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Regular isomorphism of Markov chains is almost topological

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Abstract. Two Markov chains are regularly isomorphic if and only if they have a common Markov extension by right closing block codes of degree one. A certain ideal class in the integral group ring of the ratio group associated to a Markov chain is a new invariant of regular isomorphism and some other coding relations.

1. *Introduction*

The object of this paper is to show that two Markov chains are regularly isomorphic if and only if they have a common extension by right closing block codes of degree 1. Before turning to our discussion of this result, we recall some related terminology and facts.

Let M be an irreducible stochastic matrix. Suppose M is $k\times k$ and let $\{1,\ldots,k\}$ have the discrete topology. Give $\prod_{-\infty}^{\infty}\{1,\ldots,k\}$ the product topology, and let X be the subspace consisting of those $x=(x_n)\in\prod_{-\infty}^{\infty}\{1,\ldots,k\}$ with $M(x_n,x_{n+1})>0$ for all $n\in\mathbb{Z}$. Define the left shift homeomorphism $S:X\to X$ by taking $(Sx)_n=x_{n+1}$ for all $x\in X$ and $n\in\mathbb{Z}$. The pair (X,S) is a *subshift of finite type.* Regarding the shift S as being implicit, we often refer to X as a subshift of finite type. The *alphabet* of (X,S) is $\mathscr{A}(X)=\{1,\ldots,k\}$. An element of $\mathscr{A}(X)$ is a *symbol* of X. If $i_0,i_1,\ldots,i_{l-1}\in\mathscr{A}(X)$ are such that $M(i_0,i_1)$, $M(i_1,i_2),\ldots,M(i_{l-2},i_{l-1})>0$, then the string $i_0i_1\cdots i_{l-1}$ is called a *word* (or *block*) of length l; when there is a need to emphasize X, we use the terms *X-word* and *X-block.* The closed-open sets

$$[i_0i_1\cdots i_{l-1}]_r=\{x=(x_n)\in X: x_r=i_0, x_{r+1}=i_1,\ldots,x_{r+l-1}=i_{l-1}\}$$

are *cylinders.* (When $r=0$, we drop this subscript.) A base for the topology of X is given by these sets. The *state partition* of X consists of the cylinders $[i]$, $i\in\mathscr{A}(X)$. For $x\in X$ and $n\in\mathbb{Z}$, we denote by $x_{-\infty}^{n}$ the one-sided sequence $\cdots x_{n-2}x_{n-1}x_n$. Let m be the unique probability vector with $mM=m$. The matrix M gives an S-invariant

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Borel probability measure on X. This measure, which we also denote by m, is defined by setting

$$m([i_0 i_1 \cdots i_{l-1}]_r) = m(i_0)M(i_0, i_1) \cdots M(i_{l-2}, i_{l-1}).$$

The triple (X, S, m) is the *Markov chain* defined by M. (Thus, we restrict our attention to irreducible Markov chains.) As usual, we associate to (X, S, m) a directed graph with k vertices. There is an edge from vertex i to vertex j provided $M(i,j) > 0$; the transition probability $M(i,j)$ is then assigned to this edge. We say that $j \in \mathcal{A}(X)$ is a *follower* of $i \in \mathcal{A}(X)$ whenever $M(i,j) > 0$. A Markov chain is completely described by giving its alphabet and, for each symbol, specifying its followers and the associated transition probabilities.

Fix two (irreducible) Markov chains (X, S, m) and (Y, T, p). A *block code* $\phi: (X, S, m) \to (Y, T, p)$ is a continuous surjection $\phi: X \to Y$ such that $\phi S = T\phi$ and $m \circ \phi^{-1} = p$. Up to composition with a power of the shift, every block code $\phi: (X, S, m) \to (Y, T, p)$ may be expressed as an l-block code for some $l \in \mathbb{N}$ in the following way: there exists a map, which we also denote by ϕ, from X-blocks of length l onto $\mathcal{A}(Y)$ such that $\phi(x)_n = \phi(x_n x_{n+1} \cdots x_{n+l-1})$ for all $x \in X$ and $n \in \mathbb{Z}$. The block code ϕ is *bounded-to-one* if there exists a number K with card $(\phi^{-1}(y)) \leq K$ for all $y \in Y$. In this case, there exists an integer d such that card $(\phi^{-1}(y)) = d$ for all doubly transitive $y \in Y$; this integer d is called the *degree* of ϕ. When ϕ is a 1-block code and bounded-to-one, a necessary and sufficient condition for $d = 1$ is the existence of a Y-word $j_0 j_1 \cdots j_{l-1}$ such that $\phi^{-1}[j_0 j_1 \cdots j_{l-1}] \subset [i]_r$ for some $i \in \mathcal{A}(X)$ and $r \in \mathbb{Z}$ (see [1]); $j_0 j_1 \cdots j_{l-1}$ is then called a *magic word* for ϕ. A 1-block code $\phi: (X, S, m) \to (Y, T, p)$ is called *right resolving* if, for each $i \in \mathcal{A}(X)$, the map $\phi: \mathcal{A}(X) \to \mathcal{A}(Y)$ restricts to a bijection from the followers of i onto those of $\phi(i)$. It is easy to see that a right resolving code is bounded-to-one. A block code $\phi: (X, S, m) \to (Y, T, p)$ is *right closing* if, for each $x \in X$, its image $\phi(x)$ and its past $x_{-\infty}^0 = \cdots x_{-2}x_{-1}x_0$ determine x. Right closing codes are necessarily bounded-to-one. In fact, according to [**6**] (or, see [**2**]), a block code is right closing if and only if it is topologically conjugate to a right resolving code.

We identify all measure-theoretic objects which are equal almost everywhere. We use the usual notation for refinements of partitions and σ-algebras generated by partitions (see [**17**]).

