A theorem of M. L. Urquhart's and some consequences

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1. Introduction

In an obituary of M. L. Urquhart in [1], David Elliott quotes him as claiming that Urquhart's theorem (below) is the most elementary theorem of Euclidean Geometry 'since it involves only the concepts of straight line and distance'.

**Urquhart's theorem**

Let \( AC \) and \( AE \) be two straight lines.

Let \( B \) be a point on \( AC \), \( D \) a point on \( AE \), and suppose that \( BE \) and \( CD \) intersect at \( F \).

If \( AB + BF = AD + DF \) then \( AC + CF = AE + EF \).  

(1)

Elliott suggests that the proof of this theorem by purely geometric methods is not elementary but over the succeeding eight years several proofs were given, and these were reviewed by Dan Pedoe in [2].

The proposition is shown in Figure 1, where it should be noted that point \( B \) is strictly 'in' \( AC \) and \( D \) is 'in' \( AE \). From Figure 2 it is clear that the theorem is untrue if the statement is relaxed to allow '\( B \) on \( AC \) produced', as although \( AB + BF = AD + DF \), we have \( AC + CF < AE + EF \).

![Figure 1](https://www.cambridge.org/core/images/1ed.png)

![Figure 2](https://www.cambridge.org/core/images/2ed.png)

\[
AB + BF = AD + DF \\
\Rightarrow AC + CF = AE + EF \\
\]

FIGURE 1

\[
AB + BF = AD + DF \\
\text{but } AC + CF < AE + EF.
\]

FIGURE 2

We give a proof below, and show that Urquhart’s theorem is a member of a family of equivalent propositions.

We also show (see Figure 3) that a second family arises from the corresponding theorem on differences (with the same restriction on the position of \( D \) in \( AE \)):

If \( AB - BF = AD - DF \) then \( AC - CF = AE - EF \).  

(2)
These theorems together form the basis of a geometry of confocal conic sections.

2. The two key theorems

Here is the proof of (1).

Let $D'$ on $CD$ produced be such that $DD' = AD$, and let $\angle DAD' = d$. Then $\triangle ADD'$ is isosceles with base angle $d$. Similarly create $B'$, $C'$, and $E'$ so that triangles $\triangle ABB'$, $\triangle ACC'$, and $\triangle AEE'$ are isosceles with base angles $b$, $c$ and $e$ as in Figure 4.

Since $D'F = AD + DF = AB + BF = B'F$, $\triangle BFD'$ is isosceles. Let its base angle be $f$. Now the exterior angle at $D$ of $\triangle ADD'$ is $2d$, but it is also the exterior angle of $\triangle DEF$ and so is $2(e + f)$. Thus $e = d - f = B'D'A$. Therefore $A$, $B'$, $E'$, $D'$, are concyclic. Similarly $c = b - f = D'B'A$, whence $A$, $B'$, $C'$, $D'$ are concyclic. Consequently the five points $A$, $B'$, $C'$, $E'$, $D'$ are concyclic, so $f = E'C'D' = E'B'D'$, whence $FE' = FC'$, and so $AE + EF = AC + CF$, as required.

The converse of (1) is also true, i.e. if $AC + CF = AE + EF$ then $AB + BF = AD + DF$. This may be proved on the same diagram by starting with $B'E'C' = F'F'C'$ and reversing the argument.

It is interesting that Urquhart's claim should relate to distances, while this proof relies almost entirely on angle equalities.

Figure 5 is the figure for the proof of (2), where the construction lines are now reversed in direction, i.e. $D'$ is on $DC$ produced, $B'$ is on $BE$ produced, etc, so as to make $FD' = AD - DF$, and $FB' = AB - BF$, etc.

As for (1) the proof consists in showing that $A$, $C'$, $B'$, $D'$, $E'$ are concyclic. We leave this and the proof of the converse to the reader.

A natural interpretation of these theorems is in terms of conics, which we illustrate below in Figure 6 for (1) and Figure 9 for (2). We have relabelled the diagrams so that $A$ becomes the negative focus, $F'$, with $F$ the positive focus, $FF'$ defining the $x$-axis. The points $B$, $D$, relabelled $P$, $Q$ respectively, lie on a 'basic' conic and the points $C$, $E$, relabelled $P'$, $Q'$ lie on a 'derived' conic.
The specifications of the positions of $P$, $Q$, $P'$ and $Q'$ relative to the $x$-axis are essential in analysing the possible cases, so we introduce an appropriate nomenclature.
We let $E(a, \beta), H(\alpha, \beta, \gamma)$ represent confocal ellipses and hyperbolae respectively, $\alpha$ being $S$ or $D$ depending on whether $P$ and $Q$ are on the same or different sides of $FF'$, $\beta$ being $S$ or $D$ depending on whether $P'$, $Q'$ are on the same or different sides of $FF'$, and $\gamma$ being $S$ or $D$ depending on whether $P$, $Q$ are on the same or different branches of the hyperbola. In Figure 1 we have $E(D, D)$; for Figure 3 we have $H(D, D, S)$.

