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CHARACTERISTIC CLASSES ON SYMMETRIC PRODUCTS OF CURVES

ISRAEL MORENO MEJÍA

Humboldt-Universität zu Berlin, Math.-Nat.-Fakultät II, Institut für Mathematik, Unter den Linden 6, D-10099 Berlin, Germany
e-mail: imm@dunelm.org.uk

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Abstract. Let X be a smooth complex projective curve, S^bX the b-symmetric product of X. Assume that X has an automorphism h. Our aim is to compute the characteristic classes of the fixed point set of h in S^bX.

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1. Introduction. Let h be a finite order automorphism of a smooth complex projective variety W. The components of the fixed point set of h are smooth subvarieties of W. Let N be the normal bundle of a component of the fixed point set of h. Then N has a decomposition $N = \oplus_{i=1}^{p-1} N(\nu^i)$ where $N(\nu^i)$ is a vector bundle on which h acts as ν^i, $\nu^i = e^{2\pi i/p}$ and p is the order of h. Each $N(\nu^i)$ has Chern classes and these Chern classes can be used to compute the so called *stable characteristic classes* of $N(\nu^i)$ defined as

$$\mathcal{U}(N(\nu^i)) = \prod_j \left(\frac{1 - \frac{e^{-x_j}}{\nu^i}}{1 - \frac{1}{\nu^i}} \right)^{-1}, \tag{1}$$

where $\{x_j\}$ are the Chern roots of $N(\nu^i)$. These characteristic classes are required if one wants to apply the *Holomorphic Lefschetz Theorem* (see [**2**, Theorem 4.6]). In this work we study the situation where W is the symmetric product S^bX of a complex curve X and h is an automorphism of X. The fixed point set turns out to be a disjoint union of varieties which are isomorphic to symmetric products S^kY, where Y is the quotient curve $X/\langle h\rangle$. We can compute these characteristic classes if $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of the automorphism group of X and our main result is the Theorem 3.8.

2. Fixed points in S^bX. Suppose that X is a curve with an automorphism h. Let p be the order of h. If we consider the map

$$\begin{aligned} f_{k,0} : S^kX &\to S^{pk}X \\ D &\mapsto \sum_{i=0}^{p-1} h^i D, \end{aligned}$$

then $f_{k,0}(S^k X)$ is a subset of fixed points of h in $S^{pk}X$. Let D be an effective divisor of degree d invariant under the action of h. Consider the embedding

$$\begin{aligned} \mathcal{A}_D : S^{pk}X &\hookrightarrow S^{pk+d}X \\ u &\longrightarrow u+D. \end{aligned}$$

The image of $S^k X$ under the map $f_{k,D} = \mathcal{A}_D \circ f_{k,0}$ is a subset of fixed points of h in $S^{pk+d}X$. Notice that when $k=0$ the image of $f_{0,D}$ is the divisor D.

We shall now describe the fixed point set $fix(h)$ of h in $S^b X$. Take integers $m,\ l$ such that

$$b = pm + l$$

and $m \geq 0$ and $p > l \geq 0$. For each integer k such that $m \geq k \geq 0$, let $d_k = b - kp$. Let $(S^{d_k}X)^h$ denote the fixed point set of h in $S^{d_k}X$. Define A_k as the set of divisors $D \in (S^{d_k}X)^h$ satisfying the following property: if $x \in X$ is a point in the support of D then $D - \sum_{i=0}^{p-1} h^i x$ is not an effective divisor nor the zero divisor.

Now consider the set

$$F_k = \bigcup_{D \in A_k} f_{k,D}(S^k X).$$

Notice that $F_i \cap F_j = \emptyset$ and $f_{D_1}(S^i X) \cap f_{D_2}(S^i X) = \emptyset$ for $D_1, D_2 \in A_i$. It is easy to verify the following result.

LEMMA 2.1.

$$\bigcup_{k=0}^{m} F_k = fix(h).$$

Notice that if p is a prime number then

$$A_k = \left\{ D = a_1 x_1 + \cdots + a_s x_s \mid 0 \leq a_j \leq p-1 \quad \text{and} \quad \sum_{j=1}^{s} a_j = d_k \right\},$$

where $x_1, \ldots, x_s$ are the fixed points of h in X and there are

$$\sum_{j=0}^{m-k} (-1)^j \binom{s}{j} \binom{s-1+d_k-jp}{d_k-jp} \tag{2}$$

divisors in A_k.

