J. Austral. Math. Soc. (*Series A*) **42** (1987), 84–91

SEMIDIRECT PRODUCT GROUPS WITH ABELIAN AUTOMORPHISM GROUPS

M. J. CURRAN

(Received 16 December 1983; revised 23 September 1985)

Communicated by D. E. Taylor

Abstract

Miller's group of order 64 is a smallest example of a nonabelian group with an abelian automorphism group, and is the first in an infinite family of such groups formed by taking the semidirect product of a cyclic group of order 2^m ($m \geqslant 3$) with a dihedral group of order 8. This paper gives a method for constructing further examples of non abelian 2-groups which have abelian automorphism groups. Such a 2-group is the semidirect product of a cyclic group and a special 2-group (satisfying certain conditions). The automorphism group of this semidirect product is shown to be isomorphic to the central automorphism group of the corresponding direct product. The conditions satisfied by the special 2-group are determined by establishing when this direct product has an abelian central automorphism group.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 F 28.

1. Introduction

Any group G mentioned in this paper is finite. An automorphism ϕ of G is *central* if ϕ commutes with every inner automorphism of G, or equivalently if $g^{-1}g^{\phi}$ lies in the centre $Z(G)$ of G for all $g \in G$. The central automorphisms form a normal subgroup, denoted $\operatorname{Aut}_c G$, of the full automorphism group $\operatorname{Aut} G$. If H is a subgroup of G, we let $\operatorname{Aut}(G : H) = \{\phi \in \operatorname{Aut} G | H^{\phi} = H\}$.

Struik [8] showed that Miller's group [6] of order 64 constitutes the first in an infinite family $M = \langle a, b, c | a^{2^m} = b^4 = c^2 = 1,\ a^b = a^{1+2^{m-1}},\ b^c = b^{-1},\ a^c = a\rangle$, $m \geqslant 3$, of nonabelian 2-groups having abelian automorphism groups. M is a

---

semidirect product of a cyclic group $A = \langle a \rangle$ of order 2^m by $N = \langle b, c \rangle \approx D_4$, the dihedral group of order 8. This paper provides a method of constructing M and similar semidirect product groups with abelian automorphism groups. In Section 4 we prove:

THEOREM. *Let N be a nonabelian special 2-group with a subgroup J of index 2 such that* (i) $Z(J) = Z(N)$; (ii) $\operatorname{Aut}(N\colon J) = \operatorname{Aut}_c N$. *If $G = AN$ is the semidirect product of a cyclic group A of order 2^m and a group N such that $a^n = a$ if $n \in J$ and $a^n = a^{1+2^{m-1}}$ if $n \in N \setminus J$, then* $\operatorname{Aut} G$ *is abelian.*

Section 2 gives preliminary results and Section 3 gives a criterion for $\operatorname{Aut}_c G$ to be abelian when G is a p-group with an abelian direct factor.

2. Automorphisms of semidirect products

Let $G = AN$ be a semidirect product of a group A by a group N, with multiplication given by $(a, n)(a_1, n_1) = (aa_1^{n^{-1}}, nn_1)$, where $a, a_1 \in A$ and $n, n_1 \in N$. We denote by ι_A and ι_N the inclusion maps from A and N, respectively, into G, and by π_A and π_N the projection maps from G to A and N, respectively. For $\phi \in \operatorname{End} G = \operatorname{Hom}(G, G)$, we put $\alpha = \iota_A\phi\pi_A$, $\beta = \iota_A\phi\pi_N$, $\gamma = \iota_N\phi\pi_A$ and $\delta = \iota_N\phi\pi_N$. Then $\beta \in \operatorname{Hom}(A, N)$ and $\delta \in \operatorname{End} N$.

