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INVOLUTION AND THE HAAGERUP TENSOR PRODUCT

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Abstract We show that the involution $\theta(a \otimes b) = a^* \otimes b^*$ on the Haagerup tensor product $A \otimes_{\mathrm{H}} B$ of C^*-algebras A and B is an isometry if and only if A and B are commutative. The involutive Banach algebra $A \otimes_{\mathrm{H}} A$ arising from the involution $a \otimes b \to b^* \otimes a^*$ is also studied.

Keywords: C^*-algebras; Haagerup tensor product; second dual; closed ideals

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1. Introduction

The Haagerup norm of an element u in the algebraic tensor product $A \otimes B$ of two C^*-algebras A and B is defined by

$$\|u\|_{\mathrm{H}} = \inf \left\| \sum_{j=1}^{n} a_j a_j^* \right\|^{1/2} \left\| \sum_{j=1}^{n} b_j^* b_j \right\|^{1/2} = \inf \|(a_1, a_2, \dots, a_n)\| \, \|(b_1, b_2, \dots, b_n)'\|,$$

where these infima are taken over all representations of $u = \sum_{j=1}^{n} a_j \otimes b_j$, $a_j \in A$, $b_j \in B$, and $(b_1, b_2, \dots, b_n)'$ is the transpose of the row operator. The Haagerup tensor product $A \otimes_{\mathrm{H}} B$ is the Banach space obtained by completing the algebraic tensor product $A \otimes B$ in the Haagerup norm. A direct calculation with the definition and Cauchy–Schwarz inequality shows that $A \otimes_{\mathrm{H}} B$ is a Banach algebra with the natural multiplication $(a \otimes b)(x \otimes y) = ax \otimes by$, $a, x \in A$ and $b, y \in B$ [**3**]. The Haagerup tensor product $A \otimes_{\mathrm{H}} B$ is a C^*-algebra if and only if A or B equals $\mathbb{C}$ [**4**]. This tensor product plays an important role in the theory of operator spaces [**4**–**6**, **8**, **9**] and is an injective tensor product [**13**]. The ideal structure of this Banach algebra has been studied in [**1**] and [**2**].

First we show that a natural involution $\theta : A \otimes B \to A \otimes B$ given by $\theta(a \otimes b) = a^* \otimes b^*$ lifts to a continuous map θ_{H} on $A \otimes_{\mathrm{H}} B$ if and only if either A or B is finite dimensional, or A and B are infinite dimensional and subhomogeneous. Recall that a C^*-algebra is subhomogeneous if for some $k \in N$, every irreducible representation is on a Hilbert space of dimension not greater than k. Furthermore, it has been shown that θ_{H} is an isometry if and only if A and B are commutative. It follows from the definition of the Haagerup norm that the Haagerup tensor product $A \otimes_{\mathrm{H}} A$ is an involutive Banach algebra with

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isometric involution given by $a \otimes b \to b^* \otimes a^*$. For a unital C^*-algebra A, we show that if $A \otimes_{\mathrm{H}} A$ has a faithful *-representation on a Hilbert space, then A is commutative. As a corollary it follows that $A \otimes_{\mathrm{H}} A$ is *-semi simple (Hermitian) if and only if A is commutative. Finally, the closed *-ideals of $A \otimes_{\mathrm{H}} A$ are studied.

2. Results

For a Banach space X, X^* denotes the dual of X. Let M_n be the C^*-algebra of $n \times n$ complex matrices acting on the n-dimensional complex Hilbert space $\mathbb{C}^n$. For a complex Hilbert space H, let $B(H)$ be the algebra of bounded operators on H and $K(H)$ the ideal of compact operators. The following lemma is proved in [**12**] using the Cauchy–Schwarz inequality and the action of $M_n \otimes_{\mathrm{H}} M_n$ on M_n as completely bounded operators.

