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MODEL THEORY OF EPIMORPHISMS

BY
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Given a first-order theory T, we let $\mathscr{M}(T)$ be the category of models of T and homomorphisms between them. We shall show that a morphism $A \to B$ of $\mathscr{M}(T)$ is an epimorphism if and only if every element of B is definable from elements of A in a certain precise manner (see Theorem 1), and from this derive the best possible Co-well-power Theorem for $\mathscr{M}(T)$. We shall then investigate conditions ensuring surjectivity of epimorphisms and among other results prove that if T is an equational theory with a cogenerator S, every injection of $\mathscr{M}(T)$ is an equaliser if and only if every implicit operation on S is explicit (see Theorem 4). This is a partial solution to a problem of Lawvere.

It is hoped that the general results proved here will illuminate the descriptive work on epimorphisms in particular theories, especially that of Storrer [14] on commutative rings.

This paper is a sequel to [3] but can be read independently of it (except perhaps for §5). Most of the results here were communicated in a short talk at the Cambridge Logic Conference, August 1971.

0. **Some model-theoretic definitions.** We shall require only fairly basic techniques of Model Theory: for background and standard terminology the reader is referred to Bell and Slomson [4] or Grätzer [7].

Let L be a language. For simplicity we shall always assume that L is countable. We denote L-formulas by letters $\theta, \varphi, \ldots$, finite lists of variables by $\mathbf{x}, \mathbf{y}, \ldots$, L-structures by $A, B, \ldots$ (confusing them with their underlying set in the usual way), and finite lists of elements by $\mathbf{a}, \mathbf{b}, \ldots$. The notation $A \leq B$ means that A is a substructure of B.

A formula is called *pure* if it is logically equivalent to one of the form $\exists\, \mathbf{x}(\theta_1 \wedge \cdots \wedge \theta_n)$ with each θ_i atomic.

A theory T is called *universal* if it has a set of axioms which are universal sentences, and similarly for *universal Horn*. Equivalently (see [7; Chapter 7]) T is universal just if $\mathscr{M}(T)$ is closed under substructures, and T is universal Horn just if $\mathscr{M}(T)$ is closed under substructures and products.

If A is an L-structure we let $L(A)$ be the language L plus new constants a, for $a \in A$ (denoting themselves), and define $\Delta(A)$ to be the set of atomic sentences of $L(A)$ true in A.

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Given a theory T let T^* be the theory with axioms all *universal* sentences provable in T. It is well known (see [7; Section 43]) that $A \models T^*$ just if A has an extension to a model of T. A morphism $B \to C$ of $\mathscr{M}(T)$ is epic in $\mathscr{M}(T^*)$ just if it is epic in $\mathscr{M}(T)$. Thus we might as well work in universal theories, and from now on we shall assume T is universal. Hence $\mathscr{M}(T)$ admits the standard surjection-injection factorisation.

1. **Domination is definable.** Let $B \in \mathscr{M}(T)$, $A \leq B$, $b \in B$. Following Isbell [8], we say that b is *dominated* by A in B, and write $b \in \mathrm{Dom}(A, B)$, if whenever $C \in \mathscr{M}(T)$ and $f, g: B \to C$ are morphisms of $\mathscr{M}(T)$ such that $f \mid A = g \mid A$, then $f(b) = g(b)$.

It is easy to check that $\mathrm{Dom}(A, B) \leq B$ and that the inclusions $A \to \mathrm{Dom}(A, B)$, $\mathrm{Dom}(A, B) \to B$ are the *antidominion* and *dominion* of $A \to B$ in the sense of [3]. Thus the "pointwise" terminology agrees with the categorical: in particular $A \to B$ is *epic* just if $\mathrm{Dom}(A, B) = B$ and *regular* (i.e. a generalised *equaliser*) just if $\mathrm{Dom}(A, B) = A$.

