FARTHEST POINTS IN $W^*$-COMPACT SETS

R. DEVILLE AND V.E. ZIZLER

We show that while farthest points always exist in $w^*$-compact sets in duals to Radon-Nikodym spaces, this is generally not the case in dual Radon-Nikodym spaces. We also show how to characterise weak compactness in terms of farthest points.

The purpose of this note is to find under what conditions the Edelstein-Asplund-Lau results on the existence of farthest points in weakly compact sets can be extended to $w^*$-compact sets.

To fix our notation, let $C$ be a norm closed bounded subset of a real Banach space $X$ and $x$ be an element of $X$. We define

$$ r(x) = r(x,C) = \sup\{\|x - z\| \mid z \in C\} $$

and call $r(x)$ the farthest distance from $x$ to $C$. Equivalently, $r(x)$ is the radius of the smallest ball of centre $x$, containing $C$. The function $r$ is convex as supremum of such functions, and continuous since $|r(x) - r(y)| \leq \|x - y\|$, for all $x, y \in X$. A point $z \in C$ is called a farthest point of $C$ if there exists $x \in X$ such that $\|x - z\| = r(x)$. The existence of a farthest point of $C$ is equivalent to the fact that the set

$$ D = \{x \in X \mid (\exists z \in C)(\|x - z\| = r(x))\} $$

is non-empty. Since it follows that any farthest point of a convex set $C$ in a locally uniformly rotund space is a strongly exposed point of $C$, the notion of a farthest point is widely used in the study of the extreme structure of sets. In fact, it was the first discovered method of obtaining exposed points in sets [10]. We refer the reader to [3] and [5] for unexplained notions.

Extending the results of [6] and [1], Lau showed [9] that if $C$ is a weakly compact set in a Banach space $X$, then the set $D$ defined above is dense in $X$ and thus, in particular, $C$ has farthest points. Other extensions of the results of [6] and [1] may be found in [11]. The connection between farthest points and strongly exposed points in

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mentioned above led Edelstein and Lewis [7] to ask the following question: Does the existence of (strongly) exposed points in $C$ imply the existence of farthest points in $C$?

Proposition 1 gives a negative answer to this question. Moreover, it shows that Lau’s result on weakly compact sets does not always extend to $w^*$-compact sets in a dual space $X^*$. However, our Proposition 3 shows that if $X$ has the Radon-Nikodym property then any $w^*$-compact set in $X^*$ contains a farthest point, thus extending Proposition 1 of [11]. Finally, we point out in Proposition 4 that Lau’s result actually characterises weakly compact sets.

**Proposition 1.** Let $\ell^1(N)$ be the Banach space of all real summable sequences $x = (x_n)$, equipped with its usual norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$. Let

$$
C = \{ (a_n) \in \ell^1(N) : \sum_{n=1}^{\infty} (|a_n| + |a_n|^2) \leq 1 \}. 
$$

Then $C$ is a weak*--compact convex of $\ell^1(N)$ and $C$ has no farthest points.

**Proof of Proposition 1:** To prove that $C$ has no farthest points, it is enough to show that, given $x \in \ell^1(N)$:

i) $r(x) = 1 + \|x\|$ and

ii) $\|x - z\| < 1 + \|x\|$ for all $z \in C$.

To do so, first notice that if $z \in C$ then $\|z\| < 1$. Therefore, $\|x - z\| < \|x\| + \|z\| < 1 + \|x\|$ and ii) is proved. This also shows that $r(x) \leq 1 + \|x\|$. Hence, to prove i), it is enough to construct a sequence $(u_n)$ of elements of $C$ such that

$$
\lim_{n \to \infty} \|x - u_n\| = 1 + \|x\|.
$$

For each $n \in N$, let $\delta_n > 0$ be such that $n(\delta_n + \delta_n^2) = 1$. Note that $0 < \delta_n < \frac{1}{n}$ and $\lim_{n \to \infty} n\delta_n = 1$. Let $u_n = (\delta_n, \ldots, \delta_n, 0, 0, \ldots)$ where $\delta_n$ is repeated $n$ times. By our choice of $\delta_n$, $u_n \in C$.

