AN OPEN MAPPING THEOREM
ON HOMOGENEOUS SPACES

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Abstract

We shall prove an open mapping theorem concerning a Polish group acting transitively on a complete metric space.


Let $X$ be a topological space and $G$ be a topological transformation group on $X$ which is a Polish (separable, complete metric) group acting transitively on $X$. This action is denoted by the map $\psi$ from $G \times X$ to $X$ with $\psi(g , x) = g \cdot x$ for $g \in G$ and $x \in X$.

We shall consider an open mapping theorem on the map $G \ni g \rightarrow \psi(g , x) = g \cdot x \in X$ from $G$ to $X$ for each fixed $x \in X$.

Even if there is a famous open mapping theorem on Banach spaces or topological linear spaces, the proof of this case is different from that of Banach space cases.

We shall call $\psi$ continuous if $\psi$ is continuous as the map from $G \times X$ to $X$. We shall call $\psi$ separately continuous if

1. for each fixed $g \in G$, the map $x \rightarrow g \cdot x$ from $X$ to $X$ is continuous,
2. for each fixed $x \in X$, the map $g \rightarrow g \cdot x$ from $G$ to $X$ is continuous.

Now, we have the following theorem.
THEOREM A. Let $G$ be a complete metric group acting on a metric space $X$. If the map $\Psi: G \times X \to X$ defined by $\Psi(g, x) = g \cdot x$ is separately continuous, then $\Psi$ is continuous.

**Proof.** Due to the inequality:

$$d(gx, g_0x_0) \leq d(gx, gx_0) + d(gx_0, g_0x_0),$$

the continuity of $\psi$ at $(g_0, x_0) \in G \times X$ follows from the claim that for any $\varepsilon > 0$ there exist $\delta > 0$ and a neighborhood $U$ of $g_0$ such that $d(gx, gx_0) < \varepsilon$ whenever $d(x, x_0) < \delta$ and $g \in U$. Fix $x_0 \in X$, and set

$$A_{m,n} = \left\{ g \in G : d(gx, gx_0) \leq \frac{1}{m} \text{ if } d(x, x_0) < \frac{1}{n} \right\}.$$

The separate continuity of $\psi$ yields the closedness of each $A_{m,n}$ and $G = \bigcup_{n=1}^{\infty} A_{m,n}$. Furthermore, $A_{m,n} \subset A_{m,n+1}$ by construction. Baire's category property of $G$ implies that every non-empty open subset $U$ of $G$ contains a non-empty open subset $V \subset U$ such that $V \subset A_{m,n(m)}$ for some $n(m) \in \mathbb{N}$. Thus, we can find a sequence of open subsets $O_m$ of $G$ such that

$$O_m \subset A_{m,n(m)}, \quad \text{diameter } (O_m) < 1/m,$$

$$O_m \supset \overline{O}_{m+1}.$$

By the completeness of $G$, there exists a point $g_0 \in \bigcap_{m=1}^{\infty} O_m$. It then follows that $d(gx, gx_0) \leq 1/m$ if $d(x, x_0) < 1/n(m)$ and $g \in O_m$. Hence $\psi$ is continuous at $(g_0, x_0)$ by the first observation. Therefore, for any $\varepsilon_1 > 0$ there exists $\delta > 0$ and a neighborhood $U$ of $g_0$ such that $d(gx, g_0x_0) < \varepsilon_1$ whenever $g \in U$ and $d(x, x_0) < \delta$. For an arbitrary $g_1 \in G$ and $\varepsilon > 0$, choose $\varepsilon_1 > 0$ so that $d(g_1g_0^{-1}y, g_1x_0) < \varepsilon$ whenever $d(y, g_0x_0) < \varepsilon_1$. We then set $V = g_1g_0^{-1}U$ as a neighborhood of $g_1$ and conclude that if $g \in V$ and $d(x, x_0) < \delta$ then $g_0g_1^{-1}g \in U$ so that $d(g_0g_1^{-1}gx, g_0x_0) < \varepsilon_1$ and therefore

$$d(gx, g_1x_0) < \varepsilon \quad \text{if} \quad g \in V \text{ and } d(x, x_0) < \delta.$$

Thus $\psi$ is continuous at $(g_1, x_0) \in G \times X$.

Next, we shall state an open mapping theorem in this case. In this note, a group $G$ acting transitively on a metric space $X$ means that for each $x, y \in X$ there exists an element $g \in G$ with $gx = y$.

THEOREM B. Let $G$ be a Polish group acting transitively on a complete metric space $X$. For each $x \in X$ the map $G \ni g \to \psi(g, x) = g \cdot x \in X$ is open.
PROOF. Let $B(x, \delta) = \{y; d(x, y) < \delta\}$, that is, the open ball with center $x$ and radius $\delta > 0$. Let $x_0 \in X$ be fixed. We shall prove that for any neighbourhood (nbd in short) $U$ of the unit $e$ of $G$, there exists an open ball with

$$
\overline{Ux_0} \supset B(x_0, \delta)
$$

for some $\delta > 0$.

If this is proved, then we can prove the theorem as follows: for any $x_0 \in X$ and for any nbd $U_0$ of the unit $e$ of $G$, there exist $\varepsilon_0 > 0$ and $B(x_0, \varepsilon_0) \subset \overline{U_0x_0}$. Now, by induction we will construct two types of decreasing sequences of nbds of $e$: $U_n$ and $V_n$ for a given $U_0$. We want to prove that any point $y_0$ of $B(x_0, \varepsilon_0)$ is contained in $U_0x_0$.