To state the Characterisation Theorem we require one more definition. Let us call a formula $\theta(x, \mathbf{z})$ *univalent* (for T) if $T \vdash \forall\, \mathbf{z}, x, y(\theta(x, \mathbf{z}) \& \theta(y, \mathbf{z}) \to x = y)$. Then we have:

THEOREM 1. *Let T be a universal theory, $B \in \mathscr{M}(T)$, $A \leq B$, $b \in B$. Then $b \in \mathrm{Dom}(A, B)$ with respect to T just if there is a pure formula $\theta(x, \mathbf{z})$, univalent for T, and a list $\mathbf{a} \in A$ (of length that of $\mathbf{z}$), with $B \models \theta(b, \mathbf{a})$.*

Proof. The Sufficiency direction is easy (note that pure formulas are preserved under homomorphism). For the converse we apply the Compactness Theorem to a certain theory which exactly describes the situation "$b \in \mathrm{Dom}(A, B)$", as follows.

Let L^* be the language obtained from L by adjoining constants b, b' for each $b \in B$. Let $T(A, B)$ be the theory with axioms

$$T + \Delta(B) + \Delta(B)' + \{b = b' : b \in A\},$$

where $\Delta(B)'$ is obtained from $\Delta(B)$ by replacing each b by b'. Thus models $(C, u(b), u'(b))_{b \in B}$ of $T(A, B)$ correspond bijectively to diagrams

$$A \longrightarrow B \underset{u'}{\overset{u}{\rightrightarrows}} C$$

with $u, u' \in \mathscr{M}(T)$ and $u \mid A = u' \mid A$.

Since $b \in \mathrm{Dom}(A, B)$ we have $T(A, B) \vdash b = b'$, and so by the Compactness Theorem there is a finite subset S of $T(A, B)$ with $S \vdash b = b'$. By consideration of the various kinds of formulas in S we can find a conjunction $\varphi(x, x_1, \ldots, x_k, y_1, \ldots, y_r)$ of atomic formulas of L, $b_1, \ldots, b_k \in B \backslash (A \cup \{b\})$, and $a_1, \ldots, a_r \in A$ such that $B \models \varphi(b, b_1, \ldots, b_k, a_1, \ldots, a_r)$, and

$$T + \varphi(b, b_1, \ldots, b_k, a_1, \ldots, a_r) + \varphi(b', b_1', \ldots, b_k', a_1', \ldots, a_r') + \bigwedge_{i=1}^{r} a_i = a_i' \vdash b = b'.$$

Let $\theta(x, y_1, \ldots, y_r)$ be $\exists\, x_1 \cdots x_k \varphi$. Then $B \models \theta(b, a_1, \ldots, a_r)$, and $T + \theta(x, y_1, \ldots, y_r) + \theta(x', y_1, \ldots, y_r) \vdash x = x'$.

The method of proof can be regarded as an adaptation of a lemma of G. Kreisel [10].

As an immediate consequence we obtain the best possible cardinality bound on epimorphic images. This generalises [8; Theorem 1.7].

Corollary 1. *Let $u: A \to B$ be an epimorphism of $\mathcal{M}(T)$. Then $|B| \leq |A| + \omega$.*

Proof. Since u is epic, $B = \mathrm{Dom}(uA, B)$. Each element of B is thus determined by a pair $(\theta, \mathbf{a})$ where $\theta(x, \mathbf{z})$ is a pure univalent formula and $\mathbf{a}$ a finite list in A. Clearly there are at most $|A| + \omega$ such pairs.

Previously Freyd in [6; Exercise 3.0] had shown $|B| \leq 2^{|A| + \omega}$: by the remarks at the end of §0 it follows that our result improves his.

2. Surjectivity of epimorphisms is a bounded problem. We shall show that to decide whether or not epimorphisms are surjective in $\mathcal{M}(T)$ requires at most 2^ω tests. Specifically, let $\mathcal{M}^\omega(T)$ be the full subcategory of $\mathcal{M}(T)$ determined by the countable models of T. Then we have:

Theorem 2. *Surjectivity of epimorphisms holds in both or neither of $\mathcal{M}(T)$ and $\mathcal{M}^\omega(T)$.*

Proof. This proceeds in several steps.

(1) A morphism $u: A \to B$ of $\mathcal{M}^\omega(T)$ is epic in $\mathcal{M}^\omega(T)$ just if it is epic in $\mathcal{M}(T)$. This is easy to prove after one observes that if $f, f': B \to C$ are two morphisms of $\mathcal{M}(T)$ there is a monic $u: C' \to C$ of $\mathcal{M}(T)$ with C' countable through which f and f' factor.

