

SUMS OF THE DIVISOR FUNCTION

B. GORDON AND K. ROGERS

1. Introduction. Shapiro and Warga **(2)** have proved in an elementary way that all large integers are expressible as the sum of at most 20 primes. In so doing, they proved that

$$(1) \quad \sum_{\substack{n \leq x \\ (n,u)=1 \\ n \text{ square-free}}} \frac{\tau(n)}{n} = \frac{1}{2} \cdot \prod_{p|u} \frac{p}{p+2} \cdot \prod_p \frac{(p-1)^2(p+2)}{p^3} \cdot \log^2 x \\ + O(\log x \cdot \log \log xu) + O((\log \log 3u)^2),$$

as $x \rightarrow \infty$, where u is a positive square-free integer,

$$\tau(n) = \sum_{d|n} 1,$$

and all the constants implied by O are absolute, here and throughout this paper (except for dependence on ϵ , when it occurs). We shall sum $\tau(n)$ itself over the same range and derive a refined form of (1): for every $\epsilon > 0$,

$$(2) \quad \sum_{\substack{n \leq x \\ (n,u)=1 \\ \mu(n) \neq 0}} \frac{\tau(n)}{n} = \frac{1}{2} A \log^2 x + (A + B) \log x + C + O\left(x^{-\frac{1}{2}+\epsilon} \exp \frac{c\sqrt{\log 3u}}{\log \log 3u}\right)$$

for some constant $c > 0$, where

$$(2') \quad \begin{cases} A = \prod_{p|u} \frac{p}{p+2} \cdot \prod_p \frac{(p-1)^2(p+2)}{p^3}, \\ B = A \left\{ (2\gamma - 1) + 2 \sum_{p|u} \frac{\log p}{p-1} + 6 \sum_{p \nmid u} \frac{(p-1) \log p}{p^2(p+2)} \right\}, \\ C = C(u) = O\left(\exp \frac{c\sqrt{\log 3u}}{\log \log 3u}\right), \end{cases}$$

and γ denotes Euler's constant. Note that (2) gives better error terms than (1), for fixed u as $x \rightarrow \infty$, because $A = O(1)$ and $B = O(\log \log 3u)$.

2. LEMMA 1. Let

$$F_d(x) = F_d^u(x) = \sum_{\substack{n \leq x \\ (n,u)=1 \\ d|n}} \tau(n).$$

Then for any prime p not dividing du we have, for $v = 1, 2, \dots$,

Received February 12, 1963. The first author is an Alfred P. Sloan fellow.

$$(3) \quad F_{dp^\nu}(x) = (\nu + 1)F_d\left(\frac{x}{p^\nu}\right) - \nu F_d\left(\frac{x}{p^{\nu+1}}\right).$$

Proof. Since the two sides of (3) are step functions which increase only when $dp^\nu | x$, and since they are equal at $x = 0$, it suffices to verify that they have the same increment at $x = dp^{\nu+\rho}y$, with $\rho \geq 0$ and $p \nmid y$. For $\rho = 0$, the requirement is that

$$\tau(p^\nu \cdot dy) = (\nu + 1)\tau(dy) = \tau(p^\nu) \cdot \tau(dy),$$

which is true. For $\rho \geq 1$, we require

$$\tau(dy)\tau(p^{\nu+\rho}) = (\nu + 1)\tau(dy)\tau(p^\rho) - \nu\tau(dy)\tau(p^{\rho-1}),$$

i.e.

$$\nu + \rho + 1 = (\nu + 1)(\rho + 1) - \nu\rho,$$

which is true.