If ϕ also satisfies (a) $[A^\beta, A] = 1 = [N^\gamma, N]$, then $\alpha \in \operatorname{End} A$, $\gamma \in \operatorname{Hom}(N, A)$ and $(a, n)^\phi = (a^\alpha n^\gamma, a^\beta n^\delta)$, where $a \in A$, $n \in N$. If every $\phi \in \operatorname{End} G$ satisfies (a) and (b) $[A^\alpha, N^\gamma] = 1 = [A^\beta, N^\delta]$, and if $\mathscr{R}$ denotes the matrix semigroup

$$\mathscr{R} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{array}{l} \alpha \in \operatorname{End} A,\ \beta \in \operatorname{Hom}(A, N),\ [A^\alpha, N^\gamma] = 1 \\ \gamma \in \operatorname{Hom}(N, A),\ \delta \in \operatorname{End} N,\ \ [A^\beta, N^\delta] = 1 \end{array} \right\},$$

then $f\colon \operatorname{End} G \to \mathscr{R}$ defined by $f(\phi) = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ is a monomorphism.

In particular, if $G = A \times N$, then every $\phi \in \operatorname{End} G$ satisfies (a) and (b) above. Conversely, if $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathscr{R}$ and $\phi\colon G \to G$ is defined by $(a, n)^\phi = (a^\alpha n^\gamma, n^\beta n^\delta)$, where $a \in A$, $n \in N$, then $\phi \in \operatorname{End} G$. We thus have

LEMMA 2.1. *If $G = A \times N$, then* $\operatorname{End} G \approx \mathscr{R}$.

For convenience let

$$\mathscr{L} = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \middle| \alpha \in \operatorname{Aut} A,\ \gamma \in \operatorname{Hom}(N, Z(A)),\ \delta \in \operatorname{Aut} N \right\},$$

$$\mathscr{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{array}{l} \alpha \in \operatorname{Aut} A,\ \beta \in \operatorname{Hom}(A, Z(N)) \\ \gamma \in \operatorname{Hom}(N, A),\ \delta \in \operatorname{Aut} N \end{array} \right\},$$

and let $\mathscr{L}_c$ and $\mathscr{H}_c$ be the corresponding sets of matrices, where $\alpha \in \operatorname{Aut}_c A$ and $\delta \in \operatorname{Aut}_c N$.

The following two lemmas, whose proofs are straightforward, are used in Section 3.

LEMMA 2.2. *Let* $G = A \times N$. *Then*
(i) $\operatorname{Aut}(G : A) \approx \mathscr{L}$;
(ii) $\operatorname{Aut}_c(G : A) = \{\phi \in \operatorname{Aut}_c G \mid A^\phi = A\} \approx \mathscr{L}_c$;
(iii) $\operatorname{Aut}(G : N)$ *and* $\operatorname{Aut}_c(G : N)$ *are isomorphic to analogous groups of upper triangular matrices.*

LEMMA 2.3. *Let* $G = A \times N$, *where* A *is abelian. If there is a characteristic subgroup* C *of* G *such that* $N \geqslant C \geqslant Z(N)$, *then* $f\colon \operatorname{Aut} G \to \mathscr{H}$ *is a monomorphism.*

The next two results are used in Section 4.

THEOREM 2.4. *Let* $G = AN$ *be a semidirect product, where*
(i) A *is abelian,* $N' \geqslant Z(N)$ *and* $[Z(N), A] = 1$;
(ii) *for all* $\phi \in \operatorname{Aut} G$, $A^\beta \leqslant Z(N)$ *and* $[N^\gamma, N] = 1$.
Then $f\colon \operatorname{Aut} G \to \mathscr{H}$ *is a monomorphism.*

PROOF. From (i) and (ii), every $\phi \in \operatorname{Aut} G$ satisfies (a) and (b) above. Further, if $(a, 1)^\phi = (1, n)$, where $a \in A$, $n \in N$, then $a^\beta = n \in Z(N) \leqslant N'$. But N' is characteristic in G, since $[N^\gamma, N] = 1$. Thus $(1, n)^{\phi^{-1}} = (a, 1)$ implies that $a = n = 1$, so $\alpha \in \operatorname{Aut} A$ and $\delta \in \operatorname{Aut} N$.

THEOREM 2.5. *Let* $G = A \times N$, *where* A *is abelian and* $N' \geqslant Z(N)$. *Then*
(i) $\operatorname{Aut} G \approx \mathscr{H}$, *and*
(ii) $\operatorname{Aut}_c G \approx \mathscr{H}_c$.