If p is not a prime number then the divisors in A_k are not necessarily supported on the fixed points of h in X. For instance there are situations in which h has no fixed points in X but h^2 has finitely many.

Let $f : X \to Y$ be a morphism of degree p of smooth curves. Then there is an embedding

$$i : S^k Y \to S^{pk}X$$

that sends $D \in S^k Y$ to the divisor $f^*D \in S^{pk}X$. If we take f to be the quotient map $f : X \to X/\langle h \rangle = Y$, then the map $f_{k,0}$ splits as

$$f_{k,0} : S^k X \overset{a}{\twoheadrightarrow} S^k Y \overset{i}{\hookrightarrow} S^{pk}X,$$

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where a is the natural map induced by f on the symmetric product. From this we see that the fixed point set of h in S^bX is a disjoint union of varieties which are isomorphic to symmetric products of the quotient curve Y.

We refer to [**6**] for the definition of the cohomology classes η, σ_i on the symmetric product of a curve. Let $\vartheta = \sum_{i=1}^{g} \sigma_i$. The proof of the following Lemma involves at least two different symmetric products and we shall use the same notation to represent these cohomology classes regardless of the symmetric product on which they are defined as this will be clear from the context.

LEMMA 2.2. *Consider the induced map* $i^* : H^*(S^{pk}X, \mathbb{Z}) \to H^*(S^kY, \mathbb{Z})$. *Then we have* $i^*\eta = \eta$ *and* $i^*\vartheta = p\vartheta$.

Proof. Consider the maps

$$f_{k,0}^* : H^*(S^{pk}X, \mathbb{Z}) \xrightarrow{i^*} H^*(S^kY, \mathbb{Z}) \xrightarrow{a^*} H^*(S^kX, \mathbb{Z}).$$

We will first show that a^* is injective and that $a^*\eta = p\eta$, then we will see that $f_{k,0}^*\vartheta = pa^*\vartheta$ and $f_{k,0}^*\eta = p\eta$. From this we deduce that $i^*\eta = \eta$ and $i^*\vartheta = p\vartheta$.

Notice that the natural map $f^* : H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C})$ is injective by (1.2) in [**6**]. The commutative diagram

$$\begin{array}{ccc} X^k & \longrightarrow & Y^k \\ \downarrow & & \downarrow \\ S^kX & \to & S^kY \end{array}$$

induces another commutative diagram

$$\begin{array}{ccc} H^*(S^kY, \mathbb{C}) & \xrightarrow{a^*} & H^*(S^kX, \mathbb{C}) \\ \downarrow & & \downarrow \\ H^*(Y^k, \mathbb{C}) & \longrightarrow & H^*(X^k, \mathbb{C}) \end{array}$$

where the vertical maps and the lower map are injective; therefore a^* is injective. Now fix a symplectic basis

$$\alpha_1, \ldots, \alpha_\gamma, \alpha_{\gamma+1}, \ldots, \alpha_{2\gamma}$$

for $H_1(Y, \mathbb{Z})$. Above each cycle α_i there are p cycles $r_i, hr_i, \ldots, h^{p-1}r_i$ on X, and they satisfy

$$h^m r_i h^l r_j = \begin{cases} r_i r_j = \alpha_i \alpha_j & \text{if } m = l \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The set $\mathcal{A} = \{h^m r_i \mid m = 0, \ldots, p-1 \text{ and } i = 1, \ldots, 2\gamma\}$ forms part of a symplectic basis

$$\mathcal{B} = \{\alpha'_1, \ldots, \alpha'_g, \alpha'_{g+1}, \ldots, \alpha'_{2g}\}$$

of $H_1(X, \mathbb{Z})$ in which

$$\alpha'_m = h^j r_{q+1}$$
$$\alpha'_{m+g} = h^j r_{q+1+\gamma}$$

for $m = qp + j$, where $1 \leq j \leq p$ and $0 \leq q \leq \gamma - 1$. Abusing our notation we shall write α_m instead of α'_m. Consider the map

$$f^* : H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z}).$$

Under this map we have

$$f^*\alpha_i = \sum_{j=1}^{p-1} h^j r_i = \sum_{j=0}^{p-1} \alpha_{j+p(i-1)} \tag{4}$$

and

$$f^*\alpha_{i+\gamma} = \sum_{j=0}^{p-1} h^j r_{i+\gamma} = \sum_{j=1}^{p-1} \alpha_{j+p(i-1)+g}.$$