Let α and β be the state partitions of (X, S, m) and (Y, T, p), respectively. Write $\alpha^- = \bigvee_{n=0}^{\infty} S^n\alpha$ and $\beta^- = \bigvee_{n=0}^{\infty} T^n\beta$. A *regular isomorphism* between (X, S, m) and (Y, T, p) is a measure-theoretic isomorphism $\phi: (X, S, m) \to (Y, T, p)$ such that

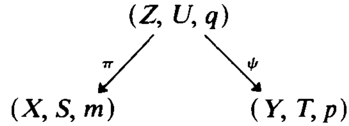
$$\phi^{-1}(\beta^-) \subset S^{-N}\alpha^- = \alpha^- \vee S^{-1}\alpha \vee \cdots \vee S^{-N}\alpha,$$

$$\phi(\alpha^-) \subset T^{-N}\beta^- = \beta^- \vee T^{-1}\beta \vee \cdots \vee T^{-N}\beta$$

for some non-negative integer N (cf. [**5**]). Our main result is the following.

THEOREM. *The Markov chains (X, S, m) and (Y, T, p) are regularly isomorphic if and only if there exists a Markov chain (Z, U, q) and right closing codes $\pi: (Z, U, q) \to (X, S, m)$ and $\psi: (Z, U, q) \to (Y, T, p)$ of degree* 1.

The situation described by the theorem may be pictured



where each of π and ψ is right closing and of degree 1. It is easy to see that the map $\psi \circ \pi^{-1} : (X, S, m) \to (Y, T, p)$, defined for all doubly transitive points, is a regular isomorphism. (See [**1, 4, 6**].) The theorem asserts, conversely, that the existence of a regular isomorphism ϕ implies the above picture. The body of the paper, in the next two sections, is devoted to the proof of this statement. First we give the construction for a common extension by right closing maps. Then we explore the choices involved in the construction to provide magic words.

We list [**3, 5, 8, 12, 15**] for examples of material on regular isomorphism. A special case of the theorem, that dealing with measures of maximal entropy, was established in [**3**] with the aid of dimension groups and other algebraic ideals. The proof we give for the general case is direct.

Our definition of regular isomorphism agrees with [**3**] and is a time-reversal of the definition used in earlier work. Much of the earlier work dealt with invariants. Our final section is a brief discussion of this topic, and includes an ideal class invariant similar to the one employed in [**3**] and [**14**].

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2. *Common right closing extensions*

Let $\phi : (X, S, m) \to (Y, T, p)$ be a regular isomorphism. We describe, in this section, the construction of a Markov chain (Z, U, q) with a right resolving map $\pi : (Z, U, q) \to (X, S, m)$ and a right closing map $\psi : (Z, U, q) \to (Y, T, p)$.

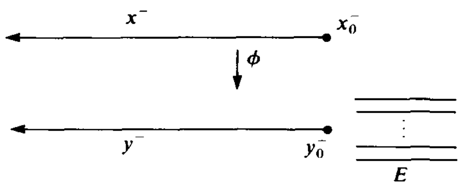
We continue to denote the state partition of (X, S, m) by α, atoms of the σ-algebra $\alpha^- = \bigvee_{n=0}^{\infty} S^n \alpha$ correspond to (allowable) one-sided sequences $x^- = \cdots x_{-2} x_{-1} x_0$; let X^- be the space of all such sequences x^-. Corresponding to a point $x^- \in X^-$ we have the set $[x^-] = \{x \in X : x_{-\infty}^0 = x^-\}$ in X. More generally, we write $[x^-]_n = \{x \in X : x_{-\infty}^n = x^-\}$. X^- is equipped with the restriction of m (to α^-). Similarly, we have the space Y^-, given by the atoms of β^-.

By the definition of a regular isomorphism, there exists a non-negative integer N with $\phi^{-1}(\beta^-) \subset S^{-N}(\alpha^-)$ and $\phi(\alpha^-) \subset T^{-N}(\beta^-)$. Thus, $S^N \circ \phi^{-1}(\beta^-) \subset \alpha^-$ and $\phi \circ S^{-N}(\alpha^-) \subset T^{-2N}(\beta^-)$. Write ϕ for $\phi \circ S^{-N}$ and N for $2N$. We then have $\phi^{-1}(\beta^-) \subset \alpha^-$ and $\phi(\alpha^-) \subset T^{-N}(\beta^-) = \beta^- \vee T^{-1}\beta \vee \cdots \vee T^{-N}\beta$, and the theory of Lebesgue spaces [**13**] implies the following

LEMMA 1. *There exist* (a.e. *defined*) *maps* $\bar{\phi} : X^- \to Y^-$ *and* $\overline{\phi^{-1}} : Y^- \to X^-$ *such that* $\bar{\phi}(x_{-\infty}^n)$, $\overline{\phi^{-1}}(y_{-\infty}^{n+N})$ *are defined and* $\phi(x)_{-\infty}^n = \bar{\phi}(x_{-\infty}^n)$, $\phi^{-1}(y)_{-\infty}^n = \overline{\phi^{-1}}(y_{-\infty}^{n+N})$ *for all* $n \in \mathbb{Z}$ *and* a.a. $x \in X$, $y \in Y$.

Without further ado, we restrict our attention to those points of X^-, X, Y^-, Y on which the maps considered in Lemma 1 are defined. In addition, we restrict to $x^- \in X^-$ with the property that the conditional measure $m_{[x^-]}$ on $[x^-]$ is sent by ϕ to the conditional measure $p_{\phi[x^-]}$ on $\phi[x^-] = \phi\{x \in X : x^0_{-\infty} = x^-\}$. Thus we work on good sets of measure 1 and do not need to make any further explicit reference to Lemma 1. Preferring to use only ϕ in our formulations, we will not make any reference to the maps $\bar{\phi}$ and $\overline{\phi^{-1}}$ either.

Consider $x^- \in X^-$. Since $\beta^- \subset \phi(\alpha^-) \subset \beta^- \vee T^{-1}\beta \vee \cdots \vee T^{-N}\beta$, there exist $y^- \in Y^-$ and a set E in $\bigvee_{n=1}^{N} T^{-n}\beta$ such that $\phi[x^-] = [y^-] \cap E$. Since E corresponds to a finite collection of Y-words of length N, this may be pictured as follows.