PROOF. (i) The conditions of Theorem 2.4 hold, so $f|_{\operatorname{Aut} G}\colon \operatorname{Aut} G \to \mathscr{H}$ is a monomorphism. Conversely, if $\phi = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathscr{H}$, then, by (2.1), $\phi \in \operatorname{End} G$ and $f(\phi) = \phi$. Further, if $(a, n)^\phi = (1, 1)$, where $a \in A$, $n \in N$, then $(a^\alpha n^\gamma, a^\beta n^\delta) = (1, 1)$. Thus $(a^{-1})^{\beta\delta^{-1}} \in Z(N) \leqslant N' \leqslant \ker\gamma$, so that $n^\gamma = 1$. Hence $a = n = 1$, and $\phi \in \operatorname{Aut} G$. Thus f is an isomorphism.

(ii) $\phi \in \operatorname{Aut}_c G \Leftrightarrow (a, n)^{-1}(a, n)^\phi \in A \times Z(N) \Leftrightarrow n^{-1}n^\delta \in Z(N) \Leftrightarrow \delta \in \operatorname{Aut}_c N$.

Corollary 2.6. *If $G = AN$ satisfies the hypotheses of Theorem* 2.4, *then* $\operatorname{Aut} G$ *is isomorphic to a subgroup of* $\operatorname{Aut}(A \times N)$.

3. Central automorphisms of groups with abelian direct factors

The following theorem is a modification of a result of Earnley (Theorem 2.3 in [3]), but the proof is given for convenience.

Theorem 3.1. *Let $G = A \times N$ be a p-group, where $1 \neq A$ is abelian, and where $1 \neq N$ is purely nonabelian* (*i.e. N has no non-trivial abelian direct factor*). *Then* $\operatorname{Aut}_c G$ *is abelian if and only if A and N satisfy*

(i) *A is cyclic of order 2^m, $m \geqslant 2$;*

(ii) *N is a* 2-*group with* $\operatorname{Aut}_c N$ *abelian;*

(iii) $\Phi(N) = N' \geqslant Z(N)$, *and $Z(N)$ has exponent* 2.

Proof. Suppose $\operatorname{Aut}_c G$ is abelian. Then $\operatorname{Aut} A$ and $\operatorname{Aut}_c N$, being subgroups of $\operatorname{Aut}_c G$, are abelian. In particular, A is cyclic. (See 3.12 in [2].) If $A = \langle a \rangle$, define $\tau \in \operatorname{Aut} A$ by $a^\tau = a^{-1}$. Thus, by Lemma 2.2, if $y \in Y = \operatorname{Hom}(N, A)$, and if $x \in X = \operatorname{Hom}(A, Z(N))$, then $\left(\begin{smallmatrix}\tau & 0\\ y & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ are in $\operatorname{Aut}_c G$ and so commute. Hence $xy = 0 = yx$ and $\tau x = x$. Now $X \neq 0$ since A and $Z(N)$ are non-trivial p-groups. Thus $|A| \neq 2$, for otherwise there is an epimorphism $y \in Y$, and the condition $yx = 0$ implies that $x = 0$ for all $x \in X$. Further, $\tau x = x$ implies that A^x has exponent dividing 2, so either $p = 2$, or, again, $x = 0$ for all $x \in X$.

Therefore $|A| = 2^m$, $m \geqslant 2$, and $Z(N)$ has exponent 2. Hence $Z(N) \leqslant \Phi(N)$, for otherwise there exists $n_1 \in Z(N) \setminus \Phi(N)$ of order 2 such that $\{n_1, \dots, n_t\}$, $t \geqslant 2$, is a minimal generating set for N. But then $N = \langle n_1 \rangle \times \langle n_2, \dots, n_t \rangle$, a contradiction.

Similarly, the commutativity of $\left(\begin{smallmatrix}1 & 0\\ y & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\tau & x\\ 0 & 1\end{smallmatrix}\right)$ gives $y\tau = y$, and hence N^y has exponent dividing 2. Since A has exponent greater than 2, N/N' has exponent 2. Thus $\Phi(N) = N'$, and A and N satisfy the given conditions.