There exists $V_0 \subset U_0$ such that $V_0 = V_0^{-1}$ (symmetric), $V_0^2 \subset U_0$ and $\overline{V_0y_0} \subset B(x_0, \varepsilon_0)$. Choose $\delta_0 > 0$ such that

$$
\overline{V_0y_0} \supset B(y_0, \delta_0) \quad \text{and} \quad \delta_0 < (1/2)\varepsilon_0.
$$

Then there exists $g_0 \in U_0$ with $g_0 \cdot x_0 = x_1 \in B(y_0, \delta_0)$. Choose a nbd $U_1$ of $e$ such that

$$
\overline{U_1x_1} \subset B(y_0, \delta_0) \quad \text{with} \quad U_1 = U_1^{-1}, \ U_1^2 \subset U_0
$$

and

$$
d(U_1g_0) \quad (= \text{diameter of } U_1g_0) \leq (1/2)\varepsilon_0.
$$

Choose further $0 < \varepsilon_1 < (1/2)\varepsilon_0$ so that

$$
\overline{U_1x_1} \supset B(x_1, \varepsilon_1).
$$

Then, there exists $h_0 \in V_0$ such that

$$
y_1 = h_0y_0 \in B(x_1, \varepsilon_1).
$$

Next, choose $V_1$ such that $V_1 = V_1^{-1}$, $V_1^2 \subset V_0$, with

$$
\overline{V_1y_1} \subset B(x_1, \varepsilon_1) \quad \text{and} \quad d(V_1h_0) < (1/2)\delta_0.
$$

Then there exists $0 < \delta_1 < (1/2)\delta_0$ such that

$$
\overline{V_1y_1} \supset B(y_1, \delta_1).
$$

Continue this process to get $\{U_n\}$, $\{V_n\}$, $\{\varepsilon_n\}$, $\{\delta_n\}$, $\{g_n\}$, $\{h_n\}$, $\{x_n\}$, and $\{y_n\}$ such that

$$
x_{n+1} = g_n \cdot x_n = \cdots = g_n \cdot g_{n-1} \cdots g_0 \cdot x_0,
$$

$$
y_{n+1} = h_n \cdot y_n = \cdots = h_n \cdot h_{n-1} \cdots h_0 \cdot y_0,
$$

$$
d(U_ng_{n-1} \cdots g_0) < (1/2)\varepsilon_{n-1}, \quad g_n \in U_n,
$$

$$
d(V_nh_{n-1} \cdots h_0) < (1/2)\delta_{n-1}, \quad h_n \in V_n,
$$

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with $0 < \varepsilon_n < (1/2)\varepsilon_{n-1}$ and $0 < \delta_n < (1/2)\delta_{n-1}$ and

$$U_n x_n \supset B(x_n, \varepsilon_n) \supset V_n y_n,$$

$x_{n+1} = g_n \cdot x_n \in B(y_n, \delta_n)$.

Let $\lim_{n \to \infty} g_n \cdot g_{n-1} \cdots g_1 = g$ and $\lim_{n \to \infty} h_n h_{n-1} \cdots h_1 = h$. Then $g \in U_0^2$, $h \in U_0^2$ and

$$d(x_n, y_n) < \varepsilon_n \quad \text{implies} \quad g \cdot x_0 = h \cdot y_0.$$

Therefore, we have $y_0 = h^{-1} g \cdot x_0 \in U^4 x_0$.

From this result, we have that for arbitrary $x_0 \in X$ and for arbitrary nbhd $U$, there exists $\varepsilon > 0$ with $U x_0 \supset B(x_0, \varepsilon)$. Hence the proof is complete, if the following lemma is proved.

**Lemma.** For any nbhd $U$ of $e$ in $G$, $\overline{U x_0}$ contains an open ball $B(x_0, \delta)$ for some $\delta > 0$.

**Proof.** Let $\{g_n\}$ be a sequence of elements in $G$ such that $\bigcup_n g_n U = G$, since $G$ is a Polish group. Hence $\bigcup_n g_n U x_0 = X$ so that there exists an $n_0$ such that $g_{n_0}^{-1} \overline{U x_0}$ contains an open ball; hence $\overline{U x_0}$ contains an open ball $B(y, \delta)$ because $\overline{U x_0} = g_{n_0}^{-1} g_{n_0} \overline{U x_0}$. Choose a nbhd $V$ of $e$ with $V = V^{-1}$ and $V^2 \subset U$. Then $\overline{V x_0} \supset B(y, \delta)$ means that there exists $g \in V$ such that $g \cdot x_0 \in B(y, \delta)$; hence there exists $\varepsilon > 0$ such that $B(g \cdot x_0, \varepsilon) \subset B(y, \delta) \subset \overline{V x_0}$. Thus $g^{-1} B(g \cdot x_0, \varepsilon) \subset g^{-1} B(y, \delta) \subset g^{-1} \overline{V x_0} \subset V^2 x_0 \subset \overline{U x_0}$. But $g^{-1} B(g \cdot x_0, \varepsilon)$ is an open set containing $x_0$, so that there exists $\delta_1 > 0$ such that $B(x_0, \delta_1) \subset g^{-1} B(g \cdot x_0, \varepsilon) \subset \overline{U x_0}$.

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**References**


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