With this nomenclature, the theorem (1) referring to Figure 1, and shown in Figure 6, may be stated as $E(D, D) \Rightarrow E(D, D)$, for $P$, $Q$ lie on the basic ellipse, $\mathcal{E}$, on opposite sides of $FF'$ as do $P'$, $Q'$ on the derived ellipse, $\mathcal{E}'$. Similarly for the theorem (2) referring to Figure 3, and shown in Figure 9, where $P'$, $Q'$ lie on the same branch of a hyperbola, $\mathcal{H}$, confocal with the basic hyperbola $\mathcal{H}'$ determined by $P$, $Q$, we may write $H(D, D, S) \Rightarrow H(D, D, S)$.

There are formally 4 outcomes of $E(*, *)$ and 8 of $H(*, *, *)$. In fact two of the $H$s, $H(S, D, S)$ and $H(D, D, D)$ and one of the $E$s, $E(D, S)$ do not exist. There are thus 9 valid forms, between which we show there are 12 relationships (6 direct and their converses) occurring in two clusters of 6 theorems each.

We set out the relations for the first cluster in Figures 6, 7, 8 and those for the second cluster in Figures 9, 10, 11.
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\[ E(D, D) \iff E(D, D) \]
\[ FP + PF' = FQ + QF' \]
\[ \iff FP' + P'F' = Q'F' + QF' \]

**FIGURE 6**

\[ E(S, D) \iff H(D, S, D) \]
\[ FP + PF' = FQ + QF' \]
\[ \iff FP' - P'F = QF' - Q'F' \]

**FIGURE 7**

\[ H(S, S, D) \iff H(S, S, D) \]
\[ FP - PF' = F'Q - QF \iff F'P' - P'F = FQ' - Q'F' \]

**FIGURE 8**

\[ F(D, D, S) \iff F(D, D, S) \]
\[ FP' - P'F' = QF' - QF' \]
\[ FP + PF' = FQ + QF' \]

**FIGURE 9**

\[ F(S, S, S) \iff H(S, S, S) \]
\[ FP - PF = FQ - QF \]
\[ FP - PF' = F'Q - QF \]

**FIGURE 10**

\[ H(S, D, D) \iff H(D, S, S) \]
\[ FP - PF' = F'Q - QF \]
\[ FP' - P'F' = Q'F' - QF' \]

**FIGURE 11**
Geometric proofs along the lines of those for (1) and (2) may be constructed for the remaining configurations of the two clusters, but these propositions are in fact implied as consequences of (1) and (2).

Figure 12 shows the relabelled Figure 1, and the lengths of the line segments in the diagram are denoted by \(a, \ldots, h\).

With this notation (1) and its converse become

\[ a + b = c + d \Leftrightarrow a + e + g = c + f + h. \]  

(3)

We show that the theorems of Figures 7 and 8 can be deduced from (3).

If we label \(A\) the property \(a + b = c + d\) and \(B\) the property \(a + e + g = c + f + h\), (1) asserts that \(A \Leftrightarrow B\). By algebraic manipulation of \(A\) we get \(C: a - c = d - b\), and from \(B\) we get \(D: a + e - c - f = h - g\). Together, \(A\) and \(B\) give \(E: e + g + d = f + h + b\) and \(F: d + g - f = h + b - e\). These are all equivalent to one another since \(A \Leftrightarrow B\).

Now \(C (a - c = d - b)\), may be interpreted as points \(F, F'\) on a hyperbola with foci \(P, Q: H(D, S, D)\), and \(E (e + g + d = f + h + b)\) represents points \(P', Q'\) lying on an ellipse also with foci \(P, Q: E(S, D)\). Hence \(H(D, S, D) \Leftrightarrow E(S, D)\).

Similarly \(D\) representing points \(F, F'\) on a hyperbola with foci \(P', Q'\): \(H(S, S, D)\), and \(F\) representing points \(P, Q\) on a hyperbola with foci \(P', Q': H(S, S, D)\) we have \(H(S, S, D) \Leftrightarrow H(S, S, D)\). This completes the proof of the relationships of the first cluster.