One can check that if $\alpha_i \in \mathcal{B} \setminus \mathcal{A}$ then $\sum_{j=0}^{p-1} h^j \alpha_i = 0$. We refer to **[6]** for the definition of the cohomology classes β, $\alpha_{i,l}$, $\beta_{i,l}$, and ξ_i.

Using the relations (3) we have

$$f^*(\beta) = (\alpha_i \alpha_{i+\gamma}) = \left(\sum_{j=0}^{p-1} h^j r_i\right)\left(\sum_{j=0}^{p-1} h^j r_{i+\gamma}\right) = p\beta. \tag{5}$$

Now under the map

$$a^* : H^*(Y^k, \mathbb{Z}) \to H^*(X^k, \mathbb{Z})$$

we have from (4) that

$$a^*(\alpha_{i,l}) = \sum_{j=0}^{p-1} \alpha_{j+p(i-1),l}$$

and

$$a^*(\beta_l) = p\beta_l.$$

Suppose that $i \leq \gamma$. Then, using the definition of ξ_i, we get

$$a^*(\xi_i) = \sum_{l=1}^{k} \sum_{j=0}^{p-1} \alpha_{j+p(i-1),l}.$$

Notice from the definition of $\mathcal{B}$ that

$$\sum_{r=0}^{p-1} h^r \alpha_{pi,l} = \sum_{j=1}^{p} \alpha_{j+p(i-1),l}.$$

Then

$$a^*(\xi_i) = \sum_{j=0}^{p-1} h^j \xi_{pi}$$

and

$$a^*(\xi_{i+\gamma}) = \sum_{l=1}^{k}\sum_{j=0}^{p-1}\alpha_{j+p(i-1)+g,l} = \sum_{j=0}^{p-1} h^j \xi_{pi+g}.$$

From the definition of η and from (5) we have

$$a^*(\eta) = \sum_{l=1}^{k} p\beta_l = p\eta.$$

Consider the map

$$f_{k,0}^* : H^*(X^{pk}, \mathbb{Z}) \to H^*(X^k, \mathbb{Z}).$$

In this case $f_{k,0} : X^k \to (X^k)^p$ is defined by the rule $D \mapsto \left(D, hD, \dots, h^{p-1}D\right)$. Now we shall compute $f_{k,0}^*(\xi_m)$. We first compute $f_{k,0}^*(\alpha_{il})$. Notice that

$$H^*(X^{pk}, \mathbb{Z}) = H^*(X, \mathbb{Z})^{\otimes pk} = H^*(X^k, \mathbb{Z})^{\otimes p}.$$

In particular

$$H^1(X^{pk}, \mathbb{Z}) = \bigoplus_{i=1}^{p} \underbrace{H^0(X^k, \mathbb{Z}) \otimes \cdots \otimes H^0(X^k, \mathbb{Z}) \otimes H^1(X^k, \mathbb{Z})}_{i^{th}\ place} \otimes H^0(X^k, \mathbb{Z}) \otimes \cdots \otimes H^0(X^k, \mathbb{Z}).$$

$$\left(\cong \bigoplus_{i=1}^{p} H^1(X^k, \mathbb{Z}).\right)$$

Suppose that $l = sk + j$, where s, j are non-negative integers and $1 \le j \le k$. We now can see that

$$f_{k,0}^*(\alpha_{il}) = h^s \alpha_{i,j}$$

and

$$f_{k,0}^*(\beta_l) = h^s(\alpha_{i,j}\alpha_{i+g,j}) = h^s \beta_j = \beta_j.$$

