Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1139–1144

A NOTE ON OPEN EXTENSION OF MAPS

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0. Introduction. In recent years there has been some interest in trying to improve the behaviour of maps by extending their domains (see Whyburn [**10**], Baur [**3**], Krolevec [**8**], Dickman [**5**], Franklin and Kohli [**6**]). It was shown in [**6**] that every map can be extended to an open map so that certain properties of the domain and range are preserved in the new domain. In [**6**] and [**7**] we also related the topological properties of the domain and range of the mapping with the new domain; also these results were then used to obtain analogues and improvements of recent theorems of Arhangelskiĭ, Čoban, Hodel, Keesling, Nagami, Okuyama, and Proizvolov. In this note we give a method of unifying the domain and range of a mapping so as to yield a meaningful open extension. This method of unifying the domain and range of a mapping is a modification of Whyburn's "unified space technique" [**10**] (also see Dickman [**5**]). But our extension space does not satisfy separation axioms, although in some special cases it can be modified so as to be Hausdorff or regular. This modified extension is useful in obtaining partial improvements of recent theorems of Arhangelskiĭ, Čoban, and Proizvolov.

1. A unified open extension. Let f be a function, not necessarily continuous, from a topological space X into a topological space Y. We call a point $x \in X$ and its image $f(x) \in Y$ *singular points* of X and Y, respectively, if there is an open set U in X containing x such that $f(U)$ is not a neighbourhood of $f(x)$. In the sequel that follows S and T will always denote the sets of singular points of X and Y respectively.

Without any loss of generality, we may assume that the spaces X and Y are disjoint. Let W denote the set theoretic union of X and Y. Define a subset Q of W to be open if it satisfies the following conditions.

(i) The sets $Q \cap X$ and $Q \cap Y$ are open in X and Y respectively.

(ii) The set $Q \cap S = \emptyset$ or else for each $x \in Q \cap S$, the set $Q \cap Y$ contains a neighbourhood of $f(x)$ in Y.

Let τ denote the collection of all open sets (i.e., all sets satisfying (i) and (ii) above). Then the following statements are easily verified.

1.1. *The collection τ is a topology for W.*

1.2. *The set X is closed in W and the set Y is open in W.*

1.3. *The retraction map r from W onto Y, defined by $r(x) = f(x)$ for $x \in X$*

Received October 22, 1971 and in revised form, June 16, 1972.

and $r(x) = x$ *for* $x \in Y$, *is an open function. If the function* f *is continuous, so also is* r.

1.4. *If the function* f *is closed (respectively compact), then the retraction map* r *as defined in* 1.3 *is a closed (respectively compact) function.*

The space (W, τ) is called a *unified open extension space* for the function f. There may exist topologies on W distinct from τ which satisfy the conditions 1.1–1.4. For example, let the condition (ii) be replaced by (ii′): the set $Q \cap S = \emptyset$ or else for each $x \in Q \cap S$, the set $Q \cap Y$ contains a deleted neighbourhood of $f(x)$. Then the conditions (i) and (ii′) define a topology τ' on W such that $\tau \subseteq \tau'$. It is easy to give examples where τ and τ' do not coincide.

In the sequel, by the space W we shall always mean the space (W, τ), unless expressly stated otherwise.

It is routine to verify the following assertions. Hence the proofs are omitted.

1.5. *The space* W *is* T_0 *if and only if* X *and* Y *are* T_0.

1.6. *The space* W *is never* T_1 *and hence never* T_2. *Furthermore, the space* W *is never regular.*

1.7. *The space* W *is connected if the spaces* X *and* Y *are connected.*

1.8. *The space* W *is locally connected if the spaces* X *and* Y *are locally connected.*

The assertion 1.6 tells us that the space W is never T_1 and hence never T_2. Nevertheless, in some special cases it is possible to reduce W to a T_1-space or to a T_2-space by deleting some points of W. (This was first observed by Professor S. P. Franklin in an example.) The next result tells us under what conditions on the function f and on the set of singular points of X and Y, W can be reduced to a T_1-space or T_2-space.