Similar considerations yield the remaining theorems of the second cluster.

The diagrams in Figures 6 to 11 have been drawn to show the essential differences between the six cases, but obviously there is a smooth transition between say Figures 6 and 10 in which \(P\) and \(Q\) have moved from being on opposite sides of \(FF'\) to being on the same side; the derived conic has changed from an ellipse to a hyperbola. Again, comparing Figures 9 and 11 we see increasing the separation of \(P'\) and \(Q'\) has put \(P\) and \(Q\) on different branches of the hyperbola \(H\), but on the same side of \(FF'\).

It is instructive to examine these transitions systematically using the graphical facilities of one of the mathematical packages available for computers.
3. **Image of a point in a conic**

Consider the four points $X_1, X_2, X_3, X_4$ where the joins of $X$ to the foci $F, F'$ cut the (basic) conic, shown in Figure 13 as an ellipse $\mathcal{E}^*$ and, in Figure 14 as a hyperbola $\mathcal{H}^*$. Let $X_{12}, X_{13}, X_{42}, X_{43}$ be determined as in the diagram. Certainly $X_{12}$ suggests itself as an image of $X$ in $\mathcal{E}^*$, but $X_{13}, X_{42}$ and $X_{43}$ equally appear as "images" of $X$. By analogy with the behaviour of the inverse of a point with respect to a circle, the properties of these four images will be investigated using the theorems of Section 2.

By constraining $X$ to a (traversed) conic confocal with the basic one, we get a consistent set of properties for the paths of the four images. In fact, in all cases, the path of the image exactly mirrors that of the point $X$, whatever the species of the basic conic. Thus for the ellipse of Figure 13 if $X$ describes an ellipse (confocal with $\mathcal{E}^*$), the four points all describe ellipses (confocal with $\mathcal{E}^*$), while if $X$ describes a hyperbola, so do all the four images.

There is a total of $2 \times 2 \times 4 (= 16)$ figures, but this number is halved as $X_{12}$ and $X_{43}$ describe conics of the same species, as do $X_{13}$ and $X_{42}$. Of the remaining 8 cases, 4 may be disposed of directly because the image point lies on the traversed conic, so its path is the traversed conic. Typical of this simple situation is Figure 13 where $X_{12}$ lies on the hyperbola containing $X$, and if $X$ is now constrained to follow this hyperbola, $X_{12}$ will do so also.
There are four cases needing more specific treatment:

1. $\mathcal{H}$ imaged in $\mathcal{E}^*$ as $X_{13}$ or $X_{42}$;
2. $\mathcal{H}$ imaged in $\mathcal{H}^*$ as $X_{13}$ or $X_{42}$;
3. $\mathcal{E}$ imaged in $\mathcal{E}^*$ as $X_{12}$ or $X_{43}$;
4. $\mathcal{E}$ imaged in $\mathcal{H}^*$ as $X_{12}$ or $X_{43}$.

To illustrate the method of proof for these problems we take case 3 shown in Figure 15.

As $X$ moves along $\mathcal{E}$, say to $Y$, the image of $X$, $X_{12}$, becomes $Y_{12}$; $X_{12}$ and $Y_{12}$ now have to be shown to be on the same ellipse. This is accomplished by noting that $S$, $T$, then $V$, $T$, then $S$, $U$ all lie on the same hyperbola ($E(S, S) \Rightarrow H(S, S, S)$). With $V$ and $U$ on a hyperbola, $X_{12}$ and $Y_{12}$ lie on an ellipse ($H(S, S, S) \Rightarrow E(S, S)$).

This is the pattern of proof for all the remaining cases.

4. An alternative image of a point in a conic

An alternative ‘image’ $X_i$ of a point $X$ with respect to a (basic) conic, $\mathcal{E}^*$, may be defined directly from the tetrad $\{X_1, X_2, X_3, X_4\}$ shown in Figure 16 (for the ellipse $\mathcal{E}^*$) without further reference to the foci; we let $X_i$ be the
intersection of the join of $X_1, X_3$ with the join of $X_2, X_4$. We seek the path of $X_i$ as $X$ moves along the conic, $\mathcal{C}$, confocal with $\mathcal{C}^*$. Here again we may distinguish four cases depending on whether the traversed conic, $\mathcal{C}$, is an ellipse or a hyperbola and whether $\mathcal{C}^*$ is an ellipse or a hyperbola. The conclusion is in line with what has been found before, i.e. that an ellipse always images into an ellipse and a hyperbola into a hyperbola no matter what the species of $\mathcal{C}^*$. We illustrate the situation by considering only the ellipse-ellipse configuration; proofs for the other configurations we leave to the reader.