Conversely, suppose that A and N satisfy (i), (ii) and (iii). Then, by Theorem 2.5, $\operatorname{Aut}_c G \approx \mathscr{H}_c$. In fact, if $\phi \in \operatorname{Aut}_c G$ and $A = \langle a \rangle$, then $A^\beta \leqslant Z(N) \leqslant N' \leqslant \ker \gamma$, and $N^\gamma \leqslant \langle a^{2^{m-1}} \rangle \leqslant \langle a^2 \rangle \leqslant \ker \beta$. Thus $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{smallmatrix}\right)$ in $\mathscr{H}_c$ will commute, provided that (1) $\alpha\beta' + \beta\delta' = \alpha'\beta + \beta'\delta$ and (2) $\gamma\alpha' + \delta\gamma' = \gamma'\alpha + \delta'\gamma$ hold.

Now $a^\alpha = a^i$, i odd, so $a^{\alpha\beta'} = a^{\beta'}$, since $Z(N)$ has exponent 2; moreover $a^{\beta\delta'} = a^\beta$, since δ', being central, fixes $N' \geqslant Z(N)$. Thus $\alpha\beta' + \beta\delta' = \beta' + \beta = \alpha'\beta + \beta'\delta$, so (1) holds. If $n \in N$, then $n^{\gamma\alpha'} = n^\gamma$, since $|N^\gamma| \leqslant 2$, and so N^γ is fixed by α'; moreover, $n^{\delta\gamma'} = n^{\gamma'}$, since δ is central, and since $Z(N) \leqslant \ker\gamma'$. Thus $\gamma\alpha' + \delta\gamma' = \gamma + \gamma' = \gamma'\alpha + \delta'\gamma$, so (2) holds. Hence $\operatorname{Aut}_c G$ is abelian.

EXAMPLE 3.2. The dihedral group N of order 2^{n+1}, $n \geqslant 2$, in which $\operatorname{Aut}_c N$ is the Klein 4-group, satisfies conditions (i) and (iii) of Theorem 3.1.

COROLLARY 3.3. *If A is cyclic of order 2^m, $m \geqslant 2$, and if N is a special 2-group, then $\operatorname{Aut}_c(A \times N)$ is abelian.*

PROOF. $\operatorname{Aut}_c N$ fixes $N' = Z(N)$ pointwise and so is abelian.

THEOREM 3.4. *Let $G = A \times N$, where A is abelian (and where N is arbitrary). If there is a characteristic subgroup C of G such that $N \geqslant C \geqslant Z(N)$, then $\operatorname{Aut} G/\operatorname{Aut}_c G \approx \operatorname{Aut} N/\operatorname{Aut}_c N$.*

PROOF. Let T_G: $\operatorname{Aut} G \to \operatorname{Aut}(G/Z(G))$ take ϕ to $\bar{\phi}$, where $\bar{\phi}$ is the automorphism induced on $G/Z(G)$ by ϕ. Then T_G is a homomorphism with kernel $\operatorname{Aut}_c G$ and image $\operatorname{Im} T_G \approx \operatorname{Aut} G/\operatorname{Aut}_c G$. If T_N is defined similarly, we shall show that $\operatorname{Im} T_G \approx \operatorname{Im} T_N$.

From Lemma 2.3, f: $\operatorname{Aut} G \to \mathscr{H}$ taking ϕ to $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is a monomorphism, and $\delta \in \operatorname{Aut} N$. Thus, define h: $\operatorname{Im} T_G \to \operatorname{Im} T_N$ taking $\bar{\phi}$ to $\bar{\delta}$. If i denotes the obvious isomorphism from $G/Z(G)$ to $N/Z(N)$, then $i\bar{\delta} = \bar{\phi}i$, so h is well defined and injective. Further, h is a surjective homomorphism, and so the result follows.

EXAMPLE 3.5. Theorem 3.4 generalizes the example of [1], which shows that there are nonabelian p-groups with nonabelian automorphism groups in which every automorphism is central.