This formula is interesting in itself. It can also be derived by inversion of a functional equation. We need only:

$$(3') \quad F_{p_1^2 \dots p_n^2}(x) = 3F_{p_1^2 \dots p_{n-1}^2}\left(\frac{x}{p_n^2}\right) - 2F_{p_1^2 \dots p_{n-1}^2}\left(\frac{x}{p_n^3}\right) \quad (\text{for } p_n \nmid x)$$

and

$$(3'') \quad \sum_{\substack{n \leq x \\ n \equiv 0(p_1 \dots p_n)}} \tau(n) = H_{p_1 \dots p_n}(x) = 2H_{p_1 \dots p_{n-1}}\left(\frac{x}{p_n}\right) - H_{p_1 \dots p_{n-1}}\left(\frac{x}{p_n^2}\right).$$

By (3'') we expect to obtain $H_{p_1 \dots p_n}$ in terms of H_1 , but all we want is an asymptotic formula. Hence we recall that **(1)**:

$$(4) \quad H_1(x) = \sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is Euler's constant.

LEMMA 2. For square-free d ,

$$H_d(x) = \sum_{\substack{n \leq x \\ d|n}} \tau(n) = \prod_{p|d} \frac{2p-1}{p^2} \cdot \left\{ x \log x + x \left(2\gamma - 1 - 2 \sum_{p|d} \frac{p-1}{2p-1} \log p \right) \right\} + O\left(3^{\nu(d)} \sqrt{\frac{x}{d}}\right),$$

where

$$\nu(d) = \sum_{p|d} 1.$$

Proof. It is evident from (3'') and (4) that some formula of the type

$$H_d(x) = a(d)x \log x + b(d)x + R_d(x)$$

must hold, with $R_d(x)$ of order $\sqrt{x}$. Substituting this in (3'') gives: for $p \nmid d$,

$$a(dp)x \log x + b(dp)x + R_{dp}(x) = 2 \left\{ a(d) \frac{x}{p} \log \frac{x}{p} + b(d) \frac{x}{p} + R_d \left(\frac{x}{p} \right) \right\} \\ - \left\{ a(d) \frac{x}{p^2} \log \frac{x}{p^2} + b(d) \frac{x}{p^2} + R_d \left(\frac{x}{p^2} \right) \right\}.$$

Hence we choose $a(d)$ and $b(d)$ so that $a(1) = 1$, $b(1) = 2\gamma - 1$, and

$$a(dp) = a(d) \left(\frac{2}{p} - \frac{1}{p^2} \right), \\ b(dp) = a(d) \left\{ -2 \frac{\log p}{p} + \frac{2 \log p}{p^2} \right\} + b(d) \left(\frac{2}{p} - \frac{1}{p^2} \right).$$

Hence,

$$a(d) = \prod_{p|d} \frac{2p-1}{p^2} \quad \text{and} \quad \frac{b(dp)}{a(dp)} = \frac{b(d)}{a(d)} - \frac{2(p-1)}{p^2} \log p \cdot \frac{p^2}{2p-1},$$

and

$$b(d) = \left(2\gamma - 1 - 2 \sum_{p|d} \frac{p-1}{2p-1} \log p \right) a(d).$$

When $R_d(x)$ is defined as $H_d(x) - a(d)x \log x - b(d)x$, we get

$$R_{dp}(x) = 2R_d \left(\frac{x}{p} \right) - R_d \left(\frac{x}{p^2} \right).$$

Thus, if

$$|R_d(x)| < K \cdot 3^{v(d)} \sqrt{\frac{x}{d}},$$

it follows that

$$|R_{dp}(x)| < 3 \cdot K \cdot 3^{v(d)} \sqrt{\frac{x}{pd}} = K \cdot 3^{v(pd)} \sqrt{\frac{x}{pd}}.$$

Naturally, if the elementary $O(\sqrt{x})$ in the divisor problem is replaced by $O(x^\theta)$ for some $\theta < \frac{1}{2}$, this improves $\sqrt{(x/d)}$ to $(x/d)^\theta$. However, this will not lead to a better result in (2) by our method.