With this definition of $X_i$, the theorems of Section 2 are no longer applicable, but we can use the properties of pole and polar as $X_i$ lies on the polar of $X$ with respect to the conic $\mathcal{C}^*$.

Let the basic ellipse be $\mathcal{C}^*$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and let $X$ traverse the confocal ellipse $\mathcal{C}$: $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, where $A^2 = a^2 + \lambda, B^2 = b^2 + \lambda, \lambda > 0$.

We take $X$ in the parametric form $\{A \cos \Theta, B \sin \Theta\}$. Then the polar of $X$ with respect to $\mathcal{C}^*$ is

$$\frac{xA \cos \Theta}{a^2} + \frac{yB \sin \Theta}{b^2} = 1.$$  \hspace{1cm} (4)

It is clear that the trajectory of $X_i$ is the envelope of this line as $\Theta$ varies from 0 to $\pi$. To find its equation, we take the partial derivative of (1) with respect to $\Theta$:

$$\frac{-xA \sin \Theta}{a^2} + \frac{yB \cos \Theta}{b^2} = 0.$$  \hspace{1cm} (5)
Squaring (4) and (5) and adding:

\[ \frac{x^2}{a^4/A^2} + \frac{y^2}{b^4/B^2} = 1. \]  

This is an ellipse, \( \mathcal{E} \), with major axis \( a^2/A \) and minor axis \( b^2/B \); its foci are at \( \left( \pm \sqrt{\frac{a^4}{A^2} - \frac{b^4}{B^2}}, 0 \right) \).

It may be noted that the two ellipses \( \mathcal{E} \) and \( \mathcal{E}^* \) are assumed to be confocal. If they are not confocal but merely similarly disposed about their axes, the envelope is still the ellipse in (6) but \( X_i \) is no longer the point of contact of the polar with the envelope; in any case \( \mathcal{E} \) is not confocal with either, but it is similarly disposed about the axes.

The other cases are dealt with analogously and yield similar conclusions for confocal ellipses and hyperbolas.

Note that if we combine Figures 13 and 16, Pappus' Theorem, applied to the hexagon \( X_3F X_2X_1F'X_4 \), shows that \( X_{43}, X_i, X_{12} \) are collinear.

5. Quantitative relations

Consider again Figure 7. Let the major and minor axes of the basic ellipse \( \mathcal{E}^* \) be \( 2a \) and \( 2b \), the foci be \( (-ae, 0), (ae, 0) \), \( P, Q \) be \( (a \cos \theta, b \sin \theta), (a \cos \phi, b \sin \phi) \), respectively, and the derived conic be the hyperbola \( \mathcal{H}' \).

Let \( FP \) cut \( F'Q \) in \( P' \), \( FQ \) cut \( FP \) in \( Q' \). We seek to determine the points \( P', Q' \) in terms of \( a, b, \theta, \phi \) in order to find the parameters of \( \mathcal{H}' \).

The equation of the line \( FP \) is \( xb \sin \theta - ya (\cos \theta + e) = -abe \sin \theta \), and the equation of the line \( F'Q \) is \( xb \sin \phi - ya (\cos \phi + e) = -abe \sin \phi \). The point \( P' \) is the intersection of these lines.

Solving the equations for \( x, y \) we have as coordinates of \( P' \)

\[ x = \frac{ae \sin (\theta + \phi) - e (\sin \theta - \sin \phi)}{D}, \quad y = \frac{2be \sin \theta \sin \phi}{D}, \]

where \( D = \sin (\phi - \theta) + e (\sin \theta - \sin \phi) \).

