LETTERS TO THE EDITOR

ON THE ATTAINED WAITING TIME

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Abstract

By using properties of up- and downcrossings of the sample functions of
the workload process and of the attained waiting-time process for a $G/G/1$
queueing model, a direct proof of a theorem proved by Sakasegawa and
Wolff is given.

WORKLOAD PROCESS; SAMPLE FUNCTIONS; STATIONARY DISTRIBUTIONS

Sakasegawa and Wolff (1990) show by using sample function arguments that for the FIFO
$G/G/1$ queueing model the workload process $v_t$ and the attained waiting-time process $\eta_t$
possess the same stationary distribution, if such distributions exist. However their proof is
somewhat artificial (see their use of preemptive LIFO).

A direct proof of their Theorem 1 proceeds as follows. Consider a busy cycle $c$ with $n$ the
number of customers served; $r_1, \ldots, r_n$ are the service times of these customers, $w_1, \ldots, w_n$
their successive actual waiting times, $i$ the idle time, so

\begin{equation}
    c = r_1 + \cdots + r_n + i.
\end{equation}

The attained service time $\eta_t$ at epoch $t$ is by definition the time between $t$ and the arrival
epoch of the customer being served at epoch $t$. In Figure 1 the sample function of the
workload process $v_t$ and the corresponding $\eta_t$-process during the busy cycle $c$ are shown, with
$n = 4$.

Define for $v \geq 0$,

\begin{align}
    d(v) &:= \text{# downcrossings of } v_t, \ 0 \leq t \leq c \text{ with level } v, (\ast) \\
    u(v) &:= \text{# upcrossings of } v_t, \ 0 \leq t \leq c \text{ with level } v, (\circ) \\
    \delta(v) &:= \text{# upcrossings of } \eta_t, \ 0 \leq t \leq c \text{ with level } v, (\ast) \\
    \omega(v) &:= \text{# downcrossings of } \eta_t, \ 0 \leq t \leq c \text{ with level } v, (\circ).
\end{align}

Note that in the figure $d(v) = 3$; the upcrossings are there indicated by $\circ$, the downcrossings
by $\ast$. It is immediately evident from the geometry of the sample functions (see Cohen (1977),
(1982)) that with probability 1, for $v \geq 0$,

\begin{align}
    d(v) &= u(v), \quad \delta(v) = \omega(v), \\
    u(v) &= \delta(v); \quad \omega(v) = \delta(v),
\end{align}

and

\begin{align}
    d(v) &= \frac{d}{dv} \int_0^c (v_t < v) dt, \quad \delta(v) = \frac{d}{dv} \int_0^c (\eta_t < v) dt,
\end{align}

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where we use the notation

\[ v < v = 1_{v < v} \quad \text{and} \quad \int_0^c (v < v) \, dt = \int_0^c (v < v, c \equiv t) \, dt, \]

for the indicator function and the integral. Since

\[ i = \left\{ \int_0^c (v < v) \, dt \right\}_{v=0^+} = \left\{ \int_0^c (\eta < v) \, dt \right\}_{v=0^+}, \]

integration of (6), using the boundary conditions (8) yields, via (4) and (5), that with probability 1

\[ \int_0^c (v < v) \, dt = \int_0^c (\eta < v) \, dt, \quad v \equiv 0. \]

Because

\[ (v < v) = 1 - (v \geq v), \]

we have from (9)

\[ \int_0^c (v \leq v) \, dt = \int_0^c (\eta \leq v) \, dt, \]

which is Theorem 1 of Sakasegawa and Wolff (1990).

References