Thus

$$f_{k,0}^*(\eta) = \sum_{l=1}^{pk} f_{k,0}^*(\beta_l) = p\sum_{j=1}^{k} \beta_j = p\eta$$

and

$$f_{k,0}^*(\xi_i) = f_{k,0}^*\left(\sum_{l=1}^{pk} \alpha_{i,l}\right) = \sum_{s=0}^{p-1} h^s \sum_{j=1}^{k} \alpha_{i,j} = \sum_{j=0}^{p-1} h^j \xi_i = \sum_{j=1}^{k}\sum_{s=0}^{p-1} h^s \alpha_{i,j}.$$

From this we see that if $\alpha_m \in \mathcal{B} \setminus \mathcal{A}$ then

$$f_{k,0}^*(\xi_m) = 0.$$

Let $m \le g$ such that $\alpha_m \in \mathcal{A}$. Write $m = qp + j$ with $1 \le j \le p$. So

$$f_{k,0}^*(\xi_m) = \sum_{j=0}^{p-1} h^j \xi_{p(q+1)} = a^*(\xi_{q+1})$$

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and

$$f^*_{k,0}(\xi_{m+g}) = \sum_{j=0}^{p-1} h^j \xi_{p(q+1)+\gamma} = a^*(\xi_{q+1+\gamma}).$$

Then we have

$$f^*_{k,0}(\vartheta) = \sum_{m=1}^{g} f^*_{k,0}(\xi_m \xi_{(m+g)}) = p \sum_{q=0}^{\gamma-1} a^*\left(\xi_{(q+1)}\xi_{(q+1)+\gamma}\right) = pa^*(\vartheta). \qquad \square$$

3. Normal bundles of the fixed point sets. Now we shall consider the normal bundles of the components of the fixed point set of h in $S^b X$. The aim will be to find a way to compute the characteristic classes of their eigenvector bundles as defined in (1). Consider the quotient map

$$f : X \to X/\langle h \rangle = Y.$$

Let g and γ be the genus of X and Y respectively and let R be the ramification divisor of f in X. From section 2 we know that a component of dimension k is the image of $S^k X$ under the map $f_{k,D}$ for some $D \in A_k$ and we identify it with $S^k Y$; the embedding of $S^k Y$ into $S^b X$ is given by the composition map

$$S^k Y \overset{i}{\hookrightarrow} S^{pk} X \overset{\mathcal{A}_D}{\hookrightarrow} S^{pk+d_k} X. \tag{6}$$

We shall use the following notation:

- N_i for the normal bundle of $S^k Y$ in $S^{pk} X$,
- $N_{\mathcal{A}_D \circ i}$ for the normal bundle of $S^k Y$ in $S^{pk+d_k} X$ and
- $N_{\mathcal{A}_D}$ for the normal bundle of $S^{pk} X$ in $S^{pk+d_k} X$.

The total Chern class of N_i is given by

$$c(N_i) = \frac{i^* c\left(S^{pk} X\right)}{c\left(S^k Y\right)}.$$

The Chern class of $S^d X$ is given by

$$(1 + \eta\, t)^{(d-g+1)}\, e^{(-\frac{\vartheta\, t}{1+\eta\, t})} \tag{7}$$

where g is the genus of X (see [**1**, p. 339]). Using formula (7) and Lemma 2.2 we obtain

$$c(N_i) = \left((1 + \eta\, t)^A\, e^{(-\frac{\vartheta\, t}{1+\eta\, t})}\right)^{p-1}, \tag{8}$$

where

$$A = \left(k + \frac{\gamma - g}{p-1}\right) = k + 1 - \gamma - \frac{\deg(R)}{2(p-1)}.$$

In particular when $k = 1$ the normal bundle has degree

$$(p-1)\left(2 - 2\gamma - \frac{\deg(R)}{2(p-1)}\right).$$

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Let $D \in S^d X$. Using (7) and that $\mathcal{A}_D^* \eta = \eta$ and $\mathcal{A}_D^* \vartheta = \vartheta$ we see that the normal bundle $N_{\mathcal{A}_D}$ has total Chern class

$$c(N_{\mathcal{A}_D}) = (1+\eta)^d. \tag{9}$$

LEMMA 3.1. *Let $x \in X$ be a fixed point of an automorphism h of X of order p. Let d be a positive integer and let $Q = dx \in S^d X$. Suppose that h acts as ν^a on the tangent space $T_{X,x}$ of X at x. Then $h_|(T_{S^d X})_Q$ has eigenvalues $\nu^a, \nu^{2a}, \dots, \nu^{da}$, where $\nu = e^{2i\pi/p}$.*

