup $A$ into a semigroup $B$, both with zero, we say that $\varphi$ is 0-restricted if $a\varphi = 0$ if and only if $a = 0$. A semigroup $S$ with zero $0$ is an *orthogonal sum* of semigroups $S_i$ with zero if $S_i$ are subsemigroups of $S$ such that $S_i \cap S_j = 0$ and $S_i S_j = 0$ if $i \neq j$, and $S = \bigcup S_i$.

**2. Preliminary results.** We first state a general extension theorem which is a slightly modified version of [**1**, Theorem 4.21].

EXTENSION THEOREM. *Let $S$ be a weakly reductive semigroup and let $T$ be a semigroup with zero, disjoint from $S$. Let $\eta$: $T^* \to \Omega(S)$ be a partial homomorphism such that $(a\eta)(b\eta) \in \Pi(S)$ if $ab = 0$. Let $V = S \cup T^*$ and define a multiplication $*$ in $V$ as follows, letting $a\eta = (\lambda^a, \rho^a)$ (note the difference between $\lambda_a$ and $\lambda^a$):*

$$a * b = \begin{cases} \lambda^a b & \text{if } a \in T^*, b \in S, \\ a\rho^b & \text{if } a \in S, b \in T^*, \\ c, \text{ where } (a\eta)(b\eta) = \pi_c, c \in S & \text{if } a, b \in T^* \quad \text{and } ab = 0 \text{ in } T, \\ ab & \text{otherwise}. \end{cases}$$

*Then $V$ is an extension of $S$ by $T$. Conversely, every extension of $S$ by $T$ is obtained in this way.*

A partial homomorphism $\eta$: $T^* \to \Omega(S)$ satisfying the condition

$$(a\eta)(b\eta) \in \Pi(S) \quad \text{if} \quad ab = 0 \qquad (a, b \in T^*)$$

will be called an *extension function*.

In our case, $S$ and $T$ are Brandt semigroups. Throughout the paper we shall take $S = \mathscr{M}^0(G; I, I; \Delta)$ and $T = \mathscr{M}^0(K; J, J; \Delta)$, where $\Delta$ is the identity matrix over $I$ and $J$, respectively. Then the projection of $\Omega(S)$ onto $\tilde{P}(S)$ is one-to-one [**8**, Theorem 8] so that

$$\Omega(S) \cong \tilde{P}(S) \cong G \text{ wr } \mathscr{I}_I.$$

Considering different cases of extensions (cf. [**2**]), the Extension Theorem splits into three parts according to properties of $\eta$.

*Case* 1. $\eta$ maps $T^*$ onto the zero of $\Omega(S)$. Then $V$ is simply the orthogonal sum of $S$ and $T$.

*Case* 2. $\eta$ maps $T^*$ into $\Pi(S) \cong S$. The extension is then determined by a partial homomorphism of $T^*$ into $S^*$. This extension is called strict in [**2**].

*Case* 3. $\eta$ maps $T^*$ into $\Omega(S)\backslash\Pi(S)$. In such a case, $\eta$ can be extended to a 0-restricted homomorphism of $T$ into $\Omega(S)/\Pi(S) \cong \tilde{P}(S)/\Delta(S)$. Since $T$ is 0-bisimple, $\eta$ maps $T^*$ into a $\mathscr{D}$-class of $\Omega(S)$ disjoint from $\Pi(S)$. This extension is called *pure* in [**2**].

In any case, $\eta$ maps $T^*$ into a $\mathscr{D}$-class of $\Omega(S) \cong \tilde{P}(S)$. The following theorem gives the $\mathscr{D}$-structure of $\tilde{P}(S)$ for a Brandt semigroup $S$ with a finite number of idempotents (cf. [**6**] for the case $|G| = 1$).

THEOREM 1. *Let $S = \mathscr{M}^0(G; I, I; \Delta)$ be a Brandt semigroup with $I$ finite, $|I| = n$. Then $\tilde{P}(S)$ has a unique principal series*

$$0 = J_0 \subset J_1 \subset \ldots \subset J_n = \tilde{P}(S). \tag{1}$$

*In $\tilde{P}(S)$ we have $\mathscr{D} = \mathscr{J}$, $\Delta(S) = J_1$, and for $k = 1, 2, \ldots, n$,*

$$J_k/J_{k-1} \cong \mathscr{M}^0(G \text{ wr } \mathfrak{S}_k; \mathbf{k}, \mathbf{k}; \Delta), \tag{2}$$

*where $\mathbf{k} = \{A \subseteq I \mid |A| = k\}$.*

The proof is based on the following.