Distinguishing the zero coordinates of x^- and y^-, we put $f(x^-) = (x_0^-, y_0^-, E)$. This defines a map $f: X^- \to \mathscr{A}(X) \times \mathscr{A}(Y) \times \bigvee_{n=1}^{N} T^{-n}\beta$. Let $\mathscr{A}$ be the essential image of f. When $(i, j, E) \in \mathscr{A}$, we write $[i, j, E]$ for the subset of X corresponding to $f^{-1}(i, j, E) \subset X^-$, and put $[i, j, E]_n = S^{-n}([i, j, E])$.

$\mathscr{A}$ will give us the alphabet of the common extension Z. To define transitions, let $(i, j, E) \in \mathscr{A}$. Fix $x^- \in f^{-1}(i, j, E)$. For each follower i' of i we define a transition from (i, j, E): Concatenate x^- and i' to consider $x^-i' \in X^-$, allow a transition from (i, j, E) to $(i', j', E') = f(x^-i')$, and assign the probability

$$Q((i, j, E), (i', j', E')) = M(i, i')$$

to this transition. Choosing an irreducible component of full topological entropy, let (Z, U, q) be the resulting Markov chain. Define 1-block maps $\pi: Z \to X$ and $\psi: Z \to Y$ by setting $\pi(i, j, E) = i$ and $\psi(i, j, E) = j$. It is clear from our definition of transitions and Q that π is then a surjective resolving map and $q \circ \pi^{-1} = m$. (See Lemma 16 of [**10**].) The following lemma will be used to show that ψ is right closing.

LEMMA 2. *If* $(i_0, j_0, E_0) \in \mathscr{A}$ *and* $j_0 j_1 \cdots j_N j_{N+1}$ *is a* Y*-word with* $[j_1 \cdots j_N]_1 \subset E_0$ *then there exist* $i_1 \in \mathscr{A}(X)$ *and a set* E_1 *in* $\bigvee_{n=1}^{N} T^{-n}\beta$ *such that* $(i_1, j_1, E_1) \in \mathscr{A}$ *and* (i_1, j_1, E_1) *is the unique follower of* (i_0, j_0, E_0) *with* $[j_2 \cdots j_{N+1}]_1 \subset E_1$.

Proof. Let $x^- \in f^{-1}(i_0, j_0, E_0)$ be the point used in the definition of transitions from (i_0, j_0, E_0), and let $y^- \in Y^-$ be such that $\phi[x^-] = [y^-] \cap E_0$. Then $x_0^- = i_0$ and $y_0^- = j_0$. Since $[j_1 \cdots j_N]_1 \subset E_0$, we have $\phi^{-1}([y^-] \cap [j_1 \cdots j_N j_{N+1}]_1) \subset [x^-]$. Moreover, because $\phi(\alpha^- \vee S^{-1}\alpha) \subset \beta^- \vee T^{-1}\beta \vee \cdots \vee T^{-N}\beta \vee T^{-(N+1)}\beta$, there exists a follower i_1 of i_0 such that $\phi^{-1}([y^-] \cap [j_1 \cdots j_N j_{N+1}]_1) \subset [x^- i_1]_1$. Hence

$$\phi[x^- i_1] = [y^- j_1] \cap E_1,$$

where E_1 is a set in $\bigvee_{n=1}^{N} T^{-n}\beta$ with $[j_2 \cdots j_{N+1}]_1 \subset E_1$. Since the followers of (i_0, j_0, E_0) are determined by letting the followers of i_0 partition the set $[x^-]$, the resulting $(i_1, j_1, E_1) \in \mathscr{A}$ is the only follower of (i_0, j_0, E_0) with $[j_2 \cdots j_{N+1}]_1 \subset E_1$. □

Now let $y \in Y$ and let $z^- \in Z^-$ be such that $\psi(z^-) = y^0_{-\infty}$. To verify that ψ is right closing, it suffices to show that there exists a unique point $z \in Z$ with $\psi(z) = y$ and $z_n = z_n^-$ for all $n \leq -N$. (See [**6**], Proposition 1 of [**2**], or (5.1) of [**4**].) Write $z_n^- = (x_n, y_n, E_n)$ for $n \leq 0$. Observe that for any Z-word $(i_0, j_0, F_0) \cdots (i_N, j_N, F_N)$ of length $N+1$ we must have $[j_1 \cdots j_N]_1 \subset F_0(*)$. In particular, we have $[y_{-N+1} \cdots y_{-1} y_0]_1 \subset E_{-N}$. Apply Lemma 2 to (x_{-N}, y_{-N}, E_{-N}) and $y_{-N} y_{-N+1} \cdots y_0 y_1$ to find the unique follower $z_{-N+1} = (i_{-N+1}, y_{-N+1}, F_{-N+1})$ of (x_{-N}, y_{-N}, E_{-N}) with $[y_{-N+2} \cdots y_0 y_1]_1 \subset F_{-N+1}$. Now apply Lemma 2 to $(i_{-N+1}, y_{-N+1}, F_{-N+1})$ and $y_{-N+1} y_{N+2} \cdots y_1 y_2$ to find the unique follower $z_{-N+2} = (i_{-N+2}, y_{-N+2}, F_{-N+2})$ of z_{-N+1} with $[y_{-N+3} \cdots y_1 y_2]_1 \subset F_{-N+2}$. Continue in this way to produce, for $n > -N$, $z_n = (i_n, y_n, F_n)$ with $[y_{n+1} \cdots y_{n+N}]_1 \subset F_n$. Putting $z_n = z_n^-$ for $n \leq -N$, the resulting point $z \in Z$ has $\psi(z) = y$ and $z_n = z_n^-$ for $n \leq -N$; $(*)$ and the uniqueness in Lemma 2 show that this is the only such point in Z.

Thus, ψ is right closing. We now prove:

LEMMA 3. $q \circ \psi^{-1} = p$.