Let $G = A \times M$, where $1 \neq A$ is an elementary abelian 2-group, and where M is the generalized Miller group of Section 1. For p odd, let $G = A \times N$, where $1 \neq A$ is any abelian p-group, and where N is the special p-group of Jonah and Konvisser [5]. Then by Theorem 3.4, taking $C = \Phi(G) = \Phi(M) = Z(M)$ in the first case and $C = G' = N' = Z(N)$ in the second, we obtain $\operatorname{Aut}_c G = \operatorname{Aut} G$. But by Theorem 3.1, $\operatorname{Aut} G$ is nonabelian.

4. Semidirect products with abelian automorphism groups

Throughout this section let $A = \langle a \rangle$ be a cyclic group of order 2^m, $m \geqslant 3$. If $G = AN$ is a nontrivial semidirect product, then any automorphism of A can be extended to an automorphism of G which acts trivially on N. Thus, if $\operatorname{Aut} G$ is abelian and ϕ: $a \to a^3$, then $\phi \in \operatorname{Aut}_c G$, so $a^2 \in Z(G)$ and is fixed by the action of N. Hence N has a subgroup J of index 2 such that

$$(*) \qquad a^n = a \quad \text{if } n \in J \quad \text{and} \quad a^n = a^{1+2^{m-1}} \quad \text{if } n \in N \setminus J.$$

Conversely, suppose for the remainder of the section that N is a nonabelian special 2-group with a subgroup J of index 2 and that G is the semidirect product defined by $(*)$. Then Theorem 4.2 gives conditions for $\operatorname{Aut} G$ to be abelian, but first we establish some properties of G.

LEMMA 4.1. *If $G = AN$ is defined as above, then*
(i) $Z(N) < J$;
(ii) *for* $\phi \in \operatorname{Aut} G$, $[N^\gamma, N] = 1$;
(iii) $\operatorname{Aut}(N: J) \geqslant \operatorname{Aut}_c G$;
(iv) $A \times J$ *is characteristic in* G.

PROOF. (iii) follows from (i), which is obvious.

(ii) Any element in G of the form $a^{2i+1}n$, $n \in N$, has order 2^m. Thus $N^\gamma \leqslant \langle a^2 \rangle$, since N has exponent 4.

(iv) $C_G(a) = A \times J$, so a has 2 conjugates. If $\phi \in \operatorname{Aut} G$ and $a^\phi = a^{2i+1}n$, $n \in N$, then $C_N(n) \geqslant J$, since a^ϕ must also have 2 conjugates. Thus $n \in Z(J)$, and so $C_G(a^\phi) = A \times J$.

THEOREM 4.2. *If $G = AN$ is defined as above, and if*
(i) $A \times Z(N)$ *is characteristics in* G, *and*
(ii) $\operatorname{Aut}(N: J) = \operatorname{Aut}_c N$,
then $\operatorname{Aut} G$ *is abelian*.

PROOF. $\operatorname{Aut} G$ is isomorphic to a subgroup of $\operatorname{Aut}(A \times N)$ by Corollary 2.6, since the conditions of Theorem 2.4 follow from (i) and from Lemma 4.1. But if $\phi \in \operatorname{Aut} G$, then $\delta \in \operatorname{Aut}_c N$, for otherwise, by (ii), there is some $j \in J$ with $j^\delta \notin J$. Thus $(1, j)^\phi = (j^\gamma, j^\phi) \notin A \times J$, contradicting Lemma 4.1(iv). Hence, by Theorem 2.5(ii), $\operatorname{Aut} G$ is isomorphic to a subgroup of $\operatorname{Aut}_c(A \times N)$, which is abelian by Corollary 3.3. (In fact from Sanders' order formula [7], $|\operatorname{Aut}_c G| = |\operatorname{Aut}_c(A \times N)|$, so $\operatorname{Aut} G \approx \operatorname{Aut}_c(A \times N)$.)