LEMMA.

$$\begin{aligned} \tilde{P}(S) &\cong \{(\beta, \psi) \mid \beta \in \mathscr{I}_I, \beta \neq 0, \psi\colon \mathbf{d}\beta \to G\} \cup 0, \\ \Delta(S) &\cong \{(\beta, \psi) \mid \beta \in \mathscr{I}_I, \text{rank } \beta = 1, \psi\colon \mathbf{d}\beta \to G\} \cup 0, \end{aligned} \tag{3}$$

*with multiplication*

$$(\beta, \psi)(\beta', \psi') = (\beta\beta', \psi'') \quad \text{if} \quad \beta\beta' \neq 0$$

*and* 0 *otherwise, where $i\psi'' = (i\psi)(i\beta\psi')$ for all $i \in \mathbf{d}(\beta\beta')$.*

*Proof.* This follows directly from [**9**, Theorem 1; **8**, Theorem 4].

*Proof of Theorem* 1. It is easy to see that in $\mathscr{I}_I$,

$$\beta \mathscr{R} \beta' \Leftrightarrow \mathbf{d}\beta = \mathbf{d}\beta', \qquad \beta \mathscr{L} \beta' \Leftrightarrow \mathbf{r}\beta = \mathbf{r}\beta',$$
$$\beta \mathscr{D} \beta' \Leftrightarrow \beta \mathscr{J} \beta' \Leftrightarrow \text{rank } \beta = \text{rank } \beta'.$$

A simple calculation shows that in $\tilde{P}(S)$,

$$\begin{aligned} &(\beta, \psi) \mathscr{R} (\beta', \psi') \Leftrightarrow \mathbf{d}\beta = \mathbf{d}\beta', \\ &(\beta, \psi) \mathscr{L} (\beta', \psi') \Leftrightarrow \mathbf{r}\beta = \mathbf{r}\beta', \\ &(\beta, \psi) \mathscr{D} (\beta', \psi') \Leftrightarrow (\beta, \psi) \mathscr{J} (\beta', \psi') \Leftrightarrow \text{rank } \beta = \text{rank } \beta'. \end{aligned}$$

Thus $\mathscr{D} = \mathscr{J}$ and letting

$$J_k = \{(\beta, \psi) | \text{ rank } \beta \leqq k\}$$

for $k = 1, 2, \ldots, n$, we obtain, by (3), the unique principal series (1). It is now clear that $\Delta(S) = J_1$ and that for $k = 1, 2, \ldots, n$,

$$J_k/J_{k-1} \cong \{(\beta, \psi) | \text{ rank } \beta = k\} \cup 0,$$

where $(\beta, \psi)(\beta', \psi') = 0$ if rank $(\beta\beta') < k$; let $D_k = J_k \backslash J_{k-1}$.

Recall that $\mathbf{k} = \{A \subseteq I | |A| = k\}$. Now fix $k$, $1 \leqq k \leqq n$, and $A \in \mathbf{k}$. For every $B \in \mathbf{k}$, fix a one-to-one map $\alpha_B: A \rightarrow B$. For $(\beta, \psi) \in D_k$, let

$$(\beta, \psi)\chi = [(\theta, a); \mathbf{d}\beta, \mathbf{r}\beta], \tag{4}$$

where $\theta = \alpha_{\mathbf{d}\beta}\psi$, $a = \alpha_{\mathbf{d}\beta}\beta\alpha_{\mathbf{r}\beta}^{-1}$, with $\mathbf{d}\beta, \mathbf{r}\beta \in \mathbf{k}$ (here we have reversed the order of $a$ and $\theta$ in keeping with the usual notation of wreath products [**7**]). Then $\theta: A \rightarrow G$, $a \in \mathfrak{S}_k$, so that $(\theta, a) \in G \text{ wr } \mathfrak{S}_k$ and hence by letting $0\chi = 0$, we have

$$\chi: D_k^0 \rightarrow \mathscr{M}^0(G \text{ wr } \mathfrak{S}_k; \mathbf{k}, \mathbf{k}; \Delta).$$