LEMMA 3. For square-free u ,

$$F_1(x) = \sum_{\substack{n \leq x \\ (n,u)=1}} \tau(n) = \left(\frac{\phi(u)}{u} \right)^2 \left\{ x \log x + x \left(2\gamma - 1 + 2 \sum_{p|u} \frac{\log p}{p-1} \right) \right\} \\ + O \left(\sum_{d|u} 3^{v(d)} \sqrt{\frac{x}{d}} \right).$$

Proof.

$$\begin{aligned}
 F_1(x) &= \sum_{n \leq x} \tau(n) \sum_{d|(n,u)} \mu(d) \\
 &= \sum_{d|u} \mu(d) \sum_{\substack{d|n \\ n \leq x}} \tau(n) \\
 &= \sum_{d|u} \mu(d) \left\{ x \log x + x \left(2\gamma - 1 - 2 \sum_{p|d} \frac{p-1}{2p-1} \log p \right) \right\} \prod_{p|d} \frac{2p-1}{p^2} \\
 &\quad + O\left(\sum_{d|u} 3^{v(d)} \sqrt{\frac{x}{d}} \right),
 \end{aligned}$$

by use of Lemma 2.

Now

$$\sum_{d|u} \mu(d) \prod_{p|d} \frac{2p-1}{p^2} = \prod_{p|u} \left(1 - \frac{2p-1}{p^2} \right) = \prod_{p|u} \left(\frac{p-1}{p} \right)^2 = \left(\frac{\phi(u)}{u} \right)^2.$$

Thus, the coefficient of x is the sum of $(2\gamma - 1)(\phi(u)/u)^2$ and

$$\begin{aligned}
 &-2 \sum_{d|u} \mu(d) a(d) \sum_{p|d} \frac{p-1}{2p-1} \log p \\
 &= -2 \sum_{p|u} \frac{p-1}{2p-1} \log p \sum_{\substack{d|u \\ p|d}} \mu(d) a(d) \\
 &= 2 \sum_{p|u} \frac{p-1}{2p-1} \log p \cdot a(p) \sum_{i|u/p} \mu(i) a(i) \\
 &= 2 \sum_{p|u} \frac{p-1}{p^2} \log p \cdot \left(\frac{\phi(u/p)}{u/p} \right)^2 \\
 &= 2 \left(\frac{\phi(u)}{u} \right)^2 \sum_{p|u} \frac{\log p}{p-1},
 \end{aligned}$$

as required. As in (2), we note that it is easy to show that

$$\sum_{p|u} \frac{\log p}{p-1} \leq O(1) + \log p_{v(u)} = O(\log \log 3u),$$

since

$$\log u \geq \sum_{p \leq p_{v(u)}} \log p = (1 + o(1)) p_{v(u)},$$

by the prime-number theorem. We could use Tchebychef's inequality instead. As to the coefficient of $\sqrt{x}$ in the O -term, we have

$$\begin{aligned}
 S &= \sum_{d|u} \frac{3^{v(d)}}{\sqrt{d}} = \prod_{p|u} \left(1 + \frac{3}{\sqrt{p}} \right) \leq \prod_{p \leq p_{v(u)}} \left(1 + \frac{3}{\sqrt{p}} \right) \\
 &\leq \prod_{p \leq (1+o(1)) \log u} \left(1 + \frac{3}{\sqrt{p}} \right).
 \end{aligned}$$

Hence,

$$\log S \leq 3 \sum_{p \leq (1+o(1)) \log u} \frac{1}{\sqrt{p}} \cdot (1 + o(1)).$$

By elementary means one can show that

$$\sum_{p \leq x} \frac{1}{\sqrt{p}} = O\left(\frac{\sqrt{x}}{\log x}\right),$$

while the prime-number theorem gives

$$\sum_{p \leq x} \frac{1}{\sqrt{p}} = \frac{\sqrt{x}}{\log x} (1 + o(1)).$$

Hence,

$$(5) \quad S = O\left(\exp\left\{\frac{c\sqrt{\log 3u}}{\log \log 3u}\right\}\right),$$

where c can be taken as $3 + o(1)$ for large u .