Lemma 2.1. *For $n \in N$, if e_{ij} for $1 \leqslant i, j \leqslant n$ are the matrix units in M_n and l_n^∞ is the diagonal algebra in M_n, then*

$$\left\|\sum_{j=1}^{n} e_{1j} \otimes e_{jj}\right\|_{\mathrm{H}} = n^{1/2} \quad \textit{and} \quad \left\|\sum_{j=1}^{n} e_{j1} \otimes e_{jj}\right\|_{\mathrm{H}} = 1$$

in $M_n \otimes_{\mathrm{H}} l_n^\infty$. Also in $l_n^\infty \otimes_{\mathrm{H}} M_n$

$$\left\|\sum_{j=1}^{n} e_{jj} \otimes e_{j1}\right\|_{\mathrm{H}} = n^{1/2} \quad \textit{and} \quad \left\|\sum_{j=1}^{n} e_{jj} \otimes e_{1j}\right\|_{\mathrm{H}} = 1.$$

Theorem 2.2. *Let A and B be C^*-algebras and θ is the map on $A \otimes B$ given by $\theta(a \otimes b) = a^* \otimes b^*$. Then the following are equivalent.*

(i) *The Haagerup norm $\|\cdot\|_{\mathrm{H}}$ is equivalent to the Banach space projective norm $\|\cdot\|_\gamma$.*

(ii) *θ lifts to a continuous map θ_{H} on $A \otimes_{\mathrm{H}} B$.*

(iii) *Either A or B is finite dimensional or A and B are infinite dimensional and subhomogeneous.*

Proof. The equivalence of (i) and (iii) is shown in [**12**]. It is trivial that (i) implies (ii). We now show that (ii) implies (iii). Suppose that θ lifts to a continuous map θ_{H} on $A \otimes_{\mathrm{H}} B$ and A and B are infinite dimensional. Then θ_{H} is a continuous map on $(A \otimes_{\mathrm{H}} B)^{**}$ which contains $A^{**} \otimes_{\mathrm{H}} B^{**}$ [**5**, **10**]. For $u \in (A \otimes_{\mathrm{H}} B)^{**}$ and $\phi \in (A \otimes_{\mathrm{H}} B)^*$, $(\theta_{\mathrm{H}} u)(\phi) = u(\phi \circ \theta_{\mathrm{H}})$.

The dual space of the Haagerup tensor product of two C^*-algebras is the space of completely bounded bilinear forms on the algebras [**9**]. So, by [**9**], for $\phi \in (A \otimes_{\mathrm{H}} B)^*$ there exist Hilbert spaces H and K, representations $\pi_1 : A \to B(H)$ and $\pi_2 : B \to B(K)$, vectors $\xi \in K$ and $\eta \in H$, and a bounded linear operator $T : K \to H$ such that

$$\phi(x \otimes y) = \langle \pi_1(x) T \pi_2(y) \xi, \eta \rangle$$

for all $x \in A$, $y \in B$. Assuming that the representations π_1 and π_2 of A and B are faithful, we can identify A with $\pi_1(A)$ and B with $\pi_2(B)$. The above expression can be rewritten as

$$\phi(x \otimes y) = \langle xTy\xi, \eta\rangle$$

for all $x \in A \subseteq B(H)$, $y \in B \subseteq B(K)$. For $v \in A^{**}$, $\omega \in B^{**}$, the element $v \otimes \omega$ of $A^{**} \otimes_{\mathrm{H}} B^{**}$ can be viewed as an element of $(A \otimes_{\mathrm{H}} B)^{**}$ by

$$v \otimes \omega(\phi) = \langle vT\omega\xi, \eta\rangle.$$

This inclusion is an isometry [**5**, **10**]. Thus $\theta_{\mathrm{H}}^{**}(v \otimes \omega) = v^* \otimes \omega^*$.

If for some $\eta \in N$, the von Neumann algebras A^{**} or B^{**} (say A^{**}) contain an isomorphic copy of M_n, then, by Lemma 2.1,

$$n^{1/2} = \left\|\sum_{j=1}^{n} e_{1j} \otimes e_{jj}\right\|_{\mathrm{H}} = \left\|\theta_{\mathrm{H}}^{**}\left(\sum_{j=1}^{n} e_{j1} \otimes e_{jj}\right)\right\|_{\mathrm{H}} \leqslant \|\theta_{\mathrm{H}}^{**}\| \left\|\sum_{j=1}^{n} e_{j1} \otimes e_{jj}\right\|_{\mathrm{H}} = \|\theta_{\mathrm{H}}^{**}\|_{\mathrm{H}},$$