We establish (2) by showing that $\chi$ is an onto isomorphism. We first show that $(\beta, \psi)\chi(\beta', \psi')\chi = [(\beta, \psi)(\beta', \psi')]\chi$. If $\mathbf{r}\beta \neq \mathbf{d}\beta'$, rank $(\beta\beta') < k$ and both sides are 0. Suppose $\mathbf{r}\beta = \mathbf{d}\beta'$; then $\mathbf{d}(\beta\beta') = \mathbf{d}\beta$, $\mathbf{r}(\beta\beta') = \mathbf{r}\beta'$. On the one hand,

$$\begin{aligned}(\beta, \psi)\chi(\beta', \psi')\chi &= [(\theta, a); \mathbf{d}\beta, \mathbf{r}\beta][(\theta', a'); \mathbf{d}\beta', \mathbf{r}\beta'] \\ &= [(\theta, a)(\theta', a'); \mathbf{d}\beta, \mathbf{r}\beta'] = [(\theta \cdot {}^a\theta', aa'); \mathbf{d}\beta, \mathbf{r}\beta'],\end{aligned}$$

and on the other,

$$[(\beta, \psi)(\beta', \psi')]\chi = (\beta\beta', \psi'')\chi = [(\theta'', a''); \mathbf{d}(\beta\beta'), \mathbf{r}(\beta\beta')],$$

where $i\psi'' = (i\psi)(i\beta\psi')$ for all $i \in \mathbf{d}(\beta\beta')$. Further, for all $i \in A$,

$$\begin{aligned}i\theta'' = i\alpha_{\mathbf{d}(\beta\beta')}\psi'' = (i\alpha_{\mathbf{d}\beta}\psi)(i\alpha_{\mathbf{d}\beta}\beta\psi') &= (i\theta(i\alpha_{\mathbf{d}\beta}\alpha_{\mathbf{d}\beta}^{-1}a\alpha_{\mathbf{r}\beta}\psi') \\ &= (i\theta)(ia\alpha_{\mathbf{d}\beta'}\psi') = (i\theta)(ia\theta') = i(\theta \cdot {}^a\theta'),\end{aligned}$$

$$a'' = \alpha_{\mathbf{d}(\beta\beta')}\beta\beta'\alpha_{\mathbf{r}(\beta\beta')}^{-1} = (\alpha_{\mathbf{d}\beta}\beta\alpha_{\mathbf{r}\beta}^{-1})(\alpha_{\mathbf{d}\beta'}\beta'\alpha_{\mathbf{r}\beta'}^{-1}) = aa',$$

which proves that $\chi$ is a homomorphism. It follows easily that $\chi$ is one-to-one and onto.

The group $G \text{ wr } \mathfrak{S}_n$ $(\cong D_n)$ is isomorphic to the $\mathscr{H}$-class of the identity of $\Omega(S)$ (pairs of linked invertible left and right translations); it appears in [**3**] under the name of "Loewy group".

**3. The main theorem.** Let $S = \mathscr{M}^0(G; I, I; \Delta)$ with $|I| = n$, $T = \mathscr{M}^0(K; J, J; \Delta)$. On $V = S \cup T^*$ we define a multiplication denoted by $*$ as follows. The zero of $S$ acts as the zero of $V$; for $s, s' \in S$, let $s * s' = ss'$, and for $t, t' \in T^*$ such that $tt' \neq 0$ in $T$, let $t * t' = tt'$. In order to express the remaining products we introduce a number of functions.

Fix a $k$, $1 \leqq k \leqq n$, and let $\mathbf{k} = \{B \subseteq I \mid |B| = k\}$. Fix $A \in \mathbf{k}$ and for each $B \in \mathbf{k}$, fix a one-to-one mapping $\alpha_B: A \to B$. By $\mathfrak{S}_k$ denote the symmetric group of degree $k$ and suppose that $\mathfrak{S}_k$ acts on $A$. Let $\sigma: J \to \mathbf{k}$ be a function satisfying $|i\sigma \cap j\sigma| \leqq 1$ if $i \neq j$. Let $\delta: J \to G \text{ wr } \mathfrak{S}_k$, where $j\delta = (\theta_j, s_j)$, and let $\omega: K \to G \text{ wr } \mathfrak{S}_k$ be a homomorphism where $a\omega = (\epsilon_a, t_a)$.

