AN INTEGRAL OVER FUNCTION SPACE

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1. Introduction. Real functions

$$x(t) = \sum_0^\infty x_n t^n (-1 \leqslant t \leqslant 1) \text{ with } ||x||_1 = \sum_0^\infty |x_n| < \infty$$

may be identified with elements $x = (x_0, x_1, x_2, \ldots)$ of the sequence space l_1. Since the unit sphere S_∞ of l_1 is compact under the weak* topology[1] = topology of co-ordinatewise convergence, a countably additive measure on S_∞ is induced by a positive linear functional E (integral) on $C(S_\infty)$, the weak* continuous real-valued functions on S_∞. There exists a natural integral over S_∞ reducing to

$$E(f) = \frac{1}{2}\int_{-1}^{1} f(x_0)dx_0$$

when f is a function of x_0 alone. The partial sums $S_n = S_n(x)$ of the power series for $x(t)$ then form a martingale and zero-or-one phenomena appear. In particular, if $R(x)$ is the radius of convergence of the series and e is the base of the natural logarithms, it turns out that $R(x) = e$ for almost all x in S_∞. Applications of the integral to the theory of numerical integration, the original motivation, will appear in a later paper.

2. The integral. The linear space of real sequences $x = (x_0, x_1, x_2, \ldots)$ admits as subspaces the Banach spaces C_0, l_1, m consisting respectively of sequences x such that $\lim x_n = 0$ and

$$||x||_\infty = \sup_n |x_n|, \; ||x||_1 = \sum_0^\infty |x_n| < \infty,$$

and

$$||x||_\infty = \sup_n |x_n| < \infty.$$

We write $< x, y > = \sum_0^\infty x_n y_n$, whenever the series converges. Then $l_1 = C_0^*$, $m = l_1^*$, where * denotes the conjugate space as usual. Now the unit sphere $S_\infty = [x : ||x||_1 \leq 1]$ of l_1 is compact under the weak* topology of l_1. It is well known and readily verified that the weak* topology of S_∞ may be identified with the topology of co-ordinatewise convergence, which is induced by the metric $\rho(x, y) = \sum_0^\infty 2^{-n} |x_n - y_n|/(1 + |x_n - y_n|)$. Let $C(S_\infty)$ denote the Banach algebra and lattice of weak* continuous real-valued functions on S_∞ with

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[1]For the standard facts on linear spaces and integration employed in this paper, see **(3)**.

$$||f||_\infty = \sup_{x \in S_\infty} |f(x)|.$$

Now the elements $e_j = (\delta_{0j}, \delta_{1j}, \delta_{2j}, \ldots)$ $(j = 0, 1, 2, \ldots)$ (Kronecker delta) are just the extreme points of S_∞ and form a naturally ordered basis for l_1. The integral we define in terms of them is similar to one defined by Banach **(1)** over the unit sphere of the Hilbert space l_2, but Banach's integral is unsatisfactory in that Hilbert space admits no distinguished basis; every point on the boundary of the unit sphere is an extreme point.

DEFINITION. *If y is a sequence set $P_n y = (y_0, y_1, \ldots, y_n, 0, 0, \ldots)$. A function f on a sequence space is a cylinder function of degree n if $f(x) = f(P_n x)$ (all x). Let L_N denote the set of cylinder functions of degree N.*

The notion of a cylinder function f of degree n is just a precise form of the statement that f "depends on the first $n + 1$ variables only." It is clear that $L_0 \subset L_1 \subset L_2 \subset \ldots$. Set $L_\infty = \cup_n L_n$.

Consider now possible integrals on $C(S_\infty)$ which are such that

$$E(f) = \frac{1}{2}\int_{-1}^{1} f\, dx_0$$

whenever $f \in L_0 \cap C(S_\infty)$. Then $E(1) = 1$. The simplest way to extend E to $L_1 \cap C(S_\infty)$ is to set

$$E(f) = \frac{1}{2^2} \iint_{|x_0|+|x_1|\leqslant 1} \frac{f\, dx_1 dx_0}{1 - |x_0|}\ (f \in L_1 \cap C(S_\infty)).$$

When f is in $L_0 \cap C(S_\infty)$ this reduces to the previous definition. To extend E to $L_2 \cap C(S_\infty)$ set

$$E(f) = \frac{1}{2^3} \iiint_{|x_0|+|x_1|+|x_2|\leqslant 1} \frac{f\, dx_2 dx_1 dx_0}{[1 - |x_0|][1 - (|x_0| + |x_1|)]}.$$

This coincides with the previous definition on $L_1 \cap C(S_\infty)$.

It is clear that for f in the general $L_n \cap C(S_\infty)$ we must set

$$E(f) = \frac{1}{2^{n+1}} \int \ldots \int_{|x_0|+\ldots+|x_n|\leqslant 1} \frac{f\, dx_0 dx_1 \ldots dx_n}{[1 - |x_0|] \ldots [1 - (|x_0| + \ldots + |x_{n-1}|)]}.$$

Then E is well defined for all f in $L_\infty \cap C(S_\infty)$, the weak* continuous cylinder functions on S_∞. Moreover, it is clear that E is a positive linear functional with $||E|| = 1$.