Now given $x \in \ell^1(N)$ and $\epsilon > 0$, let $p \in N$ be such that $\sum_{n=p+1}^{\infty} |x_i| < \frac{\epsilon}{3}$. We have for $n > p$

$$
\|x - u_n\| = \sum_{i=1}^{p} |x_i - \delta_n| + \sum_{i=p+1}^{n} |x_i - \delta_n| + \sum_{i=n+1}^{\infty} |x_i| 
\geq \sum_{i=1}^{p} |x_i| - p\delta_n - \sum_{i=p+1}^{n} |x_i| + (n - p)\delta_n 
\geq \|x\| - \frac{\epsilon}{3} - p\delta_n - \frac{\epsilon}{3} + (n - p)\delta_n.
$$
Choose \( n_0 > p \) big enough so that \((n - 2p)\delta_n > 1 - \frac{\varepsilon}{3}\) for \( n \geq n_0 \). We have that for all \( n \geq n_0 \):

\[
\|x - u_n\| \geq 1 + \|x\| - \varepsilon.
\]

Therefore,

\[
\lim_{n \to \infty} \|x - u_n\| = 1 + \|x\|.
\]

**Remarks.** (a) Since \( \ell^1(N) \) has the Radon-Nikodym property, \( C \) is the norm closed convex hull of its strongly exposed points [3]. Hence, Proposition 1 gives a negative answer to the question of Edelstein and Lewis mentioned above.

(b) The proof of Proposition 1 shows that the intersection of all the balls containing \( C \) is the unit ball of \( \ell^1(N) \). Note also that the sequence \( \{u_n\} \) in the proof does not depend on \( x \).

Before proceeding, let us recall that the subdifferential of a convex function \( f \) defined on a Banach space \( X \) is defined by

\[
\partial f(x) = \{x^* \in X^* | (\forall y \in X)((x^*, y - x) \leq f(y) - f(x))\}.
\]

It was shown in [9] that if \( C \) is a closed bounded subset of a Banach space \( X \) and \( r(x) \) is the farthest distance function for \( C \), then for any \( x \in X \) and \( x^* \in \partial r(x) \), we have that \( \|x^*\| \leq 1 \) and thus \( \sup\{(x^*, x - z) : z \in C\} \leq \|x^*\|r(x) \leq r(x) \). Moreover, Lau showed that the set

\[
G = \{x \in X | (\forall x^* \in \partial r(x))(\sup\{(x^*, x - z) : z \in C\} = r(x))\},
\]

is dense \( G_\delta \) in \( X \).

We shall need the following.

**Lemma 2.** Let \( C \) be a closed bounded subset of a Banach space \( X \) and \( r \) the farthest distance function on \( X \) associated with \( C \). Assume that \( r \) is Fréchet differentiable at \( x_0 \in X \). Then

\[
r(x_0) = \sup\{(x^*_0, x_0 - z) | z \in C\}, \quad \text{where} \ \partial r(x_0) = \{x^*_0\}.
\]

**Proof:** According to Lau's result mentioned before the statement of Lemma 2 it is enough to show that, under our assumptions, the following implication holds:

If \( r(x_0) - \sup\{(x^*_0, x_0 - z) | z \in C\} = \alpha > 0 \), then there is a neighbourhood \( U \) of \( x_0 \) such that

\[
r(y) - \sup\{(y^*, y - z) | z \in C\} > 0,
\]

\[\text{where} \ \partial r(x_0) = \{x^*_0\}.
\]
whenever \( y \in U \) and \( y^* \in \partial r(y) \).

To prove that the implication holds, take \( y \in X, y^* \in \partial r(y) \) and \( z \in C \). We have:

\[
|\langle y^*, y - z \rangle - \langle x_0^*, x_0 - z \rangle| \leq |\langle y^*, y - z \rangle - \langle y^*, x_0 - z \rangle| + |\langle y^*, x_0 - z \rangle - \langle x_0^*, x_0 - z \rangle| \\
\leq \|y^*\| \|y - x_0\| + \|y^* - x_0^*\| \|x_0 - z\|.
\]

Therefore,

\[
|\sup\{(y^*, y - z)\mid z \in C\} - \sup\{(x_0^*, x_0 - z)\mid z \in C\}| \leq \|y - x_0\| + r(x_0)\|y^* - x_0^*\|.
\]

