RINGS IN WHICH EVERY ELEMENT IS THE SUM OF TWO IDEMPOTENTS

YASUYUKI HIRANO AND HISAO TOMINAGA

Let \( R \) be a ring with prime radical \( P \). The main theorems of this paper are (1) The following conditions are equivalent: 1) \( R \) is a commutative ring in which every element is the sum of two idempotents; 2) \( R \) is a ring in which every element is the sum of two commuting idempotents; 3) \( R \) satisfies the identity \( x^3 = x \). (2) If \( R \) is a PI-ring in which every element is the sum of two idempotents, then \( R/P \) satisfies the identity \( x^3 = x \). (3) Let \( R \) be a semi-perfect ring in which every element is the sum of two idempotents. If \( R_n \) is quasi-projective, then \( R \) is a finite direct sum of copies of \( \text{GF}(2) \) and/or \( \text{GF}(3) \).

Throughout, \( R \) will represent a ring with prime radical \( P \). A Boolean ring is defined as a ring in which every element is an idempotent. As a generalisation of Boolean rings, we consider the class of rings in which every element is the sum of two idempotents. We begin with an example which shows that such a ring need not be Boolean or even commutative.

Example. Let \( A(\neq 0) \) and \( B \) be Boolean rings, and \( W(\neq 0) \) a \( B \cdot A \)-bimodule. Assume, furthermore, that \( W \) is \( s \)-unital as a right \( A \)-module, that is, for any \( w \) in \( W \), there exists an element \( e \) in \( A \) such that \( we = w \). Then every element of the non-commutative ring \( R = \begin{pmatrix} A & 0 \\
W & B \end{pmatrix} \) is the sum of two idempotents. In fact, 
\[
\begin{pmatrix} a & 0 \\
w & b \end{pmatrix} = \begin{pmatrix} e & 0 \\
w & 0 \end{pmatrix} + \begin{pmatrix} a - e & 0 \\
w & b \end{pmatrix},
\]
where \( e \) is an element of \( A \) such that \( we = w \).

Our present objective is to prove the following theorems.

THEOREM 1. The following conditions are equivalent:

1) \( R \) is a commutative ring in which every element is the sum of two idempotents.
2) \( R \) is a ring in which every element is the sum of two commuting idempotents.
3) \( R \) satisfies the identity \( x^3 = x \).
THEOREM 2. Let $R$ be a PI-ring in which every element is the sum of two idempotents. Then $R/P$ satisfies the identity $x^3 = x$.

THEOREM 3. Let $R$ be a semi-perfect ring in which every element is the sum of two idempotents. If $\mathcal{R} = R_R$ is quasi-projective, then $R$ is a finite direct sum of copies of $GF(2)$ and/or $GF(3)$.

In preparation for proving our theorems, we state four lemmas.

LEMMA 1. Let $R(\neq 0)$ be a ring in which every element is the sum of two idempotents. If $R$ contains no non-trivial idempotents, then $R$ is either $GF(2)$ or $GF(3)$.

PROOF: Since 0 and 1 are the only idempotents of $R$, we have either $R = \{0, 1\}$ or $R = \{0, 1, 2\}$. Thus $R$ is either $GF(2)$ or $GF(3)$. $lacksquare$

LEMMA 2. Let $a$ be an element of $R$ with $a^2 = 0$.

(1) If $a = e + f$ for idempotents $e, f$ then $4a = 0$.

(2) If $a = e + f$ for commuting idempotents $e, f$ then $a = 2e$ and $4e = 0$.

PROOF: (1) Obviously,

$$0 = a^3 - 2a^2 = a + 2(ef + fe) + efe + fef - 2(a + ef + fe) = efe + fef - a.$$ 

Hence $0 = ea^2 + fa^2f = a + 3(efe + fef) = 4a$.

(2) Since $0 = a^2 = a + 2ef$, we get $a = -2ef$, and so $0 = a(f - e) = f - e$.

Hence $a = 2e$ and $4e = a^2 = 0$. $lacksquare$

LEMMA 3. Let $R$ be a ring with 1, and $n$ a positive integer greater than 1. Then the $n \times n$ full matrix ring $M_n(R)$ over $R$ contains an element which cannot be written as the sum of two idempotents.

PROOF: We write $M_n(R) = \sum_{i,j=1}^n Re_{ij}$, where $e_{ij}$ are matrix units. Suppose, to the contrary, that every element of $M_n(R)$ is the sum of two idempotents. Then, by Lemma 2(1), $4e_{12} = 0$ and so $4R = 0$. Consider the element $a = e_{11} + e_{12} + e_{21}$, and choose idempotents $e = \sum r_{ij}e_{ij}$ and $f$ such that $a = e + f$. Since $a - e = f = f^2 = a^2 - ae - ea + e$, we get $a^2 = a + ae + ea - 2e$. Comparing the coefficients of $e_{11}$, $e_{12}$ and $e_{21}$ on both sides, we get $1 = r_{12} + r_{21}$, $0 = r_{11} + r_{22} - r_{12}$ and $0 = r_{11} + r_{22} - r_{21}$, and therefore $1 = 2r_{12}$. Then $4R = 0$ implies that $1 = 4r_{12}^2 = 0$, which is a contradiction. $lacksquare$
LEMMA 4. Let \( R \) be a prime ring in which every element is the sum of two idempotents. If \( R \neq Z \), the centre of \( R \), then \( \text{char } R = 2 \) and \( Z \) is either 0 or \( GF(2) \).