Proof. In a neighbourhood of $(x, x, \dots, x) \in X^d$ choose coordinates $(x_1, \dots, x_d)$ so that $(x, x, \dots, x)$ is in the origin. Then in a neighbourhood of $Q = dx \in S^d X$ there are coordinates $(\sigma_1, \dots, \sigma_d)$ (see [**1**, chap. IV, section 2]) defined by the property that the natural morphism

$$X^d \to S^d X$$

is given in a neighbourhood of $(x, x, \dots, x)$ by

$$\sigma_i(x_1, \dots, x_d) = ith \text{ symmetric function of } (x_1, \dots, x_d).$$

Now, if x is a fixed point of an automorphism h of X, we can assume that in a neighbourhood of x, the action of h is the multiplication by a scalar λ. Then in our system of coordinates

$$hx_i = \lambda x_i$$

implies

$$h\sigma_i = \lambda^i \sigma_i. \qquad \square$$

Using similar arguments one can proof the following two Lemmas.

LEMMA 3.2. *Consider the map $f_{k,0}$ of section 2. Let*

$$Q = x + hx + \cdots + h^{p-1}x$$

be a point in the image of this map (that means $x \in S^k X$). Then $h\,|_{(T_{S^{pk}X})_Q}$ has eigenvalues $1, \nu, \dots \nu^{p-1}$, $\nu = e^{2i\pi/p}$, and the eigenspace of ν^i has dimension k.

LEMMA 3.3. *Consider the divisor $D = d_1x_1 + \cdots + d_sx_s$, where x_i is a fixed point of h in X $(x_i \neq x_j)$ and d_i is a positive integer. Suppose that h acts as ν^{a_i} (notice $(a_i, p) = 1$) on the tangent space T_{X,x_i}. Consider the composition map (6). Let $Q = x + hx + \cdots + h^{p-1}x + D$ be a point in the image of this map. Then the dimension of the eigenspace for ν^i of $h \mid (T_{S^{pk+d}X})_Q$ is $k + r_i$, where r_i is the number of times that ν^i appears in the following list*

$$\begin{gathered} \nu^{a_1}, \nu^{2a_1}, \dots, \nu^{d_1a_1}, \\ \nu^{a_2}, \nu^{2a_2}, \dots, \nu^{d_2a_2}, \\ \vdots \\ \nu^{a_s}, \nu^{2a_s}, \dots, \nu^{d_sa_s}. \end{gathered}$$

The normal bundle $N_{\mathcal{A}_D \circ i}$ has a decomposition

$$N_{\mathcal{A}_D \circ i} = \bigoplus_{j=0}^{p-1} N_{\mathcal{A}_D \circ i}(\nu^j).$$

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We will need to know the Chern classes of the vector bundles $N_{\mathcal{A}_D \circ i}(\nu^j)$ in order to compute their characteristic classes. We have an exact sequence

$$0 \to N_i \to N_{\mathcal{A}_D \circ i} \to i^* N_{\mathcal{A}_D} \to 0$$

from which we obtain exact sequences

$$0 \to N_i(\nu^j) \to N_{\mathcal{A}_D \circ i}(\nu^j) \to i^* N_{\mathcal{A}_D}(\nu^j) \to 0. \tag{10}$$

DEFINITION. Given $D \in A_k$, the class of D is the vector $(r_1, \ldots, r_{p-1})$, where r_j is the rank of $i^* N_{\mathcal{A}_D}(\nu^j)$.

REMARK 3.4. Consider the exact sequence

$$0 \to T_{S^k Y} \to (\mathcal{A}_D \circ i)^* T_{S^{nk+d} X} \to N_{\mathcal{A}_D \circ i} \to 0.$$