Now choose a neighbourhood \( U \) of \( x_0 \) such that

\[
2\|y - x_0\| + r(x_0)\|y^* - x_0^*\| < \alpha
\]

whenever \( y \in U \) and \( y^* \in \partial r(y) \). This is possible due to the Fréchet differentiability of \( r \) at \( x_0 \) [2, Lemma 5]. If \( y \in U \) and \( y^* \in \partial r(y) \), then

\[
(r(x_0) - \sup\{(x_0^*, x_0 - z)\mid z \in C\}) - (r(y) - \sup\{(y^*, y - z)\mid z \in C\}) \\
\leq |r(x_0) - r(y)| + |\sup\{(y^*, y - z)\mid z \in C\} - \sup\{(x_0^*, x_0 - z)\mid z \in C\}| \\
\leq \|x_0 - y\| + \|y - x_0\| + r(x_0)\|y^* - x_0^*\| < \alpha
\]

and this implies, by the definition of \( \alpha \), that

\[
r(y) - \sup\{(y^*, y - z)\mid z \in C\} > 0
\]

whenever \( y \in U \) and \( y^* \in \partial r(y) \). Our implication is proven and the proof of Lemma 2 is finished.

**PROPOSITION 3.** Let \( X \) be a Banach space with the Radon-Nikodym property, let \( X^* \) be its dual space in its usual dual norm and \( C \) be a \( w^* \)-compact subset of \( X^* \). Then the set \( D \) of all points in \( X^* \) which have farthest points in \( C \) contains a subset \( D_1 \) dense and \( \sigma \)-compact in \( X^* \).

**PROOF:** First notice that the farthest distance function \( r \) associated with \( C \) is \( w^* \)-lower semicontinuous as supremum of such functions. Since \( X \) has the Radon-Nikodym property it follows from [4] that \( r \) is Fréchet differentiable on a dense \( G_{\delta} \) subset \( D_1 \) in \( X^* \). So, to finish the proof of Proposition 3 we show that if \( r \) is Fréchet-differentiable at \( x \), then \( x \) admits a farthest point in \( C \). If \( x \in D_1 \), denote by \( x^* \) the only element of \( \partial r(x) \). Since \( r \) is \( w^* \)-lower semicontinuous and Fréchet differentiable at \( x \), \( x^* \in X \) [2, Corollary 1]. Since \( C \) is \( w^* \)-compact, there is a \( x_0 \in C \) such that \( \langle x^*, x - x_0 \rangle = \sup\{(x^*, x - z)\mid z \in C\} \). Using Lemma 2 we have \( r(x) = \langle x^*, x - x_0 \rangle \leq \|x^*\| \cdot \|x - x_0\| \leq \|x - x_0\| \leq r(x) \). Thus \( \|x - x_0\| = r(x) \) and the proof is finished.
PROPOSITION 4. Let $X$ be a Banach space and $C$ be a closed convex bounded subset of $X$. The following are equivalent:

(i) $C$ is weakly compact;

(ii) for every equivalent norm $||| \cdot |||$ on $X$, $D = \{z \in X | r(x) = ||x - z||_1 \}

for some $z \in C \}$ is dense in $X$, where $r(x) = \sup\{|x - z||z \in C\}$.

PROOF: (i) $\Rightarrow$ (ii) was proved by Lau in [9]. Conversely, let $|| \cdot ||$ be the norm of $X$ and let $B(x,r)$ denote the $|| \cdot ||$-ball centred at $x$ and radius $r$. Assume without loss of generality that $C \subset B(0,1)$. By James' theorem [5] there is a functional $f \in X^*$, $\|f\| = 1$ which does not attain its supremum on $C$. Let $B_1 = B(0,6) \cap \{x \in X | f(x) \leq 1\}$ and $|| \cdot ||_1$ be the Minkowski functional of $B_1$. Note that if $B_1(x,r)$ is the ball centred at $x$ with radius $\delta$ with respect to the norm $|| \cdot ||_1$, then

$$B_1(x,\delta) = 6B(x,\delta) \cap \{y \in X | -\delta \leq f(x) - f(y) \leq \delta\}.$$

Pick $z_0 \in B(0,4) \cap \{x \in X | f(x) < -3\}$. We claim that if $x \in B(z_0,1)$, then $x$ has no farthest point in $C$ when $X$ is equipped with the new norm $|| \cdot ||_1$, hence $D$ is not dense in $X$.