LEMMA 4. *For square-free d , we have*

$$F_{d^2}(x) = \alpha(d)x \log x + \beta(d)x + R_d(x),$$

where

$$\alpha(d) = \left(\frac{\phi(u)}{u}\right)^2 \cdot \prod_{p|d} \frac{3p-2}{p^3},$$

$$\beta(d) = \alpha(d) \left\{ 2\gamma - 1 + 2 \sum_{p|u} \frac{\log p}{p-1} - 6 \sum_{p|d} \frac{p-1}{3p-2} \log p \right\},$$

and

$$R_d(x) = O\left(\frac{5^{v(d)}}{d} \sqrt{x} \cdot S\right),$$

with S as above.

Proof. This follows the lines of Lemma 2, but (3') replaces (3''). We find that to get $R_{dp}(x) = 3R_d(x/p^2) - 2R_d(x/p^3)$ we need

$$\alpha(dp) = \alpha(d) \cdot (3p-2)/p^3,$$

and

$$\frac{\beta(dp)}{\alpha(dp)} = \frac{\beta(d)}{\alpha(d)} - 6 \frac{p-1}{3p-2} \log p,$$

for $p \nmid d$. These give the desired values, if Lemma 3 is used for evaluating $\alpha(1)$ and $\beta(1)$. The estimation of $R_d(x)$ is now similar to that in Lemma 2.

3. Now the sieve process can be used to compute $\sum \tau(n)$.

THEOREM.

$$\sum_{\substack{n \leq x \\ (n,u)=1 \\ n \text{ square-free}}} \tau(n) = Ax \log x + Bx + R(x),$$

where

$$R(x) = O\left(x^{\frac{1}{4}+\epsilon} \cdot \exp \frac{c\sqrt{\log 3u}}{\log \log 3u}\right),$$

for every $\epsilon > 0$, with A , B , and c as in (2').

Proof.

$$\begin{aligned} \sum_{\substack{n \leq x \\ (n,u)=1 \\ \mu(n) \neq 0}} \tau(n) &= \sum_{\substack{n \leq x \\ (n,u)=1}} \tau(n) \sum_{s^2 | n} \mu(s) \\ &= \sum_{\substack{s^2 \leq x \\ (s,u)=1}} \mu(s) \cdot F_{s^2}(x) \\ &= \sum_{\substack{s^2 \leq x \\ (s,u)=1}} \mu(s) \{ \alpha(s)x \log x + \beta(s)x + R_s(x) \} \\ &= x \log x \left\{ \sum_{(s,u)=1} \mu(s)\alpha(s) - \sum_{\substack{s > \sqrt{x} \\ (s,u)=1}} \mu(s)\alpha(s) \right\} \\ &\quad + x \left\{ \sum_{(s,u)=1} \mu(s)\beta(s) - \sum_{\substack{s > \sqrt{x} \\ (s,u)=1}} \mu(s)\beta(s) \right\} \\ &\quad + O\left\{ S \cdot \sqrt{x} \cdot \sum_{\substack{s \leq \sqrt{x} \\ \mu(s) \neq 0}} \frac{5^{v(s)}}{s} \right\}; \\ \sum_{(s,u)=1} \mu(s)\alpha(s) &= \alpha(1) \cdot \prod_{p|u} \left(1 - \frac{3p-2}{p^3} \right) = \left(\frac{\phi(u)}{u} \right)^2 \prod_{p|u} \frac{(p-1)^2(p+2)}{p^3} \\ &= \prod_{p|u} \frac{p}{p+2} \cdot \prod_p \frac{(p-1)^2(p+2)}{p^3} \\ &= A; \\ \sum_{(s,u)=1} \mu(s)\beta(s) &= \sum_{(s,u)=1} \mu(s)\alpha(s) \left\{ (2\gamma-1) + 2 \sum_{p|u} \frac{\log p}{p-1} - 6 \sum_{p|s} \frac{p-1}{3p-2} \log p \right\} \\ &= \left(2\gamma-1 + 2 \sum_{p|u} \frac{\log p}{p-1} \right) A - 6 \sum_{p \nmid u} \frac{p-1}{3p-2} \log p \cdot \sum_{\substack{(s,u)=1 \\ p|s}} \mu(s)\alpha(s) \\ &= \left(2\gamma-1 + 2 \sum_{p|u} \frac{\log p}{p-1} \right) A + 6 \sum_{p \nmid u} \frac{p-1}{3p-2} \cdot \frac{3p-2}{p^3} \log p \sum_{(t,up)=1} \mu(t)\alpha(t) \\ &= A \left[2\gamma-1 + 2 \sum_{p|u} \frac{\log p}{p-1} + 6 \sum_{p \nmid u} \frac{(p-1) \log p}{p^3} \cdot \frac{p}{p+2} \right] \\ &= B. \end{aligned}$$