by the injectivity of the Haagerup norm [**13**]. It follows that A^{**} and B^{**} cannot contain a type I_n factor for $n > \|\theta_{\mathrm{H}}\|^2$. So, A^{**} and B^{**} are of the form $\oplus N_j$, $j \leqslant \|\theta_{\mathrm{H}}\|^2$, where N_j is a von Neumann algebra of type I_j. If π is an irreducible representation of A on a Hilbert space H, there is a normal representation π of A^{**} on H such that $\pi(A^{**}) = \overline{\pi(A)}$ (weak closure) $= B(H)$. Hence, $\dim H \leqslant \|\theta_{\mathrm{H}}\|^2$. Similarly, B is also subhomogeneous. □

Theorem 2.3. *Let A and B be infinite-dimensional C^*-algebras and $\theta(a \otimes b) = a^* \otimes b^*$. If θ lifts to a continuous map θ_{H} on $A \otimes_{\mathrm{H}} B$, then θ_{H} is an isometry if and only if A and B are commutative.*

Proof. If A and B are commutative, by the definition of the Haagerup norm, θ_{H} is an isometry. Conversely, if θ_{H} is an isometry on $A \otimes_{\mathrm{H}} B$, then θ_{H} lifts to an isometry θ_{H}^{**} on $(A \otimes_{\mathrm{H}} B)^{**}$. As in Theorem 2.2, $\theta_{\mathrm{H}}^{**}(v \otimes \omega) = v^* \otimes \omega^*$ for all $v \in A^{**}$ and $\omega \in B^{**}$. If at least one of the von Neumann algebras A^{**} or B^{**} is not commutative, say A^{**}, then by the decomposition of a von Neumann algebra into type I, II_1, II_∞, III, it follows that $A^{**} \supset M_n$ for some $n > 1$ [**11**]. Lemma 2.1 and the injectivity of Haagerup norm [**13**] now show that θ_{H}^{**} is not an isometry. Hence A^{**} and B^{**} are commutative, and in particular so are A and B. □

Let A be a C^*-algebra. By the definition of Haagerup norm, $A \otimes_{\mathrm{H}} A$ is a Banach *-algebra with isometric involution given by $a \otimes b \to b^* \otimes a^*$, $a, b \in A$. For a Hilbert space H, $\pi : A \otimes_{\mathrm{H}} A \to B(H)$ will be called a *-representation if π is a bounded algebraic homomorphism satisfying $\pi(b^* \otimes a^*) = (\pi(a \otimes b))^*$ for all $a, b \in A$. If, in addition, $\pi(A \otimes_{\mathrm{H}} A)$ is σ-weakly dense in $B(H)$, then π is said to be irreducible.

Theorem 2.4. *Let A be a unital C^*-algebra. If $A \otimes_{\mathrm{H}} A$ has a faithful *-representation, then A is commutative.*

Proof. Let π be a faithful *-representation of $A \otimes_{\mathrm{H}} A$ on a Hilbert space H. Putting $\pi_1(a) = \pi(a \otimes 1)$ and $\pi_2(a) = \pi(1 \otimes a)$, $a \in A$, it is easy to verify that π_1 and π_2 are bounded monomorphisms from A into $B(H)$ satisfying $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a, b \in A$ and $\pi_1(a^*) = \pi_2(a)^*$, $a \in A$. If h is a self-adjoint element of A, then $\|\exp \mathrm{i}th\| = 1$ for all $t \in \mathbb{R}$. The *-homomorphism π from the Banach *-algebra $A \otimes_{\mathrm{H}} A$ to $B(H)$ is norm reducing [**14**, Proposition 1.5.2]. Thus

$$\|\exp \mathrm{i}t\pi_1(h)\| = \|\pi(\exp \mathrm{i}t(h \otimes 1))\| \leqslant \|\exp \mathrm{i}t(h \otimes 1)\|_{\mathrm{H}} = \|\exp \mathrm{i}th\| = 1,$$

for all $t \in \mathbb{R}$. Hence, $\|\exp \mathrm{i}t\pi_1(h)\| = 1$ for all $t \in \mathbb{R}$. So $\pi_1(h)$ is a self-adjoint element of $B(H)$. Let $a = h + \mathrm{i}k$, where h and k are self-adjoint elements of A. Now

$$\pi_1(a^*) = \pi_1(h - ik) = \pi_1(h) - i\pi_1(k) = (\pi_1(h) + i\pi_1(k))^* = (\pi_1(a))^*.$$

