INVOLUTION AND THE HAAGERUP TENSOR PRODUCT

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Abstract We show that the involution \( \theta(a \otimes b) = a^* \otimes b^* \) on the Haagerup tensor product \( A \otimes_H B \) of \( C^* \)-algebras \( A \) and \( B \) is an isometry if and only if \( A \) and \( B \) are commutative. The involutive Banach algebra \( A \otimes_H A \) arising from the involution \( a \otimes b \mapsto b^* \otimes a^* \) is also studied.

Keywords: \( C^* \)-algebras; Haagerup tensor product; second dual; closed ideals

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1. Introduction

The Haagerup norm of an element \( u \) in the algebraic tensor product \( A \otimes B \) of two \( C^* \)-algebras \( A \) and \( B \) is defined by

\[
\|u\|_H = \inf \left\{ \left( \sum_{j=1}^n a_j a_j^* \right)^{1/2} \left( \sum_{j=1}^n b_j^* b_j \right)^{1/2} : a_j, b_j \in A, B \right\} \| (a_1, a_2, \ldots, a_n) \| \| (b_1, b_2, \ldots, b_n)^t \|,
\]

where these infima are taken over all representations of \( u = \sum_{j=1}^n a_j \otimes b_j, a_j, b_j \in A, B \), and \( (b_1, b_2, \ldots, b_n)^t \) is the transpose of the row operator. The Haagerup tensor product \( A \otimes_H B \) is the Banach space obtained by completing the algebraic tensor product \( A \otimes B \) in the Haagerup norm. A direct calculation with the definition and Cauchy–Schwarz inequality shows that \( A \otimes_H B \) is a Banach algebra with the natural multiplication \( (a \otimes b)(x \otimes y) = ax \otimes by, a, x \in A \) and \( b, y \in B \) [3]. The Haagerup tensor product \( A \otimes_H B \) is a \( C^* \)-algebra if and only if \( A \) or \( B \) equals \( \mathbb{C} \) [4]. This tensor product plays an important role in the theory of operator spaces [4–6, 8, 9] and is an injective tensor product [13]. The ideal structure of this Banach algebra has been studied in [1] and [2].

First we show that a natural involution \( \theta : A \otimes B \to A \otimes B \) given by \( \theta(a \otimes b) = a^* \otimes b^* \) lifts to a continuous map \( \theta_H \) on \( A \otimes_H B \) if and only if either \( A \) or \( B \) is finite dimensional, or \( A \) and \( B \) are infinite dimensional and subhomogeneous. Recall that a \( C^* \)-algebra is subhomogeneous if for some \( k \in \mathbb{N} \), every irreducible representation is on a Hilbert space of dimension not greater than \( k \). Furthermore, it has been shown that \( \theta_H \) is an isometry if and only if \( A \) and \( B \) are commutative. It follows from the definition of the Haagerup norm that the Haagerup tensor product \( A \otimes_H A \) is an involutive Banach algebra with
isometric involution given by \( a \otimes b \rightarrow b^* \otimes a^* \). For a unital C*-algebra \( A \), we show that if \( A \otimes_H A \) has a faithful \(^*\)-representation on a Hilbert space, then \( A \) is commutative. As a corollary it follows that \( A \otimes_H A \) is \(^*\)-semi simple (Hermitian) if and only if \( A \) is commutative. Finally, the closed \(^*\)-ideals of \( A \otimes_H A \) are studied.

2. Results

For a Banach space \( X \), \( X^\ast \) denotes the dual of \( X \). Let \( M_n \) be the C*-algebra of \( n \times n \) complex matrices acting on the \( n \)-dimensional complex Hilbert space \( \mathbb{C}^n \). For a complex Hilbert space \( H \), let \( B(H) \) be the algebra of bounded operators on \( H \) and \( K(H) \) the ideal of compact operators. The following lemma is proved in [12] using the Cauchy–Schwarz inequality and the action of \( M_n \otimes_H M_n \) on \( M_n \) as completely bounded operators.