For square-free s , we have

$$\alpha(s) < \prod_{p|s} \frac{3}{p^2} = \frac{3^{\nu(s)}}{s^2} < \frac{s^\epsilon}{s^2}$$

for all large s , because

$$\nu(s) = O\left(\frac{\log s}{\log \log s}\right),$$

by **(1, Theorem 317)**. Hence

$$\begin{aligned} \sum_{s > \sqrt{x}} |\alpha(s)\mu(s)| &< \sum_{s > \sqrt{x}} \frac{1}{s^{2-\epsilon}} \\ &= O(x^{-\frac{1}{2}+\frac{1}{2}\epsilon}). \end{aligned}$$

From the formula for $\beta(s)$, we have

$$\begin{aligned} \sum_{s > \sqrt{x}} |\beta(s)\mu(s)| &< O(x^{-\frac{1}{2}+\frac{1}{2}\epsilon} \log \log 2u) + \sum_{s > \sqrt{x}} |\mu(s)\alpha(s) \log s| \\ &= O(x^{-\frac{1}{2}+\frac{1}{2}\epsilon} \log \log 3u). \end{aligned}$$

Finally,

$$\sum_{\substack{s \leq \sqrt{x} \\ s \text{ square-free}}} \frac{5^{\nu(s)}}{s} = O\left(\sum_{s \leq \sqrt{x}} \frac{s^\epsilon}{s}\right) = O(x^{\frac{1}{2}\epsilon}).$$

Thus,

$$\begin{aligned} R(x) &= O\{x^{\frac{1}{2}(1+\epsilon)}(\log x + \log \log 3u + S)\} \\ &= O\left(x^{\frac{1}{2}+\epsilon} \cdot \exp \frac{c\sqrt{\log 3u}}{\log \log 3u}\right). \end{aligned}$$

To derive (2), write $G(x)$ for the sum in the theorem. Then,

$$\begin{aligned} &\sum_{\substack{n \leq x \\ (n,u)=1 \\ \mu(n) \neq 0}} \frac{\tau(n)}{n} \\ &= \int_{1-}^x \frac{dG(t)}{t} \\ &= G(x)/x + \int_1^x \frac{G(t)}{t^2} dt \\ &= A \log x + B + R(x)/x + \int_1^x \{A \log t \cdot t^{-1} + Bt^{-1} + R(t)t^{-2}\} dt \\ &= \frac{1}{2} A \log^2 x + (A+B) \log x + \int_1^\infty \frac{R(t)}{t^2} dt + \frac{R(x)}{x} - \int_x^\infty \frac{R(t)}{t^2} dt + B \\ &= \frac{1}{2} A \log^2 x + (A+B) \log x + B + \int_1^\infty \frac{R(t)}{t^2} dt \\ &\quad + O\left(x^{-\frac{1}{2}+\epsilon} \cdot \exp \frac{c\sqrt{\log 3u}}{\log \log 3u}\right), \end{aligned}$$

which gives (2).

4. We conclude by observing that our method would enable $\sum \tau(n)$ over the k th-power-free integers to be similarly treated. As to the effect of replacing (1) by (2) in (2), this does not lead to a reduction of the number 20 in their result.

REFERENCES

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed. (Oxford, 1960).
2. H. N. Shapiro and J. Warga, *On the representation of large integers as sums of primes I*, Comm. Pure Appl. Math., 3 (1950), 153–176.

*University of California
Los Angeles, California*