This implies that $\pi_1(a^*) = \pi_1(a)^* = \pi_2(a)^*$ for all $a \in A$ and, thus, $\pi_1(a) = \pi_2(a)$ for all $a \in A$. But $\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$, so $\pi(ab - ba \otimes 1) = \pi_1(ab - ba) = 0$ for all $a, b \in A$. Since π is faithful, we have $ab - ba = 0$ for all $a, b \in A$, i.e. A is commutative. □

It is well known for a C^*-algebra A, $\cap\{\ker \pi : \pi \text{ is a } {}^*\text{-representation of } A\} = \{0\}$, i.e. A is *-semi simple. An equivalent form of the above result is the following.

Corollary 2.5. *Let A be a unital C^*-algebra. Then $A \otimes_{\mathrm{H}} A$ is *-semi simple if and only if A is commutative.*

Recall that a Banach *-algebra A is said to be Hermitian if every self-adjoint element of A has real spectrum [**7**]. Moreover, in a Hermitian Banach *-algebra A, the radical of A equals the star radical of A [**7**, Theorem 4.9]. Since $\mathrm{rad}(A \otimes_{\mathrm{H}} A) = (0)$ by [**1**, Proposition 5.16], we have the following.

Corollary 2.6. *Let A be a unital C^*-algebra. Then $A \otimes_{\mathrm{H}} A$ is Hermitian if and only if A is commutative.*

A careful reading of the proof of Theorem 2.4 shows the following.

Proposition 2.7. *Let A is a unital C^*-algebra and π a *-representation of $A \otimes_{\mathrm{H}} A$, then there is a *-representation π_0 of A satisfying $\pi(a \otimes b) = \pi_0(ab)$ and $\pi_0(A)$ is abelian.*

Suppose that A is a C^*-algebra having only a finite number of closed two-sided ideals. Let K be a closed *-ideal of $A \otimes_{\mathrm{H}} A$. By [**1**, Theorem 5.3], $K = \sum_j (K_j \otimes_{\mathrm{H}} I_j)$, where K_j, I_j are closed ideals of A and, hence, *-ideals. Thus any *-ideal of $A \otimes_{\mathrm{H}} A$ is of the form

$$\sum_j (K_j \otimes_{\mathrm{H}} I_j + I_j \otimes_{\mathrm{H}} K_j).$$

In particular, the only closed proper *-ideals of $B(H) \otimes_{\mathrm{H}} B(H)$ are $B(H) \otimes_{\mathrm{H}} K(H) + K(H) \otimes_{\mathrm{H}} B(H)$ and $K(H) \otimes_{\mathrm{H}} K(H)$.

Our next result characterizes the *-ideals of $A \otimes_{\mathrm{H}} A$ annihilated by a *-representation of $A \otimes_{\mathrm{H}} A$.

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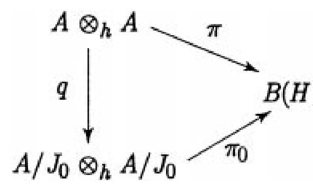
Theorem 2.8. *Let A be a unital C^*-algebra. Then a closed two-sided *-ideal J of $A \otimes_{\mathrm{H}} A$ is annihilated by a *-representation π of $A \otimes_{\mathrm{H}} A$ if and only if there is a *-representation π_0 of A with $\pi_0(A)$ abelian such that $J \subseteq J_0 \otimes_{\mathrm{H}} A + A \otimes_{\mathrm{H}} J_0$, $J_0 = \ker \pi_0$.*

Proof. Suppose that $J \subseteq \ker \pi$, where π is a *-representation of $A \otimes_{\mathrm{H}} A$ on a Hilbert space H. Let π_0 be a *-representation of A as in Proposition 2.7 and $J_0 = \ker \pi_0$. Clearly, $\ker \pi \supseteq J_0 \otimes_{\mathrm{H}} A + A \otimes_{\mathrm{H}} J_0$ and $A/J_0 \otimes_{\mathrm{H}} A/J_0$ is commutative. Let $q : A \otimes_{\mathrm{H}} A \to A/J_0 \otimes_{\mathrm{H}} A/J_0$ be the quotient map with kernel $J_0 \otimes_{\mathrm{H}} A + A \otimes_{\mathrm{H}} J_0$. The representation π induces a faithful representation π_0 of $A/J_0 \otimes_{\mathrm{H}} A/J_0$ on H. Moreover, the following diagram commutes.