(2) It suffices to test epic inclusions in each case, as $u: A \to B$ is epic just if $uA \to B$ is.

(3) There is a theory T^i such that $\mathcal{M}(T^i)$ is isomorphic to the category whose objects are inclusions $A \to B$ of $\mathcal{M}(T)$ and morphisms are commuting squares between these. For let L^i be L plus a new unary predicate H. We write L^i-structures as (B, A) with B an L-structure, $A \subseteq B$. Let θ^H be the *relativisation* of θ to H (see [16; Section 1.5]): thus $(B, A) \models \theta^H$ just if $A \models \theta$. Then let T^i be the theory with axioms $T + \{\theta^H : \theta \in T\}$.

(4) Let $A \to B$ be a proper epic inclusion of $\mathcal{M}(T)$. By the Lowenheim-Skolem Theorem (applied to L^i) there is a countable elementary submodel (B', A') of (B, A). Thus $(B', A') \models T^i$, and so $A', B' \models T$ and $A' \leq B'$. It remains to show that $A' \to B'$ is a proper epic inclusion of $\mathcal{M}(T)$.

(5) Let $b \in B'$. Thus $B \models \theta(b, \mathbf{a})$ for some pure univalent formula $\theta(x, \mathbf{z})$ of L and list $\mathbf{a} \in A$. Let $\varphi(x)$ be $\exists\, \mathbf{z}(\theta(x, \mathbf{z}) \,\&\, \bigwedge_{i=1}^{r} H(z_i))$. Now $(B, A) \models \varphi(b)$ and so $(B', A') \models \varphi(b)$: hence there is $\mathbf{c} \in A'$ such that $B' \models \theta(b, \mathbf{c})$, and so $b \in \mathrm{Dom}(A', B')$. Also $A' \to B'$ is proper, as $(B, A) \models \exists\, x \neg H(x)$.

3. **Pure injections are equalisers.** Let us recall from [2] that an inclusion $A \to B$ of $\mathscr{M}(T)$ is called *pure* if for each pure formula $\theta(\mathbf{x})$ and list $\mathbf{a} \in A$ with $B \models \theta(\mathbf{a})$, we have $A \models \theta(\mathbf{a})$. Pure injections have been investigated fairly extensively in Model Theory (for example in [2] and [15]), but the following result would seem to be new.

Lemma 3. *Any pure injection is regular.*

Proof. Let $A \to B$ be a pure inclusion, $b \in \mathrm{Dom}(A, B)$. By Theorem 1 there is a pure univalent $\theta(x, \mathbf{z})$ and list $\mathbf{a} \in A$ with $B \models \theta(b, \mathbf{a})$. Thus $A \models \theta(b', \mathbf{a})$ for some $b' \in A$ (as $A \to B$ is pure) and so $B \models \theta(b', a)$. Hence $b = b' \in A$ as θ is univalent.

From this we obtain a more general proof of [8; Corollary 1.8]: for any absolutely pure model (in the sense of [2]) is absolutely closed by Lemma 3, and every model has an absolutely pure extension by a result of [13].

We can also use Lemma 3 to prove surjectivity of epimorphisms in some cases. For example let T_{BA} be the equational theory of Boolean algebras. Then every injection of $\mathscr{M}(T_{BA})$ is pure by [2; Section 4], and so an equalizer. Hence we obtain a new proof of the well-known surjectivity of epimorphisms in Boolean algebras.

Finally let us observe that Lemma 3 is crucially dependent on Theorem 1 and hence on the Compactness Theorem. For if $\mathscr{C}$ is the category of metric spaces and contractions, the space Q of rationals is a local injective by [1; Theorem 3] and so absolutely pure by [2; Lemma 4.4]; but every absolutely closed space is complete by [3; Section 4].

4. **Explicit and implicit operations.** Let T be a universal Horn theory. We shall say that T has a *cogenerator* S if T has a model S such that for each $A \in \mathscr{M}(T)$ there is a set I and an injection $A \to S^I$ (observe that $S^I \models T$). Equivalently, S is a cogenerator if for each $A \in \mathscr{M}(T)$, predicate P of L (including $=$) and list $\mathbf{a} \in A$ with $A \models \neg P\mathbf{a}$ there is a morphism $u: A \to S$ with $S \models \neg P(u\mathbf{a})$.