**Lemma 2.1.** For \( n \in \mathbb{N} \), if \( e_{ij} \) for \( 1 \leq i, j \leq n \) are the matrix units in \( M_n \) and \( l_n^\infty \) is the diagonal algebra in \( M_n \), then

\[
\left\| \sum_{j=1}^{n} e_{1j} \otimes e_{jj} \right\|_H = n^{1/2} \quad \text{and} \quad \left\| \sum_{j=1}^{n} e_{j1} \otimes e_{jj} \right\|_H = 1
\]

in \( M_n \otimes_H l_n^\infty \). Also in \( l_n^\infty \otimes_H M_n \)

\[
\left\| \sum_{j=1}^{n} e_{jj} \otimes e_{1j} \right\|_H = n^{1/2} \quad \text{and} \quad \left\| \sum_{j=1}^{n} e_{jj} \otimes e_{jj} \right\|_H = 1.
\]

**Theorem 2.2.** Let \( A \) and \( B \) be C*-algebras and \( \theta \) is the map on \( A \otimes B \) given by \( \theta(a \otimes b) = a^* \otimes b^* \). Then the following are equivalent.

(i) The Haagerup norm \( \| \cdot \|_H \) is equivalent to the Banach space projective norm \( \| \cdot \|_\gamma \).

(ii) \( \theta \) lifts to a continuous map \( \theta_H \) on \( A \otimes_H B \).

(iii) Either \( A \) or \( B \) is finite dimensional or \( A \) and \( B \) are infinite dimensional and subhomogeneous.

**Proof.** The equivalence of (i) and (iii) is shown in [12]. It is trivial that (i) implies (ii). We now show that (ii) implies (iii). Suppose that \( \theta \) lifts to a continuous map \( \theta_H \) on \( A \otimes_H B \) and \( A \) and \( B \) are infinite dimensional. Then \( \theta_H \) is a continuous map on \( (A \otimes_H B)^{**} \) which contains \( A^{**} \otimes_H B^{**} \) [5, 10]. For \( u \in (A \otimes_H B)^{**} \) and \( \phi \in (A \otimes_H B)^* \), \( (\theta_H u)(\phi) = u(\phi \circ \theta_H) \).

The dual space of the Haagerup tensor product of two C*-algebras is the space of completely bounded bilinear forms on the algebras [9]. So, by [9], for \( \phi \in (A \otimes_H B)^* \) there exist Hilbert spaces \( H \) and \( K \), representations \( \pi_1 : A \rightarrow B(H) \) and \( \pi_2 : B \rightarrow B(K) \), vectors \( \xi \in K \) and \( \eta \in H \), and a bounded linear operator \( T : K \rightarrow H \) such that

\[
\phi(x \otimes y) = \langle \pi_1(x)T\pi_2(y)\xi, \eta \rangle
\]
for all \( x \in A, y \in B \). Assuming that the representations \( \pi_1 \) and \( \pi_2 \) of \( A \) and \( B \) are faithful, we can identify \( A \) with \( \pi_1(A) \) and \( B \) with \( \pi_2(B) \). The above expression can be rewritten as

\[
\phi(x \otimes y) = (xTy\xi, \eta)
\]

for all \( x \in A \subseteq B(H), y \in B \subseteq B(K) \). For \( v \in A^{**}, \omega \in B^{**} \), the element \( v \otimes \omega \) of \( A^{**} \otimes H B^{**} \) can be viewed as an element of \( (A \otimes H B)^{**} \) by

\[
v \otimes \omega(\phi) = (vT\omega\xi, \eta).
\]

This inclusion is an isometry  [5, 10]. Thus \( \theta^{**}_H(v \otimes \omega) = v^* \otimes \omega^* \).