So $\pi(J) = 0$ implies that $q(J) = 0$. Thus $J \subseteq J_0 \otimes_{\mathrm{H}} A + A \otimes_{\mathrm{H}} J_0$. Conversely, suppose that $J \subseteq J_0 \otimes_{\mathrm{H}} A + A \otimes_{\mathrm{H}} J_0$, $J_0 = \ker \pi_0$, π_0 is a *-representation of A with $\pi_0(A)$ abelian. Defining π by $\pi(a \otimes b) = \pi_0(ab)$ on $A \otimes_{\mathrm{H}} A$, it is easy to verify that π is a *-representation of $A \otimes_{\mathrm{H}} A$ and $J \subseteq \ker \pi$. $\square$

Let H be a separable infinite-dimensional Hilbert space, it follows from the above theorem that the *-ideal $K(H) \otimes_{\mathrm{H}} K(H)$ cannot be annihilated by a *-representation of $B(H) \otimes_{\mathrm{H}} B(H)$.

In contrast to Theorem 2.8, if the involution $a \otimes b \to b^* \otimes a^*$ is dropped, then of course for every proper closed two-sided ideal J there is a bounded algebraic homomorphism $\pi : A \otimes_{\mathrm{H}} A \to B(H)$, satisfying $\pi(a^* \otimes b^*) = \pi(a \otimes b)^*$, $a, b \in A$ such that $J \subseteq \ker \pi$. The proof of this result is implicitly contained in [**1**] (see also [**2**]), but to be more explicit, we outline the proof.

Theorem 2.9. *Let A and B be unital C^*-algebras. Then every proper closed two-sided ideal of $A \otimes_{\mathrm{H}} B$ is annihilated by a representation of $A \otimes_{\mathrm{H}} B$.*

Proof. Let J be a proper closed two-sided ideal of $A \otimes_{\mathrm{H}} B$ and $J_{\min}$ be the closure of J in $A \otimes_{\min} B$, where $A \otimes_{\min} B$ is the completion of the algebraic tensor product with $\|\cdot\|_{\min}$ norm. If $J_{\min} = A \otimes_{\min} B$, then $J_{\min}$ will contain all elementary tensors, so, by [**1**, Theorem 4.4], J will be equal to $A \otimes_{\mathrm{H}} B$. Thus, $J_{\min}$ is a proper closed two-sided ideal in the C^*-algebra $A \otimes_{\min} B$. Let π be an irreducible representation of $A \otimes_{\min} B$ on a Hilbert space H annihilating $J_{\min}$. Let $\pi_1(a) = \pi(a \otimes 1)$ and $\pi_2(b) = \pi(1 \otimes b)$, for all $a \in A$ and $b \in B$. Then π_1 and π_2 are commuting representations of A and B, respectively. Let $M = \ker \pi_1$ and $N = \ker \pi_2$. Let $q : A \otimes_{\mathrm{H}} B \to A/M \otimes_{\mathrm{H}} B/N$ be the quotient map and let $\pi_1 \cdot \pi_2 : A/M \otimes_{\mathrm{H}} B/N \to B(H)$ be the faithful representation of $A/M \otimes_{\mathrm{H}} B/N$ induced by π_1 and π_2 (see [**1**] for details). So $\pi(J) = 0$ implies that $q(J) = 0$. But $A/M \otimes_{\mathrm{H}} B/N \simeq A \otimes_{\mathrm{H}} B/M \otimes_{\mathrm{H}} B + A \otimes_{\mathrm{H}} N$, thus $J \subseteq M \otimes_{\mathrm{H}} B + A \otimes_{\mathrm{H}} N$. Since $M \otimes_{\mathrm{H}} B + A \otimes_{\mathrm{H}} N$ is primitive [**1**, Theorem 5.13], there is a representation σ of $A \otimes_{\mathrm{H}} B$ such that $J \subseteq \ker \sigma$. $\square$

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