PROOF: First, we claim that \( R \) cannot be reduced. Actually, if \( R \) is reduced, then every idempotent is central, and so \( R = Z \) by hypothesis, a contradiction. Hence \( R \) has a non-zero element \( a \) with \( a^2 = 0 \). By Lemma 2 (1), we conclude that \( \text{char } R = 2 \).

Now, let \( z \) be an arbitrary element of \( Z \). By hypothesis, we can write \( z = e + f \) for idempotents \( e, f \) in \( R \). Then it is easily observed that \( ef = fe \). Since \( \text{char } R = 2 \), we obtain that \( z^2 = e + f + 2ef = e + f = z \). Since \( R \) is prime, this implies that \( z \) is either 0 or 1. This completes the proof.

PROOF OF THEOREM 1: 1) \( \Rightarrow \) 3). It is well-known that \( R \) is a subdirect sum of subdirectly irreducible rings \( R \). Since, by Lemma 1, each \( R \) is either \( GF(2) \) or \( GF(3) \), \( R \) satisfies the identity \( x^3 = x \).

3) \( \Rightarrow \) 2). As is well-known, \( R \) is a commutative ring. Replacing \( x \) by \( 2x \) in \( x^3 = x \), we obtain \( 6x = 0 \). Further, replacing \( x \) by \( x^2 - x \) in \( x^3 = x \), we obtain \( 3x^2 = 3x \). By making use of these, we see easily that \( (2x^2)^2 = 4x^4 = -2x^4 = -2x^2 \) and \( (x + 2x^2)^2 = x^2 + 4x^4 + 4x^2 = x^2 + 4x^4 + 4x^2 = x + 2x^2 + 3(x - x^2) + 6x^2 = x + 2x^2 \). Hence \( x \) is the sum of the idempotents \(-2x^2 \) and \( x + 2x^2 \).

2) \( \Rightarrow \) 1). Let \( a \) be an element of \( R \) with \( a^2 = 0 \). Then, by virtue of Lemma 2 (2), there exists an idempotent \( e \) such that \( a = 2e \) and \( 4e = 0 \). Now, \( -e = f + g \) with some commuting idempotents \( f, g \). Then \( e = (e)^2 = -e + 2fg \), so \( 2e = 2fg = 2efg \). Noting that \( fe = ef \), we see easily that \( a = 2e = 2efg \). Hence \( R \) is a reduced ring. As is well-known, every idempotent of the reduced ring \( R \) is central, and so \( R \) is commutative.

COROLLARY 1. Let \( R \) be a semiprime ring. If \( R \) has the property that every element is the sum of two idempotents, then the centre \( Z \) of \( R \) has the same property.

PROOF: Since \( R \) is semiprime, \( R \) is a subdirect sum of prime rings \( R_\lambda (\lambda \in \Lambda) \). By Lemmas 1 and 4, the centre \( Z_\lambda \) of \( R_\lambda \) is 0, or \( GF(2) \), or \( GF(3) \). Now we may regard \( Z \) as a subring of the direct product \( \prod_{\lambda \in \Lambda} Z_\lambda \). Hence \( Z \) satisfies the identity \( x^3 = x \). Then, by Theorem 1, every element of \( Z \) is the sum of two idempotents in \( Z \).

PROOF OF THEOREM 2: In view of Lemma 1, it suffices to show that every prime factor ring of \( R \) is commutative. Suppose, to the contrary, that a prime factor ring \( R' \) of \( R \) is not commutative. By [3, Corollary 1], the ring \( Q(R') \) of central quotients of \( R' \) is a full matrix ring over a division ring. Then, by Lemma 4, we have that \( R' = Q(R') \). Now, Lemmas 1 and 3 force a contradiction that \( R' \) is either \( GF(2) \) or \( GF(3) \).

COROLLARY 2. Let \( R \) be an Azumaya \( Z \)-algebra in which every element is the
sum of two idempotents. Then $R$ satisfies the identity $x^3 = x$.

**Proof:** By [1, Lemma II.3.1], $Z$ is a $Z$-direct summand of $R$, say $R = Z \oplus T$. Then $P = (P \cap Z) \oplus (P \cap Z)T$ by [1, Corollary II.3.7]. As is well-known (see, for example, [1, Theorem II.3.4]), $R$ is a finitely generated $Z$-module, and therefore $R$ is a PI-algebra. Hence, by Theorem 2, $R/P$ is commutative. Then, by [1, Proposition II.1.11], we obtain $(P \cap Z)T = T$. Since $P \cap Z$ is a nil ideal of $Z$, and $T$ is a finitely generated $Z$-module, we conclude that $T = 0$, and hence $R = Z$. Now, by Theorem 1, $R$ satisfies the identity $x^3 = x$.

**Proof of Theorem 3:** By [2, Theorem 4.6], $R$ is the finite direct sum of full matrix rings over local rings. Hence, by Lemmas 1 and 3, $R$ is the finite direct sum of copies of $GF(2)$ and/or $GF(3)$.

**Remark.** As is shown in [5] (see also [4]), the following conditions are equivalent:

1) There exists an involution $*$ of $R$ such that $xx^*x = x^*$ for all $x$ in $R$;
2) $R$ is an anti-inverse ring, that is, every element $x$ in $R$ has an anti-inverse $x^*$; $xx^*x = x^*$ and $x^*xx^* = x$;
3) For each element $x$ of $R$ there exists $x^*$ in $R$ such that $x^2x^* = x^*$ and $x^*x^2 = x$;
4) $R$ is a (dense) subdirect sum of fields isomorphic to $GF(2)$ or $GF(3)$
5) $R$ satisfies the identity $x^3 = x$.

**References**


Department of Mathematics
Okayama University
Okayama 700
JAPAN