IDEMPOTENTS IN COMPLETELY 0-SIMPLE SEMIGROUPS

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The structure theorem for completely 0-simple semigroups established by Rees [5] in 1940 has proved a very powerful tool in the investigation of such semigroups. In this paper the theorem is applied to an investigation of the subsemigroup of a completely 0-simple semigroup generated by its idempotents. Previous work on this problem has been carried out by Kim [4], but the present note offers a more direct approach.

1. Paths and values. The notations used will be those of [3]. A completely 0-simple semigroup $S$ can, by Rees's Theorem [3, Theorem III.2.5], be identified with a Rees matrix semigroup $M^0(G; I, \Lambda; P)$ in which $G$ is a group, $I$ and $\Lambda$ are index sets and $P$ is a $\Lambda \times I$ matrix $(p_{\lambda i})$ with entries in $G^0$ and with no row or column consisting of zeros. The non-zero elements of $S$ are triples $(a, i, \lambda)$ in $G \times I \times \Lambda$ multiplying according to the rule that

$$(a, i, \lambda)(b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } p_{\lambda j} \neq 0, \\ (0, j, \mu) & \text{if } p_{\lambda j} = 0. \end{cases}$$

For the present investigation it is convenient to assume that $I$ and $\Lambda$ are disjoint. Since they are merely index sets (in one-to-one correspondence respectively with the sets of $R$-classes and $L$-classes of $S$) there is no harm in doing so. With this assumption, consider the relation $K$ on $I \cup \Lambda$ defined by the rule that $(i, \lambda) \in K$ if and only if $i \in I$, $\lambda \in \Lambda$ and $p_{\lambda i} \neq 0$, and let $\mathcal{K}$ be the equivalence relation on $I \cup \Lambda$ generated by $K$. Thus for $x, y$ in $I \cup \Lambda$ we have that $(x, y) \in \mathcal{K}$ if and only if either $x = y$ or (for some $n \geq 2$) there exist $z_1, \ldots, z_n$ in $I \cup \Lambda$ such that

(i) $z_1 = x$ and $z_n = y$,
(ii) $z_r \in I \Rightarrow z_{r+1} \in \Lambda$, $z_r \in \Lambda \Rightarrow z_{r+1} \in I$,
(iii) $(z_n, z_{n+1}) \in K \cup K^{-1}$.

The sequence $(z_1, \ldots, z_n)$ will be called a path from $x$ to $y$. Among the paths from $x$ to $x$ we shall include the null path.

The equivalence relation $\mathcal{K}$ will be called the connectivity relation, and we shall call the semigroup $S$ connected if $\mathcal{K}$ is the universal relation on $I \cup \Lambda$. Notice that connectedness is a property of the semigroup and not merely of the matrix $P$. The isomorphism theorem associated with Rees's Theorem (see [3, Theorem III.2.8]) ensures that while the sandwich matrix $P$ is not uniquely determined by $S$ the pattern of non-zero entries in $P$ is invariant. Hence the property of connectedness, which depends solely on this pattern, is either possessed by all representations of $S$ as a Rees matrix semigroup or by none.

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. Let $(x, y) \in \mathcal{H}$ (and from now on we shall for simplicity write this as $x \sim y$), and let $p = (z_1, \ldots, z_n)$, where $z_1 = x$, $z_n = y$, be a path from $x$ to $y$. The value $V(p)$ of the path $p$ is the element of $G$ defined by

$$V(p) = (z_1, z_2)_1 \cdot (z_2, z_3)_2 \cdot \ldots \cdot (z_{n-1}, z_n)_n,$$

where, for $i$ in $I$ and $\lambda$ in $\Lambda$, we define

$$(i, \lambda)_1 = p_i^{-1}, \quad (\lambda, i)_1 = p_{i\lambda}.$$

The value of the null path from $x$ to $x$ is defined to be $e$, the identity element of $G$. Thus, for example, the value of the path $(\lambda, i, \mu, j, \lambda)$ is the element $p_{i\lambda}p_{\mu j}^{-1}p_{\mu i}^{-1}$ of $G$. Let $P_{x,y}$ be the set of all paths from $x$ to $y$ and let

$$V_{x,y} = \{V(p) : p \in P_{x,y}\},$$

the set of values of paths from $x$ to $y$. By convention, define $V_{x,y} = \emptyset$ if $x \neq y$.