For $t = (a; i, j) \in T^*$, $u = (b; l, m) \in T^*$, $s = (c; p, q) \in S^*$ define

$$s * t = [c(q\alpha_{i\sigma}{}^{-1}\theta_i)(q\alpha_{i\sigma}{}^{-1}s_i\epsilon_a)(q\alpha_{i\sigma}{}^{-1}s_it_as_j{}^{-1}\theta_j)^{-1}; p, q\alpha_{i\sigma}{}^{-1}s_it_as_j{}^{-1}\alpha_{j\sigma}] \quad \text{if} \quad q \in i\sigma,$$

$$t * s = [(p\alpha_{j\sigma}{}^{-1}s_jt_a{}^{-1}s_i{}^{-1}\theta_i)(p\alpha_{j\sigma}{}^{-1}s_jt_a{}^{-1}\epsilon_a)(p\alpha_{j\sigma}{}^{-1}\theta_j)^{-1}c; p\alpha_{j\sigma}{}^{-1}s_jt_a{}^{-1}s_i{}^{-1}\alpha_{i\sigma}, q] \quad \text{if} \quad p \in j\sigma,$$

$$t * u = [d; r, v] \quad \text{if} \quad j\sigma \cap l\sigma = \{h\},$$

where

$$d = (h\alpha_{j\sigma}{}^{-1}s_jt_a{}^{-1}s^{-1}\theta_i)(h\alpha_{j\sigma}{}^{-1}s_jt_a{}^{-1}\epsilon_a)(h\alpha_{j\sigma}{}^{-1}\theta_j)^{-1}(h\alpha_{l\sigma}{}^{-1}\theta_j)(h\alpha_{l\sigma}{}^{-1}s_l\epsilon_b) \times (h\alpha_l{}^{-1}s_lt_bs_m{}^{-1}\theta_m)^{-1},$$

$$r = h\alpha_{j\sigma}{}^{-1}s_jt_a{}^{-1}s_i{}^{-1}\alpha_{i\sigma},$$

$$v = h\alpha_{l\sigma}{}^{-1}s_lt_bs_m{}^{-1}\alpha_{m\sigma};$$

in all other cases the product is 0.

THEOREM 2. *Let $S = \mathscr{M}^0(G; I, I; \Delta)$ and $T = \mathscr{M}^0(K; J, J; \Delta)$ be Brandt semigroups with the only restriction that $I$ be finite. On $V = S \cup T^*$ define a multiplication as indicated above. Then $V$ is an extension of $S$ by $T$. Conversely, every extension of $S$ by $T$, except for the orthogonal sum of $S$ and $T$, can be obtained in this way.*

*Proof.* We exclude the case of an orthogonal sum of $S$ and $T$. According to the Extension Theorem, an extension of $S$ by $T$ is determined by an extension function $\eta: T^* \to [\Omega(S)]^*$. Since $\Omega(S) \cong \tilde{P}(S)$, we must find a partial homomorphism $\xi: T^* \to \tilde{P}(S)$ satisfying $(a\xi)(b\xi) \in \Delta(S)$ for all $a, b \in T^*$ such that $ab = 0$. Since $T^*$ consists of a single $\mathscr{D}$-class, $\xi$ maps $T^*$ into a non-zero $\mathscr{D}$-class of $\tilde{P}(S)$, which by Theorem 1 implies that $T^*\xi \subseteq D_k$ for some $k$ such that $1 \leqq k \leqq n$. Thus $\xi$ defines a partial homomorphism of $T^*$ into $D_k$. With the homomorphism $\chi$ defined by (4) in § 2, letting $\bar{\chi} = \chi|_{D_k}$ and $\varphi = \xi\bar{\chi}$ we have that $\varphi: T^* \to \mathscr{M}^0(G \text{ wr } \mathfrak{S}_k; \mathbf{k}, \mathbf{k}; \Delta)$ is a partial homomorphism. According to [**1**, Theorem 3.14], $\varphi$ is given by: a function $\sigma: J \to \mathbf{k}$, a function $\delta: J \to G \text{ wr } \mathfrak{S}_k$, where $j\delta = (\theta_j, s_j)$, and a homomorphism $\omega: K \to G \text{ wr } \mathfrak{S}_k$, where $a\omega = (\epsilon_a, t_a)$. If $f$ is a function mapping a set $X$ into a group $G$, let $f^{-1}$ denote the function on $X$ defined by $xf^{-1} = (xf)^{-1}$. For any $(a; i, j) \in T^*$ we obtain