EXAMPLE 4.3. Let $M = AN$ be the generalized Miller group of Section 1. Then $A \times J = \langle a, b^2, c \rangle$ is characteristic in G, and $\langle a, b \rangle$ is also, since it is the only subgroup of index 2 in M which is of type $\langle x, y | x^{2^m} = y^4 = 1,\ x^y = x^{1+2^{m-1}} \rangle$. Thus $A \times Z(N) = \langle a, b^2 \rangle = \langle a, b^2, c \rangle \cap \langle a, b \rangle$ is characteristic in M. Further, $\operatorname{Aut}_c N = \operatorname{Inn} N$ is of index 2 in $\operatorname{Aut} N \approx D_4$, and since J is not characteristic, $\operatorname{Aut}(N: J) = \operatorname{Aut}_c N$. Thus $\operatorname{Aut} M$ is abelian (which is the result of [8]).

Corollary 4.4. *If* $G = AN$ *is defined as above, and if*
(i) $Z(J) = Z(N)$, *and*
(ii) $\operatorname{Aut}(N\colon J) = \operatorname{Aut}_c N$,
then $\operatorname{Aut} G$ *is abelian.*

Proof. As in Lemma 4.1(iv), if $\phi \in \operatorname{Aut} G$ and $a^{\phi} = a^{2^{i}+1}n$, $n \in N$, then $C_N(n) \geqslant J$, so $C_N(n) = J$ or $C_N(n) = N$. In either case, $n \in Z(C_N(n)) = Z(N)$. So $A^{\beta} \leqslant Z(N)$, and condition (i) of Theorem 4.2 follows.

Examples 4.5. We use the notation of the Hall and Senior tables [4], in which the index $|\operatorname{Aut} N\colon \operatorname{Aut}_c N|$ is denoted t_3. $\operatorname{Aut} N/\operatorname{Aut}_c N$ acts on the subgroups Ω of index 2 in N, and if the orbit of $J \in \Omega$ has length t_3, then $\operatorname{Aut}(N\colon J) = \operatorname{Aut}_c N$. In particular, this holds if t_3 is prime and J is not characteristic.

The families $32\Gamma_4$ and $64\Gamma_9$ of the tables are nonabelian special 2-groups (of orders 32 and 64, respectively). Further, if N lies in either of these families, then any non-characteristic subgroup J of index 2 is of type Γ_2, and so $Z(J) = Z(N)$. The table below lists the group N, the parameter t_3, and the choice for J (up to isomorphism) which yields a group $G = AN$ with $\operatorname{Aut} G$ abelian. Further examples can be found in the families $64\Gamma_{10}$ and $64\Gamma_{11}$.

N type Γ_4	b_1	b_2	c_1	c_2	c_3	d
t_3	2	2	2	4	4	3
J type Γ_2	c_1, a_1	c_1, c_2	c_1, c_2	c_1	c_2	c_1, c_2
N type Γ_9	b_1	b_4	c	d_1	d_2	
t_3	2	2	3	2	2	
J type Γ_2	c_1	h, c_2	c_1, c_2	h, c_1	h	

References

[1] M. J. Curran, 'A non-abelian automorphism group with all automorphisms central', *Bull. Austral. Math. Soc.* **26** (1982), 393–397.

[2] J. D. Dixon, *Problems in group theory* (Dover Publications, New York, 1973).

[3] B. E. Earnley, *On finite groups whose group of automorphisms is abelian* (Ph.D. thesis, Wayne State University, 1975).

[4] M. Hall, Jr. and J. K. Senior, *The groups of order* 2^n, ($n \leqslant 6$) (New York, 1964).

[5] D. Jonah and M. Konvisser, 'Some non-abelian p-group with abelian automorphism groups', *Arch. Math.* **26** (1975), 131–133.

[6] G. A. Miller, 'A non-abelian group whose group of isomorphisms is abelian', *Messenger of Math.* **43** (1913), 124–125.
[7] P. R. Sanders, 'The central automorphisms of a finite group', *J. London Math. Soc.* **44** (1969), 225–228.
[8] R. R. Struik, 'Some non-abelian 2-groups with abelian automorphism groups', *Arch. Math.* **39** (1982), 299–302.

Department of Mathematics
University of Otago
Dunedin
New Zealand