Since h acts trivially on $T_{S^k Y}$ we have

$$(\mathcal{A}_D \circ i)^* T_{S^{nk+d} X}(\nu^j) \cong N_{\mathcal{A}_D \circ i}(\nu^j)$$

for $\nu^j \neq 1$. Let $(r_1, \ldots, r_{p-1})$ be the class of D. Notice from Lemma 3.2 and from the exact sequence (10) that

$$rank\,((\mathcal{A}_D \circ i)^* T_{S^{nk+d} X}(\nu^j)) = k + r_j.$$

If D is supported on the fixed points of h in X, say $D = d_1 x_1 + \cdots + d_s x_s$, then from Lemma 3.3 we see that r_j is the number of times that ν^j appears in the list

$$\begin{gathered} \nu^{a_1}, \nu^{2a_1}, \ldots, \nu^{d_1 a_1}, \\ \nu^{a_2}, \nu^{2a_2}, \ldots, \nu^{d_2 a_2}, \\ \vdots \\ \nu^{a_s}, \nu^{2a_s}, \ldots, \nu^{d_s a_s}. \end{gathered}$$

LEMMA 3.5. *Let r be the rank of* $(i^* N_{\mathcal{A}_D})(\nu^j)$. *Then*

$$c((i^* N_{\mathcal{A}_D})(\nu^j)) = (1+\eta)^r.$$

Proof. It is enough to notice that the Chern class of $i^* N_{\mathcal{A}_D}$ is $(1+\eta)^d$ by (9) and Lemma 2.2. □

LEMMA 3.6. *Suppose that h is an automorphism of X of order p such that h is conjugate to* h^j *in* $Aut(X)$. *Then* $c(N_i(\nu^s)) = c(N_i(\nu^{sj}))$.

Proof. Let h be an automorphism of a variety W and suppose that $h^j = u^{-1} h u$ where $u \in Aut(W)$. If Z is a subvariety contained in the fixed point set of h at W such that u acts on Z, then the action of u on the tangent bundles of W and Z extends to an action on the normal bundle $N_{Z/W}$. From this one can see that under the isomorphism $u: N_{Z/W} \to N_{Z/W}$ the eigenvector bundle $N_{Z/W}(\nu^s)$ is mapped to $N_{Z/W}(\nu^{sj})$.

In our case the embedding of $Z = S^k Y$ into $W = S^{pk} X$ is equivariant with respect to u because $f_{k,0}(ux) = \sum_{i=0}^{p-1} u h^{ij} x$ and, since $(p, j) = 1$, this is equal to $u f_{k,0}(x)$. Notice that the composition map (6) is not necessarily equivariant with respect to u. □

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LEMMA 3.7. *Let h be an automorphism of X of prime order p. Assume that $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of $Aut(X)$. Then*

$$\mathcal{U}(N_i(\nu)) = \left(1-\frac{1}{\nu}\right)^{A}\left(1-\frac{e^{-t\eta}}{\nu}\right)^{-A} e^{t\vartheta\left(\frac{e^{-t\eta}}{\nu-e^{-t\eta}}\right)},$$

where $\nu \neq 1$ is a power of $e^{2i\pi/p}$ and A is given by (8). *In particular*

$$\begin{aligned}\prod_{j=1}^{p-1}\mathcal{U}(N_i(\nu^j)) &= p^A m(e^{-\eta t})^{-A} e^{t\vartheta q(e^{-\eta t})}\\ &= p^A m(e^{-\eta t})^{-A}\sum_{i=0}^{\gamma}\frac{\left(t\vartheta q(e^{-\eta t})\right)^i}{i!},\end{aligned} \tag{11}$$

where $m(z) = \sum_{i=0}^{p-1} z^i$, and $q(z) = -\frac{zm'(z)}{m(z)}$.

Proof. Using Lemma 3.6 and (8), the Chern class of $N_i(\nu^s)$ is given by

$$(1+t\eta)^A e^{\left(\frac{-\vartheta t}{1+\eta t}\right)}.$$

The last can be written as

$$(1+t\eta)^{A-\gamma}\prod_{i=1}^{\gamma}(1+t\eta-t\sigma_i).$$

So using (1), the characteristic class of $N_i(\nu)$ is given by

$$\mathcal{U}(N_i(\nu)) = \left(\frac{1-\frac{e^{-t\eta}}{\nu}}{1-\frac{1}{\nu}}\right)^{\gamma-A}\prod_{i=1}^{\gamma}\left(\frac{1-\frac{e^{t\sigma_i-t\eta}}{\nu}}{1-\frac{1}{\nu}}\right)^{-1}.$$