$$(5) \qquad \begin{aligned}(a; i, j)\varphi &= [(\theta_i, s_i)(\epsilon_a, t_a)(\theta_j, s_j)^{-1}; i\sigma, j\sigma] \\ &= [(\theta_i \cdot {}^{s_i}\epsilon_a, s_it_a)[({}^{s_j^{-1}}\theta_j)^{-1}, s_j{}^{-1}]; i\sigma, j\sigma] \\ &= [[\theta_i \cdot {}^{s_i}\epsilon_a \cdot {}^{s_it_a}({}^{s_j^{-1}}\theta_j)^{-1}, s_it_as_j{}^{-1}]; i\sigma, j\sigma].\end{aligned}$$

From the proof of Theorem 1, we deduce

$$(6)\qquad [(\theta, a); B, C]\chi^{-1} = (\alpha_B{}^{-1}a\alpha_C, \alpha_B{}^{-1}\theta).$$

On the other hand, $\xi = \varphi\bar{\chi}^{-1}: T^* \rightarrow D_k$ and by (5) and (6) we obtain

$$(7)\qquad (a; i, j)\xi = (a; i, j)\varphi\chi^{-1} = [\alpha_{i\sigma}{}^{-1}s_i t_a s_j{}^{-1}\alpha_{j\sigma}, \alpha_{i\sigma}{}^{-1}[\theta_i \cdot {}^{s_i}\epsilon_a \cdot {}^{s_i t_a}({}^{s_j{}^{-1}}\theta_j)^{-1}]] = (\beta, \psi)$$

for some $(\beta, \psi) \in D_k$. The condition: $ab = 0$ implies $(a\xi)(b\xi) \in \Delta(S)$ is equivalent to the condition: $i \neq j$ implies $|i\sigma \cap j\sigma| \leqq 1$. In fact, if the condition on $\xi$ is satisfied, letting 1 be the identity of $K$, for $i, j \in I$, $i \neq j$, we have

$$[(1; i, i)\xi][(1; j, j)\xi] = [(1; i, i)(1; j, j)]\xi = 0\xi = 0,$$

which together with the additional condition on $\xi$ by (7) implies

$$\operatorname{rank}[(\alpha_{i\sigma}{}^{-1}s_i t_1 s_i{}^{-1}\alpha_{i\sigma})(\alpha_{j\sigma}{}^{-1}s_j t_1 s_j{}^{-1}\alpha_{j\sigma})] \leqq 1,$$

which yields $|i\sigma \cap j\sigma| \leqq 1$.

Conversely, if $|i\sigma \cap j\sigma| \leqq 1$ whenever $i \neq j$, consider $[(a; i, j)\xi][(b; l, m)\xi]$, where $j \neq l$. By (7) we have

$$(8)\qquad [(a; i, j)\xi][(b; l, m)\xi] = [\alpha_{i\sigma}{}^{-1}s_i t_a s_j{}^{-1}\alpha_{j\sigma}, \quad][\alpha_{l\sigma}{}^{-1}s_l t_b s_m{}^{-1}\alpha_{m\sigma}, \quad] = [(\alpha_{i\sigma}{}^{-1}s_i t_a s_j{}^{-1}\alpha_{j\sigma})(\alpha_{lb}{}^{-1}s_l t_b s_m{}^{-1}\alpha_{m\sigma}), \quad],$$

where the blank spaces stand for expressions of no importance for this argument. Since $j \neq l$, by the hypothesis we have $|j\sigma \cap l\sigma| \leqq 1$. It follows that

$$\operatorname{rank}[(\alpha_{i\sigma}{}^{-1}s_i t_a s_j{}^{-1}\alpha_{j\sigma})(\alpha_{l\sigma}{}^{-1}s_l t_b s_m{}^{-1}\alpha_{m\sigma})] \leqq 1$$

which by [**8**, Theorem 4] implies that the element in (8) is contained in $\Delta(S)$. Consequently, $\xi$ satisfies the required condition and thus $\eta$ is an extension function of $T^*$ into $[\Omega(S)]^*$.