1.9. *If singular points of* X *and* Y *do not accumulate,* $f|S$ *is one-to-one and if* $\bar{X} = W - T$, *then*

(1) *the space* $\bar{X}$ *is* T_1 *if the spaces* X *and* Y *are* T_1,

(2) *the space* $\bar{X}$ *is* T_2 *if the spaces* X *and* Y *are* T_2,

(3) *the function* $\bar{f} = r|\bar{X}$ *is open, and*

(4) *the function* $\bar{f}$ *is continuous if* f *is.*

Proof. (1). Suppose that the spaces X and Y are T_1 and let x_1, x_2 be any two distinct points of $\bar{X}$. Then the following three cases arise:

Case I. Both of the points x_1 and x_2 are in X. Since X is a T_1-space, there are neighbourhoods N_1 and N_2 (in X) of x_1 and x_2 respectively such that $x_1 \notin N_2$ and $x_2 \notin N_1$. Therefore, the sets $(N_1 \cup Y) \cap \bar{X}$ and $(N_2 \cup Y) \cap \bar{X}$ are neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively such that $x_2 \notin (N_1 \cup Y) \cap \bar{X}$ and $x_1 \notin (N_2 \cup Y) \cap \bar{X}$.

Case II. One of the points x_1 and x_2 lies in X and othe other lies in Y. Suppose $x_1 \in X$. Then $x_2 \in \bar{X} - X$. Since $\bar{X} = W - T$, $x_2 \in Y - T$. If x_1 is a nonsingular point of X, then by hypothesis on the set S of singular points of X, there is a neighbourhood N(in X) of x_1 which contains no singular point of X. Thus N and $\bar{X} - X$ are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively. If x_1 is a singular point of X, then $f(x_1) \neq x_2$ and by the hypothesis on the set S of singular points of X there exists a neighbourhood N(in X) of x_1 which contains no other singular points of X. Since Y is a T_1-space, there exists a neighbourhood N_1(in Y) of $f(x_1)$ such that $x_2 \notin N_1$. Then $(N \cup N_1) \cap \bar{X}$ and $Y - T$ are neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively such that $x_2 \notin (N \cup N_1) \cap \bar{X}$ and $x_1 \notin Y - T$.

Case III. Both of the points x_1 and x_2 are in Y. Since Y is a T_1-space, there are neighbourhoods N_1 and N_2 (in Y) of x_1 and x_2 respectively such that $x_2 \notin N_1$ and $x_1 \notin N_2$. Since Y is open in W, the sets $N_1 \cap \bar{X}$ and $N_2 \cap \bar{X}$ are neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively.

(2) Suppose that the spaces X and Y are T_2. Let x_1 and x_2 be any two distinct points of $\bar{X}$. The following three cases arise:

Case I. Both of the points x_1 and x_2 are in X. Since X is T_2, there are disjoint neighbourhoods N_1 and N_2 (in X) of x_1 and x_2 respectively. By hypothesis on the set S of singular points of X, the neighbourhoods N_1 and N_2 can be so chosen that they contain no other singular point of X. If the points x_1 and x_2 are nonsingular points of x, then N_1 and N_2 are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively. If x_1 is a singular point and x_2 is a nonsingular point, then $(N_1 \cup Y) - T$ and N_2 are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively. Since $f|S$ is one-to-one, if both of the points x_1 and x_2 are singular points of X, then $f(x_1) \neq f(x_2)$. Since Y is a T_2-space, there exist disjoint neighbourhoods N_1' and N_2' (in Y) of $f(x_1)$ and $f(x_2)$ respectively. Then $(N_1 \cup N_1') \cap \bar{X}$ and $(N_2 \cup N_2') \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively.

Case II. Only one of the points x_1 and x_2 belongs to X. Suppose $x_1 \in X$. Then $x_2 \in \bar{X} - X = Y - T$. By hypothesis on the set S of singular points of X, there exists a neighbourhood N (in X) of x_1 which contains no other singular point of X. If x_1 is a nonsingular point of X, then N and $Y - T$ are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively. If x_1 is a singular point of X, then $f(x_1) \notin \bar{X}$. Hence $f(x_1) \neq x_2$. Since Y is a T_2-space, there are disjoint neighbourhoods N_1 and N_2 in Y of $f(x_1)$ and x_2 respectively. Consequently, $(N \cup N_1) \cap \bar{X}$ and $N_2 \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively.