From (5.4) in **[6]** (or see proof of Proposition 10.1 (3) in **[7]**), we have

$$1-\frac{e^{t\sigma_i-t\eta}}{\nu} = \frac{\nu-e^{-t\eta}-e^{-t\eta}t\sigma_i}{\nu} = \left(\frac{\nu-e^{-t\eta}}{\nu}\right)\left(1-\frac{e^{-t\eta}t\sigma_i}{\nu-e^{-t\eta}}\right).$$

Then

$$\begin{aligned}\mathcal{U}(N_i(\nu)) &= \left(\frac{1-\frac{e^{-t\eta}}{\nu}}{1-\frac{1}{\nu}}\right)^{\gamma-A}\left(\frac{\nu-e^{-t\eta}}{\nu\left(1-\frac{1}{\nu}\right)}\right)^{-\gamma}\prod_{i=1}^{\gamma}\left(1-\frac{e^{-t\eta}t\sigma_i}{\nu-e^{-t\eta}}\right)^{-1}\\ &= \left(1-\frac{1}{\nu}\right)^{A}\left(1-\frac{e^{-t\eta}}{\nu}\right)^{-A} e^{t\vartheta\left(\frac{e^{-t\eta}}{\nu-e^{-t\eta}}\right)}.\end{aligned}$$

□

Now the following result is clear.

THEOREM 3.8. *Let h be an automorphism of X of prime order p. Assume that $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of $Aut(X)$. Let N be the normal bundle of a component*

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of the fixed point set of h in S^bX corresponding to the divisor $D \in A_k$. Then

$$\prod_{j=1}^{p-1} \mathcal{U}(N(\nu^j)) = p^A m(e^{-\eta t})^{-A} e^{t\vartheta q(e^{-\eta t})} \prod_{j=1}^{p-1} \left(\frac{1 - \frac{e^{-\eta t}}{\nu^j}}{1 - \frac{1}{\nu^j}} \right)^{-r_j},$$

where $m(z) = \sum_{i=0}^{p-1} z^i$, $q(z) = -\frac{zm'(z)}{m(z)}$ *and* $(r_1, \ldots, r_{p-1})$ *is the class of D.*

We have no formula to compute $\mathcal{U}(N_i(\nu^j))$ for any $h \in Aut(X)$, not even in the case that h has prime order (unless it satisfies the condition that $\langle h \rangle \setminus \{1\}$ is contained in a conjugacy class of $Aut(X)$). In what follows we will explain a way to compute $\mathcal{U}(N_i(\nu^j))$ when enough information about the quotient map $f : X \to Y$ is known and for the case in which N_i is the normal bundle of the curve Y in S^pX under the embedding

$$i : Y \hookrightarrow S^pX.$$

.

Lemma 3.9. *Let $f : X \to Y$ be a degree p morphism of smooth curves. We have*

$$i^* T_{S^pX} \cong f_* f^*(K_Y^{-1}) = K_Y^{-1} \otimes f_* \mathcal{O}_X.$$

Proof. Consider the graph map

$$\Gamma : X \to X \times Y$$
$$x \mapsto (x, f(x)).$$

Let Δ be the universal divisor of degree p on X. Consider the following diagram

$$\begin{tikzcd}
 & & \Delta \arrow[d, phantom, "\cap"] \\
X \cong \Delta' \arrow[r, hook] & X \times Y \arrow[r, "Id_X \times i"] \arrow[d, "\phi"] & X \times S^pX \arrow[d, "\pi"] \\
 & Y \arrow[r, "i"] & S^pX
\end{tikzcd}$$

where $\phi : X \times Y \to Y$ and $\pi : X \times S^pX \to S^pX$ are the natural projections and Δ' denotes $(1_X \times i)^*(\Delta)$. By [**1**, IV, Lemma 2.1], we have $\Delta' = \Gamma(X) \cong X$. Thus by the adjunction formula we have

$$\mathcal{O}_{\Delta'}(\Delta') \cong f^* K_Y^{-1}.$$

Now from [**1**, IV, section 2], we have $i^*\pi_*\mathcal{O}_\Delta(\Delta) \cong \phi_*\mathcal{O}_{\Delta'}(\Delta')$ and $\pi_*\mathcal{O}_\Delta(\Delta) \cong T_{S^pX}$. □

Using the following lemma we can compute the degrees of the eigen line bundles of $i^*T_{S^nX}$ and since $i^*T_{S^nX}(\nu^j) \cong N_i(\nu^j)$ for $\nu^j \neq 1$, that is all we need to compute $\mathcal{U}(N_i(\nu^j))$. Let Z be a smooth projective variety defined over $\mathbb{C}$ and let $\mathcal{L}$ be a line bundle on Z such that a positive power $\mathcal{L}^p$ admits a global section s and its corresponding divisor D has normal crossings. Write D as $C + \sum a_j E_j$ where C denotes the components of multiplicity 1 and E_j is a component of multiplicity a_j.