By the construction, $(a; i, j)\xi = \rho^{(a;i,j)} = (\beta, \psi)$. Since we are dealing with inverse semigroups, $\lambda^{(a;i,j)}$ is unique and is given by the pair $(\alpha, \phi)$, where $\alpha = \beta^{-1}$, $\phi i = (i\beta^{-1})\psi$ for all $i \in \mathbf{r}\beta$ [**9**, Theorem 1]. Recall from [**8**] that for $\lambda = (\alpha, \phi)$, $\rho = (\beta, \psi)$, we have for $(c; p, q) \in S$,

$$(c; p, q)\rho = [c(q\psi); p, q\beta) \quad \text{if } q \in \mathbf{d}\beta, \text{ and is 0 otherwise,}$$
$$\lambda(c; p, q) = [(\phi p)c; \alpha p, q] \quad \text{if } p \in \mathbf{d}\alpha, \text{ and is 0 otherwise,}$$
$$\lambda 0 = 0\rho = 0,$$
$$(\beta, \psi)(\beta', \psi') = (\beta\beta', \psi'') \quad \text{if } \beta\beta' \neq 0, \text{ and is 0 otherwise, where } i\psi'' = (i\psi)(i\beta\psi') \text{ for all } i \in \mathbf{d}(\beta\beta')$$
$$(\alpha, \phi)(\alpha', \phi') = (\alpha\alpha', \phi'') \quad \text{if } \alpha\alpha' \neq 0, \text{ and is 0 otherwise, where } \phi''i = (\phi\alpha i)(\phi' i) \text{ for all } i \in \mathbf{d}(\alpha\alpha').$$

Using the formulae for the $*$ multiplication in the Extension Theorem and the expressions just derived, we are able to give the multiplication in $V$ determined by $\eta$. A straightforward calculation of products in different cases leads to the results indicated at the beginning of § 3.

*Remarks on Theorem* 2. (a) In the case of $k = 1$, $V$ is simply an extension determined by a partial homomorphism $\nu \colon T^* \to S^*$. This extension is given by mappings $\sigma \colon J \to I$, $\delta \colon J \to G$ and a homomorphism $\omega \colon K \to G$, with

$$\begin{aligned} s * t &= [c(i\delta)(a\omega)(j\delta)^{-1}; p, j\sigma] && \text{if} \quad i\sigma = q, \\ t * s &= [(i\delta)(a\omega)(j\delta)^{-1}c; i\sigma, q] && \text{if} \quad j\sigma = p, \\ t * u &= [(i\delta)(a\omega)(j\delta)^{-1}(l\delta)(b\omega)(m\delta)^{-1}; i\sigma, m\sigma] && \text{if} \quad j\sigma = l\sigma \quad \text{and} \quad j \neq l. \end{aligned}$$

These formulae could have been obtained directly, applying [**1**, Theorem 3.14].

(b) In the general case, even though the expressions for products are rather complicated, these extensions are in fact easy to find. For, the parameters $\sigma$, $\delta$, $\omega$ are independent, the restriction on $\sigma$ amounts to finding a family $\mathscr{A}$ of $k$-tuples of elements of $I$ indexed by the elements of $J$ such that $|B \cap C| \leqq 1$ if $B, C \in \mathscr{A}$, $B \neq C$, while $\omega$ is a monomial group representation (see, e.g., [**3**]) and $\delta$ is a function subject to no restriction. Observe that for $k > 1$, $\sigma$ is one-to-one.

(c) Note that $V$ has an identity (i.e. $V$ is a unitary extension [**5**]) if and only if $k = n$. This is possible only if $T$ is a group with 0 ($|J| = 1$).