This is a well known property for equational T (note that cogenerators are sometimes called separators): any such T is *residually small* in the sense of [15]. A non-equational example is the universal Horn theory T_{PO} of partial order where the 2-element chain is a cogenerator.

Let S be any L-structure. It seems reasonable to say that *implicit operations on S are explicit* if for all pure $\theta(x, \mathbf{z})$ such that $S \models \forall\, \mathbf{z}, x, y(\theta(x, \mathbf{z}) \& \theta(y, \mathbf{z}) \to x = y)$ there is a term $t(\mathbf{z})$ such that $S \models \forall\, \mathbf{z}, x(\theta(x, \mathbf{z}) \to x = t(\mathbf{z}))$.

Theorem 4. *Let T be a universal Horn theory with a cogenerator S. Then every injection of $\mathscr{M}(T)$ is an equalizer just if implicit operations on S are explicit.*

Proof. This proceeds by establishing two intermediate statements.

(A) If T is a universal theory, then every injection of $\mathscr{M}(T)$ is an equaliser precisely if for each pure univalent $\theta(x, \mathbf{z})$ there are terms $t_1(\mathbf{z}), \ldots, t_n(\mathbf{z})$ such that

$$(R) \qquad T \vdash \forall \mathbf{z}, x\left(\theta(x, \mathbf{z}) \to \bigvee_{i=1}^{n} x = t_i(\mathbf{z})\right).$$

For Sufficiency is easy to establish, and conversely we argue as follows. Let $\theta(x, \mathbf{z})$ be pure and univalent. Then $T+\theta(x, \mathbf{z}) \vdash \bigvee_t x = t(\mathbf{z})$, as if $(B, b, \mathbf{a}) \models T+\theta(b, \mathbf{a})$ then $b \in \mathrm{Dom}(\mathbf{a}, B)$, and so $b = t(\mathbf{a})$ for some term $t(\mathbf{z})$. The result follows by the Compactness Theorem.

(B) If T is a universal Horn theory, then every injection of $\mathscr{M}(T)$ is an equaliser precisely if for each pure $\theta(x, \mathbf{z})$ such that

$$(U) \qquad T \vdash \forall \mathbf{z}, x, y(\theta(x, \mathbf{z}) \ \& \ \theta(y, \mathbf{z}) \to x = y)$$

there is a term $t(\mathbf{z})$ such that

$$(R') \qquad T \vdash \forall \mathbf{z}, x(\theta(x, \mathbf{z}) \to x = t(\mathbf{z})).$$

For this follows from (A), as if (R) holds then (R') holds for some $t \in \{t_1, \ldots, t_n\}$, by applying [12; Theorem 1].

The theorem is an easy consequence of (B) and the observation that the sentences (U) and (R') are universal Horn sentences, and so hold in S precisely if they follow from T.

An algebra S is called *primal* if it is finite and every function $S^n \to S$ is given by a term. Now if T is the equational theory of S it is easy to prove (see [5]) that S is a cogenerator for $\mathscr{M}(T)$. It now follows from Theorem 4 that in $\mathscr{M}(T)$ every injection is an equaliser. Hence primality is related to epimorphisms, as well as to injectives (compare [5; Corollary 4.7]). Also since the Boolean algebra 2 is primal, we obtain yet another proof that epimorphisms are surjective in Boolean algebras.

We have not solved the original problem of Lawvere [11]: "Characterise those equational T with epimorphisms surjective". However Theorem 4 gives a characterisation of the rather smaller but closely related class of equational T with injections regular. We shall make some remarks on the general case in the next section.

5. **When epimorphisms are well-behaved.** It should be clear from Theorem 1 that the bad behaviour of epimorphisms in a general universal theory T is due to the fact that the formulas θ contain existentially quantified variables over which one has no control. As epimorphisms are well-behaved if *injections are transferable* in $\mathscr{M}(T)$, the next result is reassuring.

THEOREM 1*. *Let T be a universal theory with injections transferable, $B \in \mathscr{M}(T)$, $A \leq B$, $b \in B$. Then $b \in \mathrm{Dom}(A, B)$ just if there is a univalent $\theta(x, \mathbf{z})$ which is a conjunction of atomic formulas and a list* $\mathbf{a} \in A$ with $B \models \theta(b, \mathbf{a})$.