Case III. Both of the points x_1 and x_2 are in Y. Since Y is a T_2-space there exist disjoint neighbourhoods N_1 and N_2 in Y of x_1 and x_2 respectively. Since Y is open in W, $N_1 \cap \bar{X}$ and $N_2 \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively.

(3) Since the singular points of Y do not accumulate, the set T is closed in Y. Since Y is open in W, the set $Y - T$ is open in W. In order to show that the function $\bar{f}$ is open, let V be an open set in $\bar{X}$. Then there exists an open set U in W such that $V = U \cap (W - T)$. The set

$$\bar{f}(V) = \bar{f}(V \cap X) \cup \bar{f}(V \cap (Y - T)) = f(V \cap X) \cup (V \cap (Y - T)).$$

If $f(V \cap X)$ is open in Y, the set $\bar{f}(V)$, being the union of two open sets, is an open set. Suppose $f(V \cap X)$ is not a neighbourhood of its point y. Then $y = f(x)$ for some $x \in V \cap X$ and x is a singular point of X. Since U is open in W, there is a neighbourhood N in Y of y such that $N \subset U \cap Y$. By the hypothesis on the set T of singular points of Y, N can be so chosen that it contains no singular points of Y other than y. Then $N - \{y\} \subset V \cap (Y - T)$. Therefore, $N \subset f(V \cap X) \cup (V \cap (Y - T)) = \bar{f}(V)$. Since the set $\bar{f}(V)$ contains a neighbourhood of each of its points, it is an open set.

(4) Since f is continuous, the retraction map r is continuous and the function $\bar{f}$ being the restriction of r to $\bar{X}$ is continuous. The proof of the assertion 1.9 is now complete.

1.10. *If the singular points of X and Y do not accumulate, $f|S$ is one-to-one and if $\bar{X} = W - T$, then the space $\bar{X}$ is locally compact Hausdorff if X and Y are locally compact Hausdorff.*

Proof. Since the spaces X and Y are Hausdorff and since the hypothesis of 1.9 is satisfied, it follows that the space $\bar{X}$ is Hausdorff. Let $x \in \bar{X}$ be any point. Now the following two cases arise:

Case I. The point $x \in X$. Since the space X is locally compact, there is a compact neighbourhood N (in X) of x. By hypothesis on the set S of singular points of X, the neighbourhood N can be chosen so that it contains no other singular point of X. If x is a nonsingular point of X, then N is a compact neighbourhood of x in W. If x is a singular point of X, then $f(x)$ is a singular point of Y. Since Y is locally compact and since the singular points of Y do not accumulate, there is a compact neighbourhood N_x (in Y) of $f(x)$ which contains no other singular points of Y. Then $N \cup N_x$ is a compact neighbourhood of x in W and $(N \cup N_x) \cap \bar{X} = (N \cup N_x) - \{f(x)\}$ is a neighbourhood of x in $\bar{X}$. We shall show that the set $(N \cup N_x) - \{f(x)\}$ is compact. Let $\{x_\alpha\}_{\alpha \in D}$ be a net in $(N \cup N_x) - \{f(x)\}$. Since $N \cup N_x$ is compact, the net $\{x_\alpha\}_{\alpha \in D}$ has a cluster point $p \in N \cup N_x$. If $p \neq f(x)$ then $p \in (N \cup N_x) - \{f(x)\}$. If $p = f(x)$, then x is also a cluster point of the net $\{x_\alpha\}_{\alpha \in D}$. In either case the net $\{x_\alpha\}_{\alpha \in D}$ has a cluster point in $(N \cup N_x) - \{f(x)\}$. So $(N \cup N_x) - \{f(x)\}$ is a compact neighbourhood of x (in $\bar{X}$).

Case II. The point $x \in \bar{X} - X$. Since Y is locally compact and since the singular points of Y do not accumulate, there is a compact neighbourhood N of x in Y which contains no singular points of Y. Then $N = N \cap \bar{X}$ is a compact neighbourhood of $x \in \bar{X}$.