Let x in S_∞ be arbitrary. Then

$$\rho(x, P_n x) = \sum_{j=n+1}^{\infty} 2^{-j}|x_j|/(1 + |x_j|) \leqslant 2^{-n}.$$

Since a continuous function on a compact metric space is uniformly continuous it follows that for every f in $C(S_\infty)$ the cylinder functions $f_n(x) = f(P_n x)$ approach $f(x)$ uniformly in x—that is,

$$\lim_n ||f - f_n||_\infty = 0.$$

Since $||E|| = 1$ on $L \cap C(S_\infty)$, it is clear that

$$\lim_n E(f_n)$$

exists and may be taken as the definition of $E(f)$. It follows that $E(f)$ is properly defined for all f in $C(S_\infty)$ by the formula

$$E(f) = \lim_{n\to\infty} \frac{1}{2^{n+1}} \int \ldots \int_{|x_0|+\ldots+|x_n|\leqslant 1} \frac{f(x_0, x_1, \ldots, x_n, 0, 0, \ldots)dx_0 \ldots dx_n}{[1 - |x_0|] \ldots [1 - (|x_0| + \ldots + |x_{n-1}|)]}.$$

It is convenient to introduce the notation $E(f) = \int_{S_\infty} f(x)\, d_E x$.

The integral E may now be extended in standard fashion and induces a countably additive measure. It is clear that the above formula serves to define E for bounded Baire functions f on S_∞.

3. Some integral formulae. We show first that the measure is concentrated on the (strong) boundary $[x:||x||_1 = 1]$ of S_∞. Let

$$Q_n^K = \frac{1}{2^{n+1}} \int \ldots \int_{|x_0|+\ldots+|x_n|\leqslant 1} \frac{[1 - (|x_0| + \ldots + |x_n|]^K dx_0 \ldots dx_n}{[1-|x_0|] \ldots [1 - (|x_0| + \ldots + |x_{n-1}|)]} \quad (K > -1).$$

It follows by induction that $Q_n^K = 1/(K+1)^{n+1}$. For

$$Q_0^K = \frac{1}{2}\int_{-1}^{1} (1 - |x_0|)^K dx_0 = \int_0^1 (1 - x_0)^K dx_0 = 1/(K+1),$$

while

$$Q_{m+1}^K = \frac{1}{2^{m+1}} \int \ldots \int_{|x_0|+\ldots+|x_m|\leqslant 1} \left\{ \int_0^{1-(|x_0|+\ldots+|x_m|)} \frac{[1 - (|x_0| + \ldots + |x_m|) - x_{m+1}]^K}{[1 - |x_0|] \ldots [1 - (|x_0| + \ldots + |x_m|)]} dx_{m+1} \right\} dx_0 \ldots dx_m$$

$$= \frac{1}{K+1} \frac{1}{2^{m+1}} \int \ldots \int_{|x_0|+\ldots+|x_m|\leqslant 1} \frac{[1 - (|x_0| + \ldots + |x_m|)]^K dx_0 \ldots dx_m}{[1 - |x_0|] \ldots [1 - (|x_0| + \ldots + |x_{m-1}|]}$$

$$= \frac{1}{K+1} Q_m^K = \left(\frac{1}{K+1}\right)^{m+2}.$$

Since $||x||_1$ is a bounded Baire function and

$$\int_{S_\infty} [1 - ||x||_1]^K d_E x = \lim_n Q_n^K,$$

THEOREM.

$$\int_{S_\infty} [1 - ||x||_1]^K d_E x = 0 \qquad (K > 0).$$

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Now $[1 - ||x||_1] > 0$ on the Borel set $[x:||x||_1 < 1]$. It follows that the measure is concentrated on the boundary $[x:||x||_1 = 1]$.

Consider now the projections x_n.

THEOREM.

$$\int_{S_\infty} |x_n|^K d_E x = \left(\frac{1}{K+1}\right)^{n+1} \qquad (K > -1).$$

Proof. The verification is direct if $n = 0$. If $n \geq 1$,

$$\int_{S_\infty} |x_n|^K d_E x = \frac{1}{2^{n+1}} \int \underset{|x_0|+\ldots+|x_n|\leqslant 1}{\ldots} \int \frac{|x_n|^K dx_0 \ldots dx_{n-1} dx_n}{[1 - |x_0|] \ldots [1 - (|x_0| + \ldots + |x_{n-1}|)]}$$

$$= \frac{1}{K+1} \cdot \frac{1}{2^n} \int \underset{|x_0|+\ldots+|x_{n-1}|\leqslant 1}{\ldots} \int \frac{[1 - (|x_0| + \ldots + |x_{n-1}|)]^K dx_0 \ldots dx_{n-1}}{[1 - |x_0|] \ldots [1 - (|x_0| + \ldots + |x_{n-2}|)]}$$

$$= \frac{1}{K+1} Q_{n-1}^K = \left(\frac{1}{K+1}\right)^{n+1}.$$

It is clear that $\int_{S_\infty} x_n d_E x = 0$. Now the expression $\langle x, y\rangle$ $(x \in S_\infty)$ is bounded and in the first Baire class for all $y \in m$. (It is well known that it is continuous in x if and only if $y \in C_0$.) Then