Proof. We first observe, by a direct argument using [3; Lemma 1.1], or by Theorem 1.2 and Lemma 1.3 of [3], that $b \in \mathrm{Dom}(A, B)$ just if $b \in \mathrm{Dom}(A, A(b))$. Thus we can assume that $B = A(b)$. Now we apply the method of Theorem 1, but observe that (using the same notation) $b_1, \ldots, b_k \in A(b)$ and so there are terms $t_1(x, \mathbf{u}), \ldots, t_k(x, \mathbf{u})$ and a list $\mathbf{c} \in A$ with $b_i = t_i(b, \mathbf{c})$, for $i=1, \ldots, k$. Then let $\theta^*(x, \mathbf{y}, \mathbf{u})$ be $\varphi(x, t_1(x, \mathbf{u}), \ldots, t_k(x, \mathbf{u}), \mathbf{y})$. Thus θ^* is univalent (in x) and as $\mathbf{a}, \mathbf{c} \in A$, the result follows.

In fact it is not hard to show that the following are equivalent for $\mathscr{M}(T)$:

(1) antidominions are epic and epimorphisms are closed under restriction by injections, and

(2) dominions can be described by positive open formulas, or more precisely, whenever $\theta(x, \mathbf{z})$ is pure and univalent, there are $\theta_1(x, \mathbf{z}), \ldots, \theta_k(x, \mathbf{z})$ which are univalent conjunctions of atomic formulas, such that

$$T \vdash \forall x, \mathbf{z}\left(\theta(x, \mathbf{z}) \rightarrow \bigvee_{i=1}^{k} \theta_i(x, \mathbf{z})\right).$$

We conclude with some results on less well-behaved epimorphisms. Let $\mathrm{Dom}^2(A, B) = \mathrm{Dom}(A, \mathrm{Dom}(A, B))$. In general, $\mathrm{Dom}^2(A, B) \neq \mathrm{Dom}(A, B)$, and in fact the descent to the *stable dominion* $\mathrm{Dom}^\infty(A, B)$ can be arbitrarily long (see Isbell [8] for results on this). However for finite n we can describe the elements of $\mathrm{Dom}^n(A, B)$. The case $n=2$ will illustrate the technique.

Let F_2 be the set of all formulas $\theta(x, \mathbf{z})$ of form

$$\exists\, u_1 \cdots u_n\left(\bigwedge_{i=1}^{n} \theta_i(u_i, \mathbf{z}) \,\&\, \varphi(x, \mathbf{u}, \mathbf{z})\right)$$

where φ is a conjunction of atomic formulas, $\theta_1, \ldots, \theta_n$ are pure, and $\theta_1, \ldots, \theta_n$, $\exists\, \mathbf{u}\varphi$ are univalent. Then by applying Theorem 1 twice (first to $A \rightarrow \mathrm{Dom}(A, B)$, then to $A \rightarrow B$) the reader should be able to establish the following:

$b \in \mathrm{Dom}^2(A, B)$ just if there is $\theta(x, \mathbf{z}) \in F_2$ and $\mathbf{a} \in A$ with $B \models \theta(b, \mathbf{a})$.

From this, in the manner of Theorem 4(A), we can obtain:

THEOREM 5. *For universal T, the following are equivalent:*

(1) *regular injections are composable*

(2) *every strong injection is regular*

(3) *antidominions are epic*

(4) *dominions can be described by F_2-formulas.*

Proof. In view of the above remarks, it suffices to observe that the equivalence of (1), (2) and (3) was proved (dually) in Kelly [9].

As n increases, the restrictions on the quantifiers in the pure univalent formulas describing elements of $\mathrm{Dom}^n(A, B)$ beome stronger. Unfortunately for infinite α, the formulas describing elements of $\mathrm{Dom}^\alpha(A, B)$ become infinitary in character and thus not so amenable to model-theoretic treatment. As epimorphisms in $\mathscr{M}(T)$ are surjective just if for every inclusion $A \rightarrow B$ of $\mathscr{M}(T)$ $\mathrm{Dom}^\infty(A, B) = A$, the possibility of obtaining a characterisation of this property by the methods used here seems small.

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