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For every real number x, $[x]$ represents the integral part of x, defined as the only integer such that

$$[x] \leq x < [x]+1.$$

Consider the line bundles

$$\mathcal{L}^{(i)} = \mathcal{L}^{i} \otimes \mathcal{O}_Z\left(-\sum_j \left[\frac{ia_j}{p}\right] E_j\right). \tag{12}$$

The sheaf of $\mathcal{O}_Z$ modules

$$\bigoplus_{i=0}^{p-1} \mathcal{L}^{-i}$$

admits a structure of $\mathcal{O}_Z$-algebra, given by the inclusion

$$s^{\vee} : \mathcal{L}^{-p} \hookrightarrow \mathcal{O}_Z.$$

Let

$$Z' = Spec_Z\left(\bigoplus_{i=0}^{p-1} \mathcal{L}^{-i}\right),$$

let $\tau' : Z' \to Z$ be the associated morphism and $n : \overline{Z} \to Z'$ the normalization of Z' and τ the composition of n and τ'.

LEMMA 3.10. *With the previous notation we have*

$$\tau_* \mathcal{O}_{\overline{Z}} = \bigoplus_{i=0}^{p-1} \mathcal{L}^{(i)^{-1}}.$$

Moreover τ is a Galois cyclic cover of degree p, then we have an automorphism h of $\overline{Z}$ which acts on $\tau_*\mathcal{O}_{\overline{Z}}$ and h acts as multiplication by ν^i on $\mathcal{L}^{(i)^{-1}}$, where $\nu = e^{2\pi I/p}$. If $\overline{Z}$ is irreducible then $\overline{Z}$ is nothing but the normalization of Z in $K(Z)(\sqrt[p]{f})$, where $K(Z)$ is the function field of Z and f is a rational function giving the section s.

Proof. See [**4**, Lemma 2]. □

EXAMPLE 1. Let X be the Klein quartic curve (see [**3**]). If h is an automorphism of order 7 then we have that h, h^2, h^4 are in the same conjugacy class whereas h^3, h^5, h^6 belong to another conjugacy class of $Aut(X) = \mathbb{P}SL_2(\mathbf{F}_7)$. In this case we have

$$X/\langle h\rangle \cong \mathbb{P}^1$$

and if we consider the normal bundle N of

$$i : \mathbb{P}^1 \hookrightarrow S^7X$$

then we see that the eigenvector bundles of N do not have the same degree because the number A in formula (8) is not an integer.

Let $G_2 = \langle h\rangle$ and let G_1 be the normalizer of G_2 in $\mathbb{P}SL_2(\mathbf{F}_7)$. One can imitate the steps followed by Macbeath in [**5**] to compute the equation of X and then one notices that X can be constructed by adding a seventh root of a polynomial

$q(z) = (z - a)^4(\omega z - a)^2(\omega^2 z - a)$ (where $\omega = e^{2\pi i/3}$) to $\mathbb{C}(z)$ (one can also use the formula (2.2) from [**3**]). The divisor defined by $q(z)$ in $\mathbb{P}^1$ has the form $4p_0 + 2p_1 + p_2$, then by Lemma 3.9 and by Lemma 3.10 we have that

$$N \cong K_{\mathbb{P}^1}^{-1} \otimes \bigoplus_{i=1}^{6} \mathcal{L}^{(i)^{-1}}.$$

Using (12) we see that

$$\mathcal{L}^{(i)^{-1}} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) & \text{for} \quad i = 1, 2, 4 \\ \mathcal{O}_{\mathbb{P}^1}(-2) & \qquad\; i = 3, 5, 6. \end{cases}$$

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