**4.** Using the Preston-Schützenberger representation of an inverse semigroup [**1**, Theorem 3.21], $V$ in Theorem 2 can be faithfully represented by matrices as the direct sum of Schützenberger representations relative to its non-zero $\mathscr{D}$-classes. The value of the Schützenberger representation of an element $(a; i, j) \in T^*$ relative to the $\mathscr{D}$-class $S^*$ is $\rho^{(a;i,j)} = (a; i, j)\xi = (\beta, \psi)$ written in matrix notation while the other representations either follow from [**1**, Theorem 3.17] or are 0. As a result we obtain matrices of the form

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\qquad\quad}^{n} & \overbrace{\qquad\quad}^{|J|} \end{array} \\ \begin{array}{r} n\left\{\vphantom{\Big|}\right. \\ |J|\left\{\vphantom{\Big|}\right. \end{array} & \left(\begin{array}{c|c} \qquad\quad & \qquad\quad \\ \hline & \end{array}\right) \end{array}$$

under the usual multiplication of matrices. Using the notation as in Theorem 2, we obtain the following matrices. For $s = (c; p, q) \in S^*$:

$$\begin{array}{cc} & \begin{array}{cc} q & \phantom{0} \end{array} \\ p & \left(\begin{array}{c|c} c & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}.$$

For $t = (a; i, j) \in T^*$:

$$\text{(i)}\quad \begin{array}{cc} \begin{array}{cc} \phantom{(m_{lk})} & j \end{array} & \\ \left(\begin{array}{c|c} (m_{lk}) & 0 \\ \hline 0 & a \end{array}\right) & i \end{array}, \qquad \text{where } \rho^{(a;i,j)} = (\beta, \psi), \quad m_{lk} = \begin{cases} l\psi & \text{if } l\beta = k, \\ 0 & \text{otherwise.} \end{cases}$$

In the particular case when $V$ is the orthogonal sum of $S$ and $T$, the matrix representing $t = (a; i, j)$ has the form (ii) below. In the case $k = 1$ (extension determined by a partial homomorphism $T^* \rightarrow S^*$) this matrix has the form (iii) with $g = (i\delta)(a\omega)(j\delta)^{-1}$:

$$\text{(ii)}\quad \begin{array}{cc} \begin{array}{cc} \phantom{0} & j \end{array} & \\ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & a \end{array}\right) & i \end{array}, \qquad \text{(iii)}\quad \begin{array}{ccc} & \begin{array}{cc} j\sigma & j \end{array} & \\ i\sigma & \left(\begin{array}{c|c} g & 0 \\ \hline 0 & a \end{array}\right) & i \end{array}.$$

From the condition that $\sigma$ be one-to-one for $k > 1$ and the hypothesis that $I$ be finite, it follows that $J$ is also finite so that the matrices in (i) are of finite format, whereas those in (ii) and (iii) need not be.

Theorem 2 can be easily generalized to yield all extensions of a Brandt semigroup $S$ with a finite number of idempotents by an orthogonal sum of Brandt semigroups $T_\mu$ (see [**4**, 5.17]) as follows. One constructs an extension $V_\mu$ of $S$ by $T_\mu$ for each $\mu$ with the requirement that if $\mu \neq \nu$ and the extensions $V_\mu$ and $V_\nu$ are pure, then $|i\sigma_\mu \cap j\sigma_\nu| \leqq 1$ (notation as in Theorem 2). For each $\mu$, multiplication in $V_\mu = S \cup T_\mu{}^*$ is already given, while the product of an element in $T_\mu{}^*$ by an element in $T_\nu{}^*$ for $\mu \neq \nu$ is defined similarly as at the beginning of § 3 (the case $t, u \in T^*$, $tu = 0$ in $T$). The conditions on $\sigma_\mu$ indicated here and the one in Theorem 2 can be combined to yield $|i\sigma_\mu \cap j\sigma_\nu| \leqq 1$ if $i \neq j$ or $\mu \neq \nu$. Conversely, every extension of $S$ by an orthogonal sum of Brandt semigroups can be obtained in this way. We omit the details.

We conclude by deriving certain properties of extensions concerning covering of idempotents and the existence of extensions. Let $S$ and $T$ be as in Theorem 2, where $|I| = n$ is kept fixed. For each $k$, $1 < k \leqq n$, let $\hat{k}$ be the greatest number of unordered $k$-tuples which can be placed in a set of $n$ elements such that two distinct $k$-tuples intersect in at most one element. The importance of $\hat{k}$ follows from the condition on $\sigma$ in Theorem 2. The following results are in fact corollaries to Theorem 2. The notation is the same as in Theorem 2 and its proof.