If for some \( \eta \in N \), the von Neumann algebras \( A^{**} \) or \( B^{**} \) (say \( A^{**} \)) contain an isomorphic copy of \( M_n \), then, by Lemma 2.1,

\[
n^{1/2} = \left\| \sum_{j=1}^n e_{1j} \otimes e_{jj} \right\|_H = \left\| \theta^{**}_H \left( \sum_{j=1}^n e_{1j} \otimes e_{jj} \right) \right\|_H \leq \left\| \theta^{**}_H \right\| \left\| \sum_{j=1}^n e_{1j} \otimes e_{jj} \right\|_H = \left\| \theta^{**}_H \right\|_H,
\]

by the injectivity of the Haagerup norm [13]. It follows that \( A^{**} \) and \( B^{**} \) cannot contain a type \( I_n \) factor for \( n > \|\theta_H\|^2 \). So, \( A^{**} \) and \( B^{**} \) are of the form \( \oplus N_j \), \( j \leq \|\theta_H\|^2 \), where \( N_j \) is a von Neumann algebra of type \( I_j \). If \( \pi \) is an irreducible representation of \( A \) on a Hilbert space \( H \), there is a normal representation \( \pi \) of \( A^{**} \) on \( H \) such that \( \pi(A^{**}) = \pi(A) \) (weak closure) = \( B(H) \). Hence, \( \dim H \leq \|\theta_H\|^2 \). Similarly, \( B \) is also subhomogeneous.

**Theorem 2.3.** Let \( A \) and \( B \) be infinite-dimensional \( C^* \)-algebras and \( \theta(a \otimes b) = a^* \otimes b^* \). If \( \theta \) lifts to a continuous map \( \theta_H \) on \( A \otimes H B \), then \( \theta_H \) is an isometry if and only if \( A \) and \( B \) are commutative.

**Proof.** If \( A \) and \( B \) are commutative, by the definition of the Haagerup norm, \( \theta_H \) is an isometry. Conversely, if \( \theta_H \) is an isometry on \( A \otimes H B \), then \( \theta_H \) lifts to an isometry \( \theta^{**}_H \) on \( (A \otimes H B)^{**} \). As in Theorem 2.2, \( \theta^{**}_H(v \otimes \omega) = v^* \otimes \omega^* \) for all \( v \in A^{**} \) and \( \omega \in B^{**} \). If at least one of the von Neumann algebras \( A^{**} \) or \( B^{**} \) is not commutative, say \( A^{**} \), then by the decomposition of a von Neumann algebra into type \( I \), \( II_1 \), \( II_\infty \), \( III \), it follows that \( A^{**} \supset M_n \) for some \( n > 1 \) [11]. Lemma 2.1 and the injectivity of Haagerup norm [13] now show that \( \theta^{**}_H \) is not an isometry. Hence \( A^{**} \) and \( B^{**} \) are commutative, and in particular so are \( A \) and \( B \).

Let \( A \) be a \( C^* \)-algebra. By the definition of Haagerup norm, \( A \otimes H A \) is a Banach \( * \)-algebra with isometric involution given by \( a \otimes b \rightarrow b^* \otimes a^* \), \( a, b \in A \). For a Hilbert space \( H \), \( \pi : A \otimes H A \rightarrow B(H) \) will be called a \( * \)-representation if \( \pi \) is a bounded algebraic homomorphism satisfying \( \pi(b^* \otimes a^*) = (\pi(a \otimes b))^* \) for all \( a, b \in A \). If, in addition, \( \pi(A \otimes H A) \) is \( \sigma \)-weakly dense in \( B(H) \), then \( \pi \) is said to be irreducible.

**Theorem 2.4.** Let \( A \) be a unital \( C^* \)-algebra. If \( A \otimes H A \) has a faithful \( * \)-representation, then \( A \) is commutative.
Let $\pi$ be a faithful *-representation of $A \otimes H A$ on a Hilbert space $H$. Putting $\pi_1(a) = \pi(a \otimes 1)$ and $\pi_2(a) = \pi(1 \otimes a)$, $a \in A$, it is easy to verify that $\pi_1$ and $\pi_2$ are bounded monomorphisms from $A$ into $B(H)$ satisfying $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a, b \in A$ and $\pi_1(a^*) = \pi_2(a)^*$, $a \in A$. If $h$ is a self-adjoint element of $A$, then $\|\exp ith\| = 1$ for all $t \in \mathbb{R}$. The *-homomorphism $\pi$ from the Banach *-algebra $A \otimes H A$ to $B(H)$ is norm reducing [14, Proposition 1.5.2]. Thus