To see it, we first show that $r(x) = \alpha$, where $\alpha = \sup\{|f(z)|z \in C\} - f(x)$. Indeed, let $y \in C$; we have $|f(y) - f(x)| \leq \sup\{|f(z)|z \in C\} - f(x)$ and $\|y - x\| \leq \|y\| + \|z_0\| + \|x_0 - x\| \leq 6 \leq 6\alpha$ (note that for any $y \in C$ and $z \in B(x_0,1)$ we have $f(y) \geq -1$ and $f(x) \leq f(z_0) + f(x - x_0) < -2$ and thus $\alpha > 1$). Thus $y \in B(z,\alpha)$.

Therefore $C \subset B_1(x,\alpha)$ and this shows that $r(x) \leq \alpha$. Conversely if $\delta < \alpha$, there exists $y \in C$ such that $f(y) - f(x) > \delta$ and so $y \notin B_1(x,\delta)$. This shows that $r(x) \geq \alpha$.

Now for $y \in C$, we have $\|y - x\| \leq 6 < 6\alpha$ and $|f(y) - f(x)| < \sup\{|f(z)|z \in C\} - f(x) = \alpha$, hence $\|y - x\|_1 < \alpha = r(x)$ and so $y$ is not a farthest point from $x$ in $C$.

Remark. We would like to point out that, while differentiability properties can be used to show that the set $D$ of points which have at least one farthest point in a $w^*$-compact $C$ of a dual Banach space $X$ is dense in $X$, convexity properties can be used to show the uniqueness of farthest points. More precisely, using standard rotundity arguments, we have

PROPOSITION 5. Let $X$ be a strictly convex Banach space, and $C$ be a norm closed bounded subset of $X$ such that the corresponding set $D$ is dense in $X$.

Then the set

$$U = \{y \in X | y \text{ has a unique farthest point in } C\}$$

is also dense in $X$. 

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PROOF: First the strict convexity of $X$ implies that for all $x \in X$ and $\lambda > 1$, 
\[ B(x, ||x||) \cap S(\lambda x, ||\lambda x||) = \{0\} \]
where $B(x, r)$ (respectively, $S(x, r)$) denotes the ball (respectively, the sphere) of centre $x$ and radius $r$.

Now let $x \in D$ and let $z$ be a farthest point from $x$ in $C$. By translation, there is no loss of generality in assuming $z = 0$. It is enough to show that for any $\lambda > 1$, $y = \lambda x$ has a unique farthest point in $C$, namely 0. We have 
\[ B(\lambda x, ||\lambda x||) \supseteq B(x, ||x||) \supseteq C \]
and hence $r(\lambda x) \leq ||\lambda x||$. On the other hand, 
\[ \{0\} \subset C \cap S(\lambda x, ||\lambda x||) \subset B(x, ||x||) \cap S(\lambda x, ||\lambda x||) = \{0\}. \]
Therefore $r(\lambda x) = ||\lambda x||$ and the only farthest point from $y$ in $C$ is 0.

Note that if $X$ is not strictly convex, the set $U$ of points which admit unique farthest points can be empty. Indeed, let $X = c_0(N)$ and 
\[ C = \{(x_n) \in c_0(N) | 0 \leq z_n \leq \frac{1}{n} \text{ for all } n\}. \]
$C$ is a norm compact convex subset of $c_0(N)$, and therefore $D = c_0(N)$. On the other hand, if $x = (x_n) \in c_0(N)$, we have 
\[ r(x) = \max\{\max\{|z_n|, |z_n - \frac{1}{n}|\} | n \in N\} \]
so $r(x) = |z_{n_0}|$ or $r(x) = |x_{n_0} - \frac{1}{n_0}|$, for some $n_0 \in N$. Define:
\[ C_1 = \{(x_n) \in C | z_{n_0} = 0\} \]
\[ C_2 = \{(x_n) \in C | z_{n_0} = \frac{1}{n_0}\}. \]
If $r(x) = |z_{n_0}|$ (respectively, $r(x) = |x_{n_0} - \frac{1}{n_0}|$), $C_1$ (respectively, $C_2$) is included in the set of farthest points from $x$ in $C$. In both cases $x \notin U$.

REFERENCES

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