COROLLARY 1. *If the extension is pure and is determined by* $\xi: T^* \rightarrow D_k$, *then* $|J| \leqq \hat{k}$.

*Proof.* This follows from the condition on $\sigma$ in Theorem 2.

COROLLARY 2. *If the extension is pure, then* $1 \leqq |J| \leqq \binom{n}{2}$.

*Proof.* This follows from Theorem 2 where $k > 1$, Corollary 1, and

$$\binom{n}{2} = \hat{2} \geqq \hat{3} \geqq \ldots \geqq \hat{n} = 1.$$

COROLLARY 3. *Let* $1 < k \leqq n + 1$. *In order that every extension of* $S$ *by* $T$ *be of one of the two types* (i) *strict or* (ii) *pure and determined by* $\xi: T^* \to D_i$ *with* $i < k$, *it is necessary and sufficient that* $|J| > \hat{k}$.

*Proof. Necessity.* If $|J| \leqq \hat{k}$, then there is a pure extension determined by $\xi: T^* \to D_k$; for in Theorem 2, $\sigma$ can be found since $|J| \leqq \hat{k}$, while $\delta$ and $\omega$ can be taken at will.

*Sufficiency.* If the extension is pure and determined by $\eta: T^* \to D_i$ with $i \geqq k$, then $\hat{\imath} \leqq \hat{k}$. By Corollary 1, $|J| \leqq \hat{\imath}$ so that $|J| \leqq \hat{k}$, contradicting the hypothesis $|J| > \hat{k}$.

COROLLARY 4. *In order that every extension of* $S$ *by* $T$ *be strict* (*i.e., determined by a partial homomorphism of* $T^*$ *into* $S$), *it is necessary and sufficient that* $|J| > \binom{n}{2}$.

*Proof.* Take $k = 2$ in Corollary 3 and note that $\hat{2} = \binom{n}{2}$.

From the multiplication formulae in Theorem 2, we easily see that for idempotents $(f; j, j) \in T^*$ and $(e; i, i) \in S^*$, in $V$: $(f; j, j)$ covers $(e; i, i)$ if and only if $i \in j\sigma$ ($i = j\sigma$ if the extension is strict). Recall that in a partially ordered set $A$, *a covers b* if $a > b$ and $a > x > b$ for no $x \in A$ (see, e.g., [**5**] where this notion is extensively used). From Theorem 2 and the above corollaries, we easily deduce the following.

COROLLARY 5. *In* $V$, *every idempotent in* $T^*$ *covers the same number of idempotents in* $S^*$. *If this number is* $k$, *then*

(i) $k = 0$ *if and only if* $V$ *is an orthogonal sum of* $S$ *and* $T$;

(ii) $k \leqq 1$ *if and only if* $V$ *is a strict extension of* $S$ *by* $T$;

(iii) $k > 1$ *if and only if* $V$ *is a pure extension of* $S$ *by* $T$ *determined by* $\xi: T^* \to D_k$.

COROLLARY 6. *Let* $1 \leqq k \leqq n + 1$. *In every extension of* $S$ *by* $T$, *every idempotent in* $T^*$ *covers less than* $k$ *idempotents in* $S^*$ *if and only if* $T^*$ *has more than* $\hat{k}$ *idempotents.*

COROLLARY 7. *The following are equivalent:*

(i) *every extension of* $S$ *by* $T$ *is strict;*

(ii) *in every extension of* $S$ *by* $T$, *every idempotent in* $T^*$ *covers at most one idempotent in* $S^*$

(iii) $T^*$ *has more than* $\binom{n}{2}$ *idempotents.*

Note that for $k > \frac{1}{2}n$ if $n$ is even, and $k > \frac{1}{2}(n+1)$ if $n$ is odd, we have $\hat{k} = 1$. On the other hand, $\hat{2} = \binom{n}{2}$ so that $1 \leqq \hat{k} \leqq \binom{n}{2}$. The determination of $\hat{k}$ as a function of $k$ and $n$ appears to be a difficult combinatorial problem (cf. balanced incomplete block designs).

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