THEOREM.

$$\int_{S_\infty} \langle x, y\rangle d_E x = 0 \qquad (y \in m).$$

Consider now

$$\int_{S_\infty} \langle x, y\rangle^2 d_E x.$$

Since clearly

$$\int_{S_\infty} x_m x_n d_E x = 0 \quad \text{if} \quad m \neq n,$$

we have

$$\int_{S_\infty} \left[\sum_0^n x_m y_m\right]^2 d_E x = \sum_{m,l=0}^n y_m y_l \int_{S_\infty} x_m x_l d_E x = \sum_{m=0}^n y_m^2 \int_{S_\infty} x_m^2 d_E x = \sum_{m=0}^n \frac{y_m^2}{3^{m+1}}.$$

THEOREM.

$$\int_{S_\infty} \langle x, y\rangle^2 d_E x = \sum_{n=0}^\infty \frac{y_n^2}{3^{n+1}}$$

(*whenever the series converges*).

4. A martingale theorem. Now identify the elements $x = (x_0, x_0, x_1, \ldots)$ of S_∞ with the power series $x(t) = \sum_0^\infty x_n t^n$ converging absolutely on the unit circle. Then the partial sums $S_n = S_n(x)$ form a martingale: the defining conditions (cf. **2**) that

$$\int_{S_\infty} x_0 d_E x = 0$$

and

$$\int_{S_\infty} \varphi(x_0, x_1 t, \ldots, x_n t^n) x_{n+1} t^{n+1} d_E x = 0$$

for every bounded Baire function φ are clearly satisfied. It is natural to expect the appearance of zero-or-one phenomena. We single out the most striking. Let $R(x)$ be the radius of convergence of the power series for $x(t)$. Then

$$R(x) = 1/\overline{\lim_n} |x_n|^{1/n}$$

and $R(x) \geq 1$ by hypothesis.

THEOREM. *$R(x) = e$ for almost all x in S_∞.*

Proof.

$$\int [|x_n|^{1/n} - e^{-1}]^2 d_E x = \int_{S_\infty} [|x_n|^{2/n} - 2e^{-1}|x_n|^{1/n} + e^{-2}] d_E x$$

$$= \frac{1}{(1+2/n)^{n+1}} - \frac{2e^{-1}}{(1+1/n)^{n+1}} + e^{-2} \rightarrow e^{-2} - 2e^{-2} + e^{-2} = 0 \ (n \rightarrow \infty).$$

Thus

$$\lim_n |x_n|^{1/n} = e^{-1}$$

in $L^2(E)$. But then there exists a subsequence (n_j) such that

$$\lim_{n_j} |x_{n_j}|^{1/n_j} = e^{-1}$$

for almost all x in S_∞, implying that

$$\overline{\lim_n} |x_n|^{1/n} \geqslant e^{-1}$$

for almost all x. Hence $R(x) \leq e$ for almost all x.

To establish that $R(x) \geqq e$ for almost all x in S_∞ let $0 < r < e$. Since

$$e = \lim_n \left(1 + \frac{1}{n}\right)^n$$

there exists an integer M such that

$$r < \left(1 + \frac{1}{M}\right)^M$$

or

$$r^{1/M} < \left(1 + \frac{1}{M}\right).$$

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Set

$$f_n(x) = \left[\sum_0^n |x_m| r^m\right]^{1/M}.$$

Then

$$\int_{S_\infty} f_n(x) d_E x = \int_{S_\infty} \left[\sum_0^n |x_m| r^m\right]^{1/M} d_E x$$

$$\leqslant \int_{S_\infty} \left[\sum_0^n |x_m|^{1/M} r^{m/M}\right] d_E x$$

$$= \sum_0^n \frac{r^{m/M}}{(1 + 1/M)^{m+1}} = \frac{1}{(1 + 1/M)} \sum_0^n \left(\frac{r^{1/M}}{1 + 1/M}\right)^m$$

$$\leqslant \frac{1}{(1 + 1/M)} \sum_0^\infty \left(\frac{r^{1/M}}{1 + 1/M}\right)^m = A < \infty.$$

It follows from Fatou's lemma that

$$\left\{\sum_0^\infty |x_m| r^m\right\}^{1/M} = \lim_n f_n(x)$$

exists for almost all x in S_∞ and is integrable. Applying the above argument to a sequence $r_n \uparrow e$ and discarding a countable number of exceptional sets of measure 0, one for each r_n, we find that $R(x) \geq e$ for almost all x in S_∞.

References

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Correction to the Paper "Submethods of Regular Matrix Summability Methods"*

It has been pointed out to the authors by Dr. F. R. Keogh that the construction for the matrix C in Theorem III is incorrect.

*Casper Goffman and G. M. Petersen, Can. J. Math., *8* (1956), 40–46.