$$\|\exp it\pi_1(h)\| = \|\exp it(h \otimes 1)\| \leq \|\exp it(h \otimes 1)\|_H = \|\exp ith\| = 1,$$

for all $t \in \mathbb{R}$. Hence, $\|\exp it\pi_1(h)\| = 1$ for all $t \in \mathbb{R}$. So $\pi_1(h)$ is a self-adjoint element of $B(H)$. Let $a = h + ik$, where $h$ and $k$ are self-adjoint elements of $A$. Now

$$\pi_1(a^*) = \pi_1(h - ik) = \pi_1(h) - i\pi_1(k) = (\pi_1(h) + i\pi_1(k))^* = (\pi_1(a))^*.$$

This implies that $\pi_1(a^*) = \pi_1(a)^* = \pi_2(a)^*$ for all $a \in A$ and, thus, $\pi_1(a) = \pi_2(a)$ for all $a \in A$. But $\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$, so $\pi(ab - ba \otimes 1) = \pi_1(ab - ba) = 0$ for all $a, b \in A$. Since $\pi$ is faithful, we have $ab - ba = 0$ for all $a, b \in A$, i.e. $A$ is commutative. \hfill \square

It is well known for a $C^*$-algebra $A$, $\cap \{\ker \pi : \pi$ is a *-representation of $A\} = \{0\}$, i.e. $A$ is *-semi simple. An equivalent form of the above result is the following.

**Corollary 2.5.** Let $A$ be a unital $C^*$-algebra. Then $A \otimes H A$ is *-semi simple if and only if $A$ is commutative.

Recall that a Banach *-algebra $A$ is said to be Hermitian if every self-adjoint element of $A$ has real spectrum [7]. Moreover, in a Hermitian Banach *-algebra $A$, the radical of $A$ equals the star radical of $A$ [7, Theorem 4.9]. Since $\text{rad}(A \otimes H A) = (0)$ by [1, Proposition 5.16], we have the following.

**Corollary 2.6.** Let $A$ be a unital $C^*$-algebra. Then $A \otimes H A$ is Hermitian if and only if $A$ is commutative.

A careful reading of the proof of Theorem 2.4 shows the following.

**Proposition 2.7.** Let $A$ be a unital $C^*$-algebra and $\pi$ a *-representation of $A \otimes H A$, then there is a *-representation $\pi_0$ of $A$ satisfying $\pi(a \otimes b) = \pi_0(ab)$ and $\pi_0(A)$ is abelian.

Suppose that $A$ is a $C^*$-algebra having only a finite number of closed two-sided ideals. Let $K$ be a closed *-ideal of $A \otimes H A$. By [1, Theorem 5.3], $K = \sum_j (K_j \otimes H I_j)$, where $K_j, I_j$ are closed ideals of $A$ and, hence, *-ideals. Thus any *-ideal of $A \otimes H A$ is of the form

$$\sum_j (K_j \otimes H I_j + I_j \otimes H K_j).$$

In particular, the only closed proper *-ideals of $B(H) \otimes H B(H)$ are $B(H) \otimes H K(H) + K(H) \otimes H B(H)$ and $K(H) \otimes H K(H)$.

Our next result characterizes the *-ideals of $A \otimes H A$ annihilated by a *-representation of $A \otimes H A$.
Theorem 2.8. Let $A$ be a unital $C^*$-algebra. Then a closed two-sided $^*$-ideal $J$ of $A \otimes_H A$ is annihilated by a $^*$-representation $\pi$ of $A \otimes_H A$ if and only if there is a $^*$-representation $\pi_0$ of $A$ with $\pi_0(A)$ abelian such that $J \subseteq J_0 \otimes_H A + A \otimes_H J_0$, $J_0 = \ker \pi_0$.