1.11. *Suppose that singular points of X and Y do not accumulate and suppose $f|S$ is one-to-one. If X and Y are T_3, then the space $\bar{X}$ is T_3.*

Proof. Let F be a closed set in $\bar{X}$ and let $x \notin F$ be any point of $\bar{X}$. The following two cases arise:

Case I. The point x is a singular point of X. Since F is closed in $\bar{X}$, the set $X \cap F$ is closed in X and the set $F \cap Y$ is closed in $Y - T$. Now the point $f(x)$ cannot be a limit point (in Y) of $F \cap Y$. For, if $f(x)$ is a limit point (in Y) of $F \cap X$, then since x is a limit point in W of $\{f(x)\}$, the point x is a limit point of F in $\bar{X}$. Therefore, the set $(F \cap Y) \cup (T - \{f(x)\})$ is closed in Y. Since Y is T_3, there exist disjoint open sets N_1 and N_2 in Y containing $f(x)$ and $(F \cap Y) \cup (T - \{f(x)\})$ respectively. Since X is T_3, there exist disjoint open sets N_3 and N_4 of X containing x and $F \cap X$ respectively. Then $(N_1 \cup N_3) \cap \bar{X}$ and $(N_2 \cup N_4) \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x and F respectively.

Case II. The point x is not a singular point of X. If $x \in X$, then since X is T_3, there are disjoint open sets N_1 and N_2 of X containing x and $F \cap X$ respectively (in case $F \cap X = \emptyset$, let $N_2 = \emptyset$). By the hypothesis on the set S of singular points of X, N_1 can be chosen so that it contains no singular point of X. Then N_1 and $(N_2 \cup Y) \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x and F respectively. If $x \in (\bar{X} - X)$, then since $F \cap Y$ is closed in $(Y - T)$ and since Y is T_3, there are disjoint neighbourhoods N_1 and N_2 in Y of x and $T \cup (F \cap Y)$ respectively. Then $N_1 \cap \bar{X}$ and $(X \cup N_2) \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x and F respectively.

In 1966, Proizvolov [**9**] showed that in the class of locally compact spaces weight and metrizability are inversely preserved under open finite-to-one maps. Later that year, Arhangelskiĭ [**1**; **2**] proved that they were always inversely preserved under clopen finite-to-one maps. In 1967, Čoban [**4**] showed that hereditary paracompactness (metacompactness, Lindelöf property) are inversely preserved under open-finite-to-one maps. (Some separation axioms are required for all these results.) In [**6**], we showed that in all the above mentioned results it is sufficient to require that f be open except at finitely many points (i.e. if the set S of singular points is finite). But the methods of [**6**] and [**7**] are not applicable if the set S of singular points of X is infinite. Here we show that in some special cases the result still holds even if the set S is infinite.

If f is finite-to-one, the singular points of X and Y do not accumulate and if $f|S$ is one-to-one, then $\bar{f} : \bar{X} \to Y$ is an open finite-to-one map and hence $\bar{X}$ will inherit properties from Y by the results quoted in the above paragraph (1.9, 1.10 and 1.11 ensure that the needed separation axioms also lift properly). Since these properties are all hereditary, X must also enjoy them. For the convenience of the reader we list the precise statements of the improved theorems (assume all spaces to be Hausdorff and all maps to be continuous onto).

1.12. *Let f be a finite-to-one map of X onto Y such that singular points of X and Y do not accumulate, and let $f|S$ be one-to-one. Then the following statements are true.*

(a) (Proizvolov) *If X and Y are locally compact, then* weight $X \leqq$ weight Y. *If Y is metrizable, so is X.*

(b) (Arhangelskiĭ) *If X and Y are locally compact and if f is a closed map, then* weight $X \leqq$ weight Y. *Further, if Y is metrizable, then so is X.*

(c) (Čoban) *If X is T_3-space and if Y is hereditarily paracompact, then X is hereditarily paracompact.*

(d) (Čoban) *If Y is hereditarily metacompact (respectively, Lindelöf), then X is hereditarily metacompact (respectively, Lindelöf).*

Let Y be the real line with its usual topology and let X be Y with the integers as an open discrete set. Then the identity mapping of X onto Y satisfies the hypotheses of 1.12 and the set S of singular points of X is infinite.

In conclusion, the author wishes to express his thanks to Professor Richard A. Alo for his interest and encouragement in writing this note and to Professor S. P. Franklin for many useful discussions.

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