Proof. The statement $q \circ \psi^{-1} = p$ is equivalent to the existence of $v(i, j, E) > 0$, $(i, j, E) \in \mathscr{A}$, such that

$$\frac{P(j, j')v(i', j', E')}{v(i, j, E)} = Q((i, j, E), (i', j', E')) = M(i, i')$$

whenever transition from (i, j, E) to (i', j', E') is allowed. (See [**15**], and Lemma 15 of [**10**].) For $(i, j, E) \in \mathscr{A}$, put $v(i, j, E) = p(E \mid [j])$, the conditional probability of the set E in $\bigvee_{n=1}^{N} T^{-n}\beta$, given the set $[j]$ of β. To show that this choice does the job, suppose we have a transition from (i, j, E) to (i', j', E'). As before, let $x^- \in X^-$ be the point used to define transitions from (i, j, E) and let $y^- \in Y^-$ be such that $\phi[x^-] = [y^-] \cap E$. Recall that x^- is such that $m_{[x^-]} = p_{\phi[x^-]} \circ \phi$. In particular:

$$m_{[x^-]}([x^- i']_1) = p_{\phi[x^-]}(\phi[x^- i']_1). \qquad (**)$$

Since $x_0^- = i$,

$$m_{[x^-]}([x^- i']_1) = M(i, i').$$

On the other hand, by the definition of transitions from (i, j, E),

$$\phi([x^- i']_1) = [y^- j']_1 \cap T^{-1}(E').$$

Using this fact and $\phi[x^-] = [y^-] \cap E$ we have

$$p_{\phi[x^-]}(\phi[x^- i']_1) = P(j, j') p_{[y^- j']}(E') / p_{[y^-]}(E).$$

By the Markov property of p we have $p_{[y^-]}(E) = p(E \mid [j]) = v(i, j, E)$ and $p_{[y^- j']}(E') = p(E' \mid [j']) = v(i', j', E')$, so the equation $(**)$ shows

$$M(i, i') = P(j, j') v(i', j', E') / v(i, j, E),$$

as desired. □

We have now established the existence of a Markov chain (Z, U, q) with a right resolving map $\pi: (Z, U, q) \to (X, S, m)$ and a right closing map $\psi: (Z, U, q) \to$

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(Y, T, p). At this point, the degrees of π and ψ may be greater than 1. We obtained Z by choosing, for each $(i, j, E) \in \mathcal{A}$, a point $x^- \in f^{-1}(i, j, E)$ and using x^- to define the transitions out of (i, j, E). The extension (Z, U, q) and the maps π, ψ depend critically on the x^- chosen. We will exploit this choice to obtain degree 1 maps. Before we can do this, however, we need to have a little more room, which we create with a slight generalization of the construction of (Z, U, q):

For $k \geq 0$, let $\mathcal{A}_k$ be the set of all triples $(i_{-k} \cdots i_{-1} i_0, j_{-k} \cdots j_{-1} j_0, E)$, where $i_{-k} \cdots i_{-1} i_0$ is an X-word, $j_{-k} \cdots j_{-1} j_0$ is a Y-word and $(i_0, j_0, E) \in \mathcal{A}$. We refer to $i_{-k} \cdots i_{-1} i_0$ as the X-coordinate of $(i_{-k} \cdots i_{-1} i_0, j_{-k} \cdots j_{-1} j_0, E)$. Similarly, $j_{-k} \cdots j_{-1} j_0$ is the Y-coordinate. We can define a Markov chain as follows. For each $(i_{-k} \cdots i_{-1} i_0, j_{-k} \cdots j_{-1} j_0, E) \in \mathcal{A}_k$, choose a point $x^- \in f^{-1}(i_0, j_0, E)$ and use x^- to determine the transitions out of $(i_{-k} \cdots i_0, j_{-k} \cdots j_0, E)$. (We will refer to this procedure as the selection of a transition rule for $(i_{-k} \cdots i_0, j_{-k} \cdots j_0, E)$.) Note that we do not require $x^-_{-k} \cdots x^-_{-1}$ to equal $i_{-k} \cdots i_{-1}$. If we let (Z, U, q) be the Markov chain given by an irreducible component of full entropy and put $\pi(i_{-k} \cdots i_0, j_{-k} \cdots j_0, E) = i_0$, $\psi(i_{-k} \cdots i_0, j_{-k} \cdots j_0, E) = j_0$ then, by the arguments given above (for the case $k = 0$), we obtain a right resolving block code $\pi : (Z, U, q) \to (X, S, m)$ and a right closing one $\psi : (Z, U, q) \to (Y, T, p)$. In the next section we show that, for large k, the transition rules may be chosen so that there exist magic words for π and ψ.

3. *Magic words*

Fix an element $(x_0, y_0, F_0) \in \mathcal{A}$.

LEMMA 4. *For every sufficiently large n, there exists an X-word* $x_{-n} \cdots x_0$ *such that*

$$m([x_{-n}, j, E]_{-n} \cap [x_{-n} \cdots x_0]_{-n} \cap [x_0, y_0, F_0]) > 0$$

whenever $(x_{-n}, j, E) \in \mathcal{A}$.

Proof. Consider an X-word $x_{-n} \cdots x_0$ terminating at x_0. Observe that, for $(x_{-n}, j, E) \in \mathcal{A}$ we have

$$\frac{m([x_{-n}, j, E] \cap [x_{-n} \cdots x_0])}{m[x_{-n} \cdots x_0]} = \frac{m[x_{-n}, j, E]}{m[x_{-n}]},$$

since $[x_{-n}, j, E] \in \alpha^-$. Hence, as $x_{-n} \cdots x_0$ and j, E run through all possibilities, the above ratios yield finitely many positive numbers. Let $\varepsilon > 0$ be such that 2ε is less than each of these numbers. Approximating $[x_0, y_0, F_0] \in \alpha^-$ by cylinders, now pick $x_{-n} \cdots x_0$ such that

$$m([x_{-n} \cdots x_0]_{-n} \cap [x_0, y_0, F_0]) / m([x_{-n} \cdots x_0]_{-n}) > 1 - \varepsilon.$$

Considering the conditional probability on $[x_{-n} \cdots x_0]_{-n}$, we then have