Proof. Suppose that $J \subseteq \ker \pi$, where $\pi$ is a $^*$-representation of $A \otimes_H A$ on a Hilbert space $H$. Let $\pi_0$ be a $^*$-representation of $A$ as in Proposition 2.7 and $J_0 = \ker \pi_0$. Clearly, $\ker \pi \supseteq J_0 \otimes_H A + A \otimes_H J_0$ and $A/J_0 \otimes_H A/J_0$ is commutative. Let $q : A \otimes_H A \to A/J_0 \otimes_H A/J_0$ be the quotient map with kernel $J_0 \otimes_H A + A \otimes_H J_0$. The representation $\pi$ induces a faithful representation $\pi_0$ of $A/J_0 \otimes_H A/J_0$ on $H$. Moreover, the following diagram commutes.

So $\pi(J) = 0$ implies that $q(J) = 0$. Thus $J \subseteq J_0 \otimes_H A + A \otimes_H J_0$. Conversely, suppose that $J \subseteq J_0 \otimes_H A + A \otimes_H J_0$, $J_0 = \ker \pi_0$, $\pi_0$ is a $^*$-representation of $A$ with $\pi_0(A)$ abelian. Defining $\pi$ by $\pi(a \otimes b) = \pi_0(ab)$ on $A \otimes_H A$, it is easy to verify that $\pi$ is a $^*$-representation of $A \otimes_H A$ and $J \subseteq \ker \pi$. □

Let $H$ be a separable infinite-dimensional Hilbert space, it follows from the above theorem that the $^*$-ideal $K(H) \otimes_H K(H)$ cannot be annihilated by a $^*$-representation of $B(H) \otimes_H B(H)$.

In contrast to Theorem 2.8, if the involution $a \otimes b \to b^* \otimes a^*$ is dropped, then of course for every proper closed two-sided ideal $J$ there is a bounded algebraic homomorphism $\pi : A \otimes_H A \to B(H)$, satisfying $\pi(a^* \otimes b^*) = \pi(a \otimes b)^*$, $a, b \in A$ such that $J \subseteq \ker \pi$. The proof of this result is implicitly contained in [1] (see also [2]), but to be more explicit, we outline the proof.

Theorem 2.9. Let $A$ and $B$ be unital $C^*$-algebras. Then every proper closed two-sided ideal of $A \otimes_H B$ is annihilated by a representation of $A \otimes_H B$.

Proof. Let $J$ be a proper closed two-sided ideal of $A \otimes_H B$ and $J_{\min}$ be the closure of $J$ in $A \otimes_{\min} B$, where $A \otimes_{\min} B$ is the completion of the algebraic tensor product with $\| \cdot \|_{\min}$ norm. If $J_{\min} = A \otimes_{\min} B$, then $J_{\min}$ will contain all elementary tensors, so, by [1, Theorem 4.4], $J$ will be equal to $A \otimes_H B$. Thus, $J_{\min}$ is a proper closed two-sided ideal in the $C^*$-algebra $A \otimes_{\min} B$. Let $\pi$ be an irreducible representation of $A \otimes_{\min} B$ on a Hilbert space $H$ annihilating $J_{\min}$. Let $\pi_1(a) = \pi(a \otimes 1)$ and $\pi_2(b) = \pi(1 \otimes b)$, for all $a \in A$ and $b \in B$. Then $\pi_1$ and $\pi_2$ are commuting representations of $A$ and $B$, respectively. Let $M = \ker \pi_1$ and $N = \ker \pi_2$. Let $q : A \otimes_B B/N$ be the quotient map and let $\pi_1 \cdot \pi_2 : A/M \otimes_H B/N \to B(H)$ be the faithful representation of $A/M \otimes_H B/N$ induced by $\pi_1$ and $\pi_2$ (see [1] for details). So $\pi(J) = 0$ implies that $q(J) = 0$. But $A/M \otimes_B B/N \simeq A \otimes_H B/M \otimes_H B + A \otimes_H N$, thus $J \subseteq M \otimes_H B + A \otimes_H N$. Since $M \otimes_H B + A \otimes_H N$ is primitive [1, Theorem 5.13], there is a representation $\sigma$ of $A \otimes_H B$ such that $J \subseteq \ker \sigma$. □
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References