We can now find \( P'F \) and \( P'F' \) thus:

\[ P'F^2 = \left( x + ae \right)^2 + y^2 = \frac{4a^2 e^2 \sin^2 \phi [((\cos \theta + e)^2 + (1 - e^2) \sin^2 \theta]}{D^2}, \]

\[ \therefore \quad P'F = \left| \frac{2ae \sin \phi (1 + e \cos \theta)}{D} \right| = \left| s \right|, \text{ say.} \]

Now \( P'F' \) is obtained from \( P'F \) by interchanging \( \theta \) and \( \phi \) and changing the sign of \( e \):

\[ P'F' = \left| \frac{2ae \sin \theta (1 - e \cos \phi)}{D} \right| = \left| t \right|, \text{ say.} \]
Let us consider \( |s - t| = \left| \frac{2ae[\sin\phi(1 + e \cos\theta) - \sin\theta(1 - e \cos\phi)]}{D} \right| \)

\[
= 2ae \left| \frac{\cos\frac{1}{2}(\theta + \phi)}{\cos\frac{1}{2}(\theta - \phi)} \right| = 2A, \text{ say.}
\]

If \( s, t \) are of the same sign, i.e. if \( st > 0 \), \( |P'F - P'F'| = 2A \), but if \( st < 0 \), \( |P'F + P'F'| = 2A \).

Turning to \( Q' \) we find \( Q'F, Q'F' \) from \( P'F, P'F' \) respectively by interchange of \( \theta \) and \( \phi \). Thus

\[
Q'F = \left| \frac{2ae \sin \theta (1 + e \cos \phi)}{D'} \right| = |u|, \text{ say,}
\]

\[
Q'F' = \left| \frac{2ae \sin \phi (1 - e \cos \theta)}{D'} \right| = |v|, \text{ say,}
\]

where \( D' = \sin(\theta - \phi) + e(\sin \phi + \sin \theta) \).

Consider now

\[
|u - v| = \left| \frac{2ae[\sin\theta(1 + e \cos\phi) - \sin\phi(1 - e \cos\theta)]}{D'} \right| \]

\[
= 2ae \left| \frac{\cos\frac{1}{2}(\theta + \phi)}{\cos\frac{1}{2}(\theta - \phi)} \right| = 2A.
\]

As \( e < 1 \), it is clear that \( uv \) has the same sign as \( st \), so if \( st > 0 \), \( |Q'F - Q'F'| = 2A \), but if \( st < 0 \), \( |Q'F + Q'F'| = 2A \). Thus \( P', Q' \) are on the same conic, an ellipse if \( st < 0 \), and a hyperbola if \( st > 0 \). Appropriate values of \( \theta, \phi \) yield the theorems of Figures 6 and 10.

The derived conic is determined by its foci and its major axis \( 2A \). If we write \( B \) for the minor axis of the derived conic, and \( E \) for its eccentricity, the latter is determined from \( ae = AE \) and the former from \( B^2 = A^2|E^2 - 1| \). Similar treatment of the case with a basic hyperbola yields the theorems of Figures 8, 9 and 11.

**Urquhart points**

We now apply the results of this section to define ‘Urquhart points’ for a basic and derived conic system. We give details for a basic ellipse \( \mathcal{E} \) and induced hyperbola \( \mathcal{H} \), connected by points \( P, Q \) on \( \mathcal{E} \) and \( P', Q' \) on \( \mathcal{H} \), as in Figure 17.

Let \( U_1 \) be the point on the axis of \( \mathcal{E} \) where \( PQ \) cuts it. If \( P \) is \((a \cos \theta, b \sin \theta)\) and \( Q \) is \((a \cos \phi, b \sin \phi)\), we have \( OU_1 = a\frac{\cos\frac{1}{2}(\phi - \theta)}{\cos\frac{1}{2}(\phi + \theta)} = \) constant for all lines through \( U_1 \). But \( \mathcal{H} \), the hyperbola determined by \( P', Q' \), has major axis \( 2A = 2a\frac{\cos\frac{1}{2}(\phi - \theta)}{\cos\frac{1}{2}(\phi + \theta)} \), which is thus the same whatever the line through \( U_1 \). This shows how to find sets of points on \( \mathcal{E} \) which yield points \( P', Q' \) always on the hyperbola \( \mathcal{H} \). We refer to \( U_1 \) as the ‘first Urquhart point’.
The intersection of $P'Q'$ with the $x$-axis yields a ‘second Urquhart point’, $U_2$, where all lines $P'Q'$ are concurrent.

The tangents at the point of intersection of $\mathcal{C}$ and $\mathcal{H}'$ go through $U_1$ and $U_2$, showing that $U_1$ and $U_2$ divide $FF'$ harmonically.

For non-intersecting (confocal) conics, $PQ$ cuts $P'Q'$ on the line $x = A/a = a/E$. (This is the $x$-coordinate of the imaginary line of intersection of the two conics.)

Urquhart points may be defined for all the configurations 6 to 11. Of course $U_1$ and $U_2$ are unique for each of the related conics $\mathcal{C}$ and $\mathcal{C}^*$. 

References


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