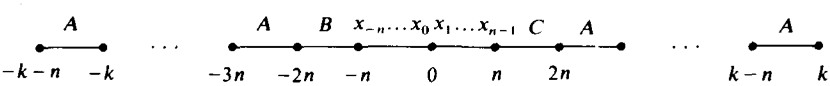
$$m([x_{-n}, j, E]_{-n} \cap [x_{-n} \cdots x_0]_{-n} \cap [x_0, y_0, F_0]) / m([x_{-n} \cdots x_0]_{-n}) > \varepsilon$$

whenever $(x_{-n}, j, E) \in \mathcal{A}$. Furthermore, for any $l > 0$, we can extend $x_{-n} \cdots x_0$ to find a word $x_{-n-l} \cdots x_{-n-1} x_{-n} \cdots x_0$ for which the analogous statement is valid. □

Suppose it happened, in Lemma 4, that the symbols appearing in $x_{-n}\cdots x_0$ were distinct. Then we could use $\mathscr{A}$ for the alphabet of the extension and turn $x_{-n}\cdots x_0$ into a magic word for π: We order the finite set $\{(i,j,E)\in\mathscr{A}: i=x_{-n}\}$ and, going through its elements (x_{-n},j,E) in order, choose transition rules for (x_{-n},j,E) and, perhaps, some other elements of $\mathscr{A}$. We make a list of the elements of $\mathscr{A}$ that we consider (choose a transition rule for); initially, this list is empty. For each (x_{-n},j,E), pick x lying in the non-trivial intersection in Lemma 4, and put $x^-=x_{-\infty}^{-n}$. Let x^- determine the transition rule for (x_{-n},j,E) and let x^-x_{-n+1}, $x^-x_{-n+1}x_{-n+2},\ldots$ determine the transition rule for $f(x^-x_{-n+1}), f(x^-x_{-n+1}x_{-n+2}),\ldots$ respectively, until either $(x_0,y_0,F_0)=f(x^-x_{-n+1}\cdots x_{-1}x_0)$ or a previously considered element of $\mathscr{A}$ is reached. Since the symbols of $x_{-n}\cdots x_0$ are distinct, whenever a previously considered state is reached before (x_0,y_0,F_0), we are in phase with the transition rules already selected and these take us to (x_0,y_0,F_0). Thus $x_{-n}\cdots x_0$ becomes a magic word for π.

In general, of course, the symbols of $x_{-n}\cdots x_0$ will not be distinct. We will use the alphabet $\mathscr{A}_k$ in dealing with the general case. For convenience, we will take k to be a multiple of n, and make use of cycles of length n. We assume that n itself is a large enough multiple of the common period of X and Y to guarantee the existence of the cycles (and words) we use.

Let $A=x_{-n}i_1\cdots i_{n-1}$ be such that $x_{-n}i_1\cdots i_{n-1}x_{-n}$ is an X-word and the corresponding periodic orbit A^∞ has least period n. Similarly, let B, C be X-words starting with x_{-n}, and of length n, such that B^∞, C^∞ give periodic orbits. Make sure that A^∞, B^∞, C^∞ are distinct orbits. Letting $k=an$, extend $x_{-n}\cdots x_0$ to $x_{-n-k}\cdots x_{-n}\cdots x_0\cdots x_k$ by putting $x_k=x_{-n}$, $x_{-2n}\cdots x_{-n-1}=B$, $x_n\cdots x_{2n-1}=C$, $x_{bn-n}\cdots x_{bn-1}=A$ for $-a\le b\le a$, $b\ne -1, 0, 1, 2$, and choosing a connecting word $x_1\cdots x_{n-1}$. The word $x_{-n-k}\cdots x_k$ may be pictured as follows.



Note that our definition ensures for sufficiently large k that $(k+1)$-subwords of $x_{-k-n}\cdots x_k$ are distinct: if $-k-n\le b$, $c\le 0$ and $b\ne c$ then $x_b\cdots x_{b+k}\ne x_c\cdots x_{c+k}$. We now select transition rules for elements of $\mathscr{A}_k$.

(i) Consider $\{(i_{-n-k}\cdots i_{-n},\ j_{-n-k}\cdots j_{-n},\ E)\in\mathscr{A}_k: i_{-n-k}\cdots i_{-n}=x_{-n-k}\cdots x_{-n}\}$. For an element $(x_{-n-k}\cdots x_{-n}, j_{-n-k}\cdots j_{-n}, E)$ of this set, use Lemma 4 to pick $x\in[x_{-n},j_{-n},E]_{-n}\cap[x_{-n}\cdots x_0]_{-n}\cap[x_0,y_0,F_0]$, and put $x^-=x_{-\infty}^{-n}$. We then have $f(x^-x_{-n+1}\cdots x_0)=(x_0,y_0,F_0)$. For $-n<b<0$, let j_b, E_b be such that $f(x^-x_{-n+1}\cdots x_b)=(x_b,j_b,E_b)$. Let x^- determine the transition rule for $(x_{-n-k}\cdots x_{-n}, j_{-n-k}\cdots j_{-n}, E)$. Continuing, let x^-x_{-n+1}, $x^-x_{-n+1}x_{-n+2},\ldots$ determine the transition rule for $(x_{-n-k+1}\cdots x_{-n}x_{-n+1},\ j_{-n-k+1}\cdots j_{-n+1},\ E_{-n+1})$, $(x_{-n-k+2}\cdots x_{-n}x_{-n+1}x_{-n+2}, j_{-n-k+2}\cdots j_{-n+2}, E_{-n+2}),\ldots$ respectively, until either $(x_{-k}\cdots x_0,\ j_{-k}\cdots j_{-1}y_0,\ F_0)$ or a previously considered element of $\mathscr{A}_k$ is reached.

(ii) Fix $\hat{x}^-\in f^{-1}(x_0,y_0,F_0)$ and, for $1\le b\le k$, let $\hat{y}_b$, F_b be such that $f(\hat{x}^-x_1\cdots x_b)=(x_b,\hat{y}_b,F_b)$. For $0\le b<k$, let $\hat{x}^-x_1\cdots x_b$ determine the transition rule

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for all elements of $\mathscr{A}_k$ which have the form $(x_{-k+b}\cdots x_0\cdots x_b, j_{-k+b}\cdots j_{-1}y_0\hat{y}_1\cdots\hat{y}_b, F_b)$.

Since $(k+1)$-subwords of $x_{-k-n}\cdots x_k$ are distinct, the transition rules chosen in (ii) are disjoint from these chosen in (i). Moreover, in (i), whenever we reach a previously considered state before $(x_{-k}\cdots x_0, j_{-k}\cdots j_{-1}y_0, F_0)$, we are in phase with previously selected transition rules and these take us to $(x_{-k}\cdots x_0, j_{-k}\cdots j_{-1}y_0, F_0)$. For the remaining elements of $\mathscr{A}_k$, select the transition rules arbitrarily. In the resulting labelled graph, every path labelled $x_{-k-n}\cdots x_k$ terminates at the state $(x_0\cdots x_k, y_0\hat{y}_1\cdots\hat{y}_k, F_k)$. Hence, $x_{-k-n}\cdots x_k$ is a magic word for the resulting map π onto (X, S, m). It also follows that the graph has only one irreducible component of full topological entropy.

Notice. At this point π has degree 1, but ψ may not. The rest of this section is concerned with getting π and ψ which simultaneously have degree 1. The argument we give is quite technical, but has the advantage of being direct and not relying on invariants. In addition, our description of this argument is concise. An alternative and less technical approach, which makes use of some invariants and Ashley's recent work [**18**], is described in the Postscript at the end of the paper.

The transitions defined in (i) and (ii) are sufficient to make $x_{-k-n}\cdots x_k$ a magic word for the map onto (X, S, m). Hence:

(iii) If the transition rules are altered for some elements of $\mathscr{A}_k$ and the X-coordinates of these elements do not contain the periodic word $AA\cdots A$ of length $[k/3n]n$, then $x_{-k-n}\cdots x_k$ is a magic word for the resulting map onto (X, S, m).

We will create a magic word for the map onto (Y, T, p) by altering some transition rules, and rely on (iii) to retain the magic word $x_{-k-n}\cdots x_k$. The proof of the following lemma is similar to that of Lemma 4. (See, also, Lemma 2.)

LEMMA 5. *For every sufficiently large n, there exists a Y-word $y_{-n}\cdots y_0\cdots y_N$ such that*

$$p(\phi[i, y_{-n}, E]_{-n}\cap[y_{-n}\cdots y_N]_{-n}\cap\phi[x_0, y_0, F_0])>0$$

whenever $(i, y_{-n}, E)\in\mathscr{A}$ and $[y_{-n+1}\cdots y_{-n+N}]_1\subset E$.

Consider the (possibly reducible) extension we have on the alphabet $\mathscr{A}_k$, with the maps π, ψ. For the periodic orbit A^∞ of X, $\psi(\pi^{-1}(A^\infty))$ consists of finitely many periodic orbits of Y. Choose a Y-word $\bar{A}$ which gives a periodic orbit $\bar{A}^\infty$ distinct from those in $\psi(\pi^{-1}(A^\infty))$. Then there exists $h\in\mathbb{N}$ such that, for all large k' and every word $j_{-k'}\cdots j_0$ appearing in $\bar{A}^\infty$, no subword of A^∞ longer than h can occur in $\pi\circ\psi^{-1}(j_{-k'}\cdots j_0)$.

Let $y_{-n}\cdots y_N$ be the Y-word given by Lemma 5. Let $\bar{B}$ be a Y-word such that $\bar{B}$ gives a periodic orbit $\bar{B}^\infty$ distinct from $\bar{A}^\infty$ and $\bar{A}\bar{B}y_{-n}$ is a Y-word. We will choose a large c and put $y_{-l}\cdots y_{-n-1}=\bar{A}^c\bar{B}$ to extend $y_{-n}\cdots y_N$ to a Y-word

$$y_{-l}\cdots y_{-n}\cdots y_N\cdots y_{k+N}=\bar{A}^c\bar{B}y_{-n}\cdots y_{k+N}$$

with the following properties.

(iv) $l\geq n+k$ and the $(k+1)$-subwords of $y_{-n-k}\cdots y_{k+N}$ are distinct.

(v) No $(k+1)$-subword of $y_{-n-k}\cdots y_{k+N}$ equals one of $\bar{A}^\infty$.

(vi) Let $\hat{x}^- \in f^{-1}(x_0, y_0, F_0)$ and $\hat{y}^- \in Y^-$ be such that $\phi[\hat{x}^-] = [\hat{y}^-] \cap F_0$. Consider the X-word $\hat{x}_1 \cdots \hat{x}_k$ with the property that $\phi^{-1}[\hat{y}^- y_1 \cdots y_{k+N}] \subset [\hat{x}^- \hat{x}_1 \cdots \hat{x}_k]$, and make sure that A does not appear in $\hat{x}_1 \cdots \hat{x}_k$.

To be sure of the existence of such $\bar{A}^c \bar{B} y_{-n} \cdots y_{k+N}$ and to then be able to use (iii) while turning it into a magic word, we describe certain recodings of the present extension on the alphabet $\mathscr{A}_k$. For $k' \geq k$, we can pass to a conjugate extension with $\mathscr{A}_{k'}$ for its alphabet by letting each $(i_{-k'} \cdots i_0, j_{-k'} \cdots j_0, E) \in \mathscr{A}_{k'}$ inherit the transition rule of $(i_k \cdots i_0, j_k \cdots j_0, E) \in \mathscr{A}_k$. Using this procedure to recode, we assume that k is large enough for the word $\bar{A}^c \bar{B} y_{-n} \cdots y_{k+N}$ to exist and have the required properties. Also assume that $k/3$ is much larger than h and n. After we increase k to pass to the conjugate systems, we again have 1-block right closing maps onto (X, S, m) and (Y, T, p); we continue to write π and ψ for these maps. The word $x_{-k-n} \cdots x_k = A^{a-1} B x_{-n} \cdots x_{n-1} C A^{a-1}$ is a magic word for π and (iii) continues to apply. We denote by $\bar{\mathscr{A}}_k$ the set of elements of $\mathscr{A}_k$ which actually appear in (doubly infinite) points of the extension.

Now let $y_{-l} \cdots y_{k+N} = \bar{A}^c \bar{B} y_{-n} \cdots y_{k+N}$ be as specified above. We mimic (i) and (ii) to make $\bar{A}^c \bar{B} y_{-n} \cdots y_{k+N}$, for large enough c, into a magic word: We use $y_{-k-n} \cdots y_{k+N}$ in place of $x_{-k-n} \cdots x_k$. We consider the set

$$\{(i_{-k-n} \cdots i_{-n}, j_{-k-n} \cdots j_{-n}, E) \in \bar{\mathscr{A}}_k : j_{-k-n} \cdots j_{-n} = y_{-k-n} \cdots y_{-n} \text{ and } [y_{-n+1} \cdots y_{-n+N}]_1 \subset E\}.$$

Starting with each element of this set, we re-select the transition rules along a path labelled $y_{-n+1} \cdots y_{k+N}$, using Lemma 5 in place of Lemma 4 and $\hat{y}^-$ of (vi) to play the role of $\hat{x}^-$ of (ii).

As a result of (v), the above re-selection of transition rules does not change the pre-image of $\bar{A}^\infty$; this set equals $\psi^{-1}(\bar{A}^\infty)$ even after the re-selection. Hence, for large enough c, every pre-image of $y_{-l} \cdots y_{-n-k-1}$ will terminate at an element of $\bar{\mathscr{A}}_k$, even after the re-selection, and the re-selected transition rules then make sure that the word $y_{-l} \cdots y_{k+N} = \bar{A}^c \bar{B} y_{-n} \cdots y_{k+N}$ is a magic word for the map onto (Y, T, p). In addition, our choice of $\bar{A}$, the fact that $k/3$ is larger than h and (vi) ensure that, for any element of $\mathscr{A}_k$ whose transition rule we altered, the X-coordinate does not contain the periodic word $AA \cdots A$ of length $[k/3n]n$. By (iii), $x_{-k-n} \cdots x_k$ continues to be a magic word for the map onto (X, S, m). As remarked earlier, the existence of magic words implies that, in the graph, there is only one irreducible component of full topological entropy. We restrict to this component to obtain the desired irreducible Markov chain (Z, U, q) with right closing block codes of degree 1 onto (X, S, m) and (Y, T, p).

4. *An invariant ideal class*

We discuss some invariants of regular isomorphism.

For a non-negative matrix M and $t \in \mathbb{R}$, let M^t denote the matrix compatible with M and having $M^t(a, b) = M(a, b)^t$ whenever $M(a, b) \neq 0$. Writing $\beta_M(t)$ for the spectral radius of M^t, we have the β-function $\beta_M : \mathbb{R} \to \mathbb{R}^+$ of M. The β-function of

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(X, S, m) equals that of its defining stochastic matrix; this is an invariant of regular isomorphism [**15**].

Consider all pairs of X-words $ii_1 \cdots i_l i'$, $ii'_1 \cdots i'_l i'$ which start with the same symbol, end with same symbol and have the same length. The ratios $m[ii_1 \cdots i_l i']/m[ii'_1 \cdots i'_l i']$ form a multiplicative group, which is denoted by Δ_m. In addition, letting d be the period of (X, S, m), consider two words, $ii_1 \cdots i_l i'$ and $ii'_1 \cdots i'_{l+d} i'$, which start with the same symbol, end with the same symbol and differ by d in length, and put $c_m = m[ii'_1 \cdots i'_{l+d} i']/m[ii_1 \cdots i_l i']$. It is easy to see that the set $c_m \Delta_m$ is independent of the choice of $ii_1 \cdots i_l i'$ and $ii'_1 \cdots i'_{l+d} i'$. The pair $(\Delta_m, c_m \Delta_m)$ was studied in [**9**]. (Δ_m had earlier appeared in [**7**] and [**16**].) Although regular isomorphism was not considered in [**9**], the invariance of $(\Delta_m, c_m, \Delta_m)$ under regular isomorphism is an immediate consequence of the work in [**9**].

The fact that regular isomorphism of (X, S, m) and (Y, T, p) implies $\beta_M = \beta_P$, $\Delta_m = \Delta_p$ and $c_m \Delta_m = c_p \Delta_p$ may also be easily deduced from the main result of the present paper; write β, Δ and $c\Delta$ for these objects.

According to [**9**], there exists a diagonal matrix δ such that the non-zero entries of $\bar{M} = \delta M \delta^{-1}/c^{1/d}$ belong to Δ. Clearly, $\bar{\beta} = \beta/c^{1/d}$ is the β-function of $\bar{M}$. Let $R = \mathbb{Z}[\Delta]$ be the integral group ring of Δ, presented as all finite sums of exponentials, $\sum_n a_n u'_n$, with $a_n \in \mathbb{Z}$ and $u_n \in \Delta$. Then $A = \bar{M}'$ may be viewed as a matrix over R, and $\bar{\beta}$ is an integer over R (it satisfies a monic polynomial whose coefficients lie in R, namely, the characteristic polynomial of A). Moreover, we can find a vector r_A over $R[\bar{\beta}]$ such that $Ar_A = \bar{\beta} r_A$. Let $\mathcal{I}_m$ be the ideal of $R[\frac{1}{\bar{\beta}}]$ generated by the entries of r_A, and consider the ideal class $[\mathcal{I}_m]$ of $\mathcal{I}_m : [\mathcal{I}_m]$ is the equivalence class consisting of ideals $\mathcal{I}$ which satisfy $u\mathcal{I} = v\mathcal{I}_m$ for some non-zero $u, v \in R[\frac{1}{\bar{\beta}}]$. It is not hard to see that $[\mathcal{I}_m]$ does not depend on the choice of the above matrix $\bar{M}$ over Δ or, by the Perron-Frobenius theorem, on the choice of the eigenvector r_A of $A = \bar{M}'$. $[\mathcal{I}_m]$ is an invariant of regular isomorphism, because of our main result and the following.

PROPOSITION. *If $\pi : (Z, U, q) \to (X, S, m)$ is a right closing block code of degree* 1, *then $[\mathcal{I}_q] = [\mathcal{I}_m]$.*

One way to see this is as follows. Note that $\beta = \beta_m = \beta_q$, $\Delta = \Delta_m = \Delta_q$ and $c\Delta = c_m \Delta_m = c_q \Delta_q$. Conjugating π to a right resolving map, find a Markov chain (Z', U', q') and a commutative diagram

$$\begin{array}{ccc} (Z', U', q') & \xrightarrow{\theta} & (Z, U, q) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & (X, S, m) & \end{array}$$

where π' is a right resolving block code and θ is a block isomorphism. (See [**6**], [**15**].) In particular, $A = \bar{Q}'$ and the analogous matrix $A' = \bar{Q}''$ are shift equivalent over R (see [**11**], [**16**]), so $[\mathcal{I}_q] = [\mathcal{I}_{q'}]$. Moreover, since π' is right resolving, the eigenvectors r_A and $r_{A'}$ may be chosen so that each entry of $r_{A'}$ equals some entry of r_A, perhaps multiplied by an exponential u', $u \in \Delta$; and vice versa. Since u' are units in R, we have $[\mathcal{I}_{q'}] = [\mathcal{I}_m]$ also.

The invariant $[\mathcal{I}_m]$ we just described generalizes the ideal class introduced by Trow [**14**] and further studied in [3]. $[\mathcal{I}_m]$ is closely tied to the dimension module [**16**] of (X, S, m). This connection, and other related material, will be discussed elsewhere. We close the paper with an example in which two Markov chains with the same β, Δ and $c\Delta$ are distinguished by the ideal class.

It is easy to see that Δ is finitely generated and, being a multiplicative subgroup of the positive reals, free. It follows that R is isomorphic to a Laurent polynomial ring $\mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$; it is often convenient to work with this polynomial ring. The following example concerns the case $R \cong \mathbb{Z}[x, x^{-1}]$.

Example. Let $a, b > 0$, $a + b = 1$, and consider the Markov chains defined by the stochastic matrices

$$M = \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & \dfrac{a^2(1+b)}{a+b^2} & \dfrac{b^2(1+a)}{a+b^2} \\ \dfrac{a+b^2}{1+b} & 0 & \dfrac{b(1+a)}{1+b} \\ \dfrac{a+b^2}{1+a} & \dfrac{a(1+b)}{1+a} & 0 \end{bmatrix}.$$

It is easy to see that $\beta = a^t + b^t$, $\Delta = \{(a/b)^n : n \in \mathbb{Z}\}$ and we may take $c = b$ for both of these. We can take δ to be the 3×3 diagonal matrix with $(a + b^2, a(1+b), b(1+a))$ as its diagonal and $\bar{M} = b^{-1}M$, $\bar{P} = b^{-1}\delta P \delta^{-1}$. Writing $x = (a/b)^t$, we have $R \cong \mathbb{Z}[x, x^{-1}]$, $\bar{\beta} = 1 + x$, and

$$A = \begin{bmatrix} x & 1 \\ x & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & x & 1 \\ x & 0 & x \\ 1 & 1 & 0 \end{bmatrix}.$$

The transposes of $(1, 1)$ and $(x^2 + x + 1, x^2 + 2x, 2x + 1)$ give right eigenvectors of A and B. Thus, $[\mathcal{I}_m]$ consists of principal ideals. We will argue by contradiction to show that

$$\mathcal{I}_p = \langle x^2 + x + 1, x^2 + 2x, 2x + 1 \rangle = \langle x^2 + x + 1, x + 2, 3 \rangle$$

is not a principal ideal of $R[1/(1+x)]$: Suppose $\mathcal{I}_p$ is principal. Since x and $1 + x$ are invertible in $R[1/(1+x)]$, this means $\mathcal{I}_p = uR[1/(1+x)]$ for some $u \in \mathbb{Z}[x]$ such that $x, 1 + x \nmid u$. As $3 \in \mathcal{I}_p$, we have $u | 3$ in $\mathbb{Z}[x]$. So, $u = \pm 3$ or $u = \pm 1$. We cannot have $u = \pm 3$, because $x + 2 \in \mathcal{I}_p$. If $u = \pm 1$, then the equation

$$(x^2 + x + 1)v_1(x) + (x + 2)v_2(x) + 3v_3(x) = x^k(1 + x)^l$$

must hold for some v_1, v_2, $v_3 \in \mathbb{Z}[x]$ and positive integers k, l. Put $x = -2$. The left hand side becomes $3v_1(-2) + 3v_3(-2)$, so it is divisible by 3, while the right hand side equals $\pm 2^k$, which is absurd.

Postscript. After this paper was written, Ashley [**18**] proved the following beautiful result. Suppose there exists a right closing code $\psi : (Z, U, q) \to (Y, T, p)$ between Markov chains with the same period $d_q = d_p$ and with $\Delta_q = \Delta_p$, (see § 4 for the definitions of these objects). Then according to [**18**], there also exists a right closing code $\bar{\psi} : (Z, U, q) \to (Y, T, p)$ which has degree 1. We use this result to describe an alternative completion of the proof of our theorem. At the point in § 3 where the

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Notice of this postscript appears, we have an irreducible Markov chain (Z, U, q) and right closing codes $\pi:(Z, U, q)\to(X, S, m)$, $\psi:(Z, U, q)\to(Y, T, p)$, and we know that π has degree 1. We find $d_q = d_m$, $\Delta_q = \Delta_m$ because π has degree 1, and $d_m = d_p$, $\Delta_m = \Delta_p$ as a result of our hypothesis that (X, S, m) and (Y, T, p) are regularly isomorphic. Thus, $d_q = d_p$, $\Delta_q = \Delta_p$ and Ashley's result may be applied to replace ψ by a right resolving code $\bar{\psi}:(Z, U, q)\to(Y, T, p)$ of degree 1.

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