**Lemma 1.** If $x, y, z \in I \cup \Lambda$ and $x \sim y \sim z$ then

(i) $V_{y,x} = V_{x,y}^{-1}$;  
(ii) $V_{x,y} V_{y,z} = V_{x,z}$.

**Proof.** Let $a \in V_{y,x}$. Then $a = V(p)$ where $p = (z_1, \ldots, z_n)$ is a path from $y$ to $x$. Then $(z_n, \ldots, z_1)$ is a path from $x$ to $y$ whose value is $a^{-1}$. Thus

$$a = (a^{-1})^{-1} \in V_{x,y}^{-1},$$

and so $V_{y,x} \subseteq V_{x,y}^{-1}$. It follows that

$$V_{y,x}^{-1} \subseteq (V_{y,x})^{-1} = V_{x,y};$$

hence, relabelling by interchanging $x$ and $y$, we have $V_{x,y}^{-1} \subseteq V_{y,x}$. This establishes part (i).

Let $p = (x, z_2, \ldots, z_{m-1}, y) \in P_{x,y}$ and $q = (y, t_2, \ldots, t_{n-1}, z) \in P_{y,z}$. Then $(x, z_2, \ldots, z_{m-1}, y, t_2, \ldots, t_{n-1}, z) \in P_{x,z}$: Since the value of this last path is evidently $V(p) V(q)$, it is clear that

$$V_{x,y} V_{y,z} \subseteq V_{x,z}.$$  

(1)

Conversely, if $a \in V_{x,z}$ then for every $b$ in $V_{x,y}$ we have (using part (i) and formula (1))

$$a = bb^{-1} a \in V_{x,y} V_{y,x} V_{x,z} \subseteq V_{x,y} V_{y,z}.$$

Thus $V_{x,z} \subseteq V_{x,y} V_{y,z}$ as required.

**Theorem 1.** Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. Let $E$ be the set of idempotents in $S$ and $\langle E \rangle$ the subsemigroup of $S$ generated by the idempotents. Then

$$\langle E \rangle = \{(a, i, \lambda) \in S : i \sim \lambda \quad \text{and} \quad a \in V_{i\lambda} \} \cup \{0\}.$$

**Proof.** It is well-known, and in any event easy to verify, that the non-zero idempotents of $S$ are the elements $(p_{i\lambda}^{-1}, i, \lambda)$ for which $p_{i\lambda} \neq 0$. Let $(a, i, \lambda) \in \langle E \rangle \setminus \{0\}$. Then there exist $i_2, \ldots, i_n$ in $I$ and $\lambda_1, \ldots, \lambda_{n-1}$ in $\Lambda$ such that

$$(a, i, \lambda) = (p_{i\lambda_1}^{-1}, i, \lambda_1)(p_{\lambda_1 i_2}^{-1}, i_2, \lambda_2) \ldots (p_{\lambda_{n-1} i_n}^{-1}, i_n, \lambda) \neq 0.$$
Hence \( i \sim \lambda_1 \sim i_2 \sim \lambda_2 \sim \ldots \sim i_n \sim \lambda \) and so \( i \sim \lambda \). Also
\[
a = p_{\lambda_1, i}^{-1} p_{\lambda_2, i_2} p_{\lambda_3, i_3} \cdots p_{\lambda_n, i_n} p_{\lambda, \lambda},
\]
the value of the path \((i, \lambda_1, i_2, \lambda_2, \ldots, i_n, \lambda)\) from \( i \) to \( \lambda \), and so \( a \in V_{i, \lambda} \).

Conversely, let \( i \sim \lambda \) and \( a \in V_{i, \lambda} \). Then there exists a path \((i, \lambda_1, i_2, \lambda_2, \ldots, i_n, \lambda)\) whose value
\[
p_{\lambda_1, i}^{-1} p_{\lambda_2, i_2} p_{\lambda_3, i_3} \cdots p_{\lambda_n, i_n} p_{\lambda, \lambda}^{-1}
\]
is equal to \( a \). Hence
\[
(a, i, \lambda) = (p_{\lambda_1, i}^{-1}, i, \lambda_1)(p_{\lambda_2, i_2}^{-1}, i_2, \lambda_2) \cdots (p_{\lambda_n, i_n}^{-1}, i_n, \lambda) \in \langle E \rangle.
\]
This completes the proof.

We shall say that \( S = M_0[G; I, \Lambda; P] \) is \textit{replete} if it is connected and \( V_{x, x} = G \) for some \( x \) in \( I \cup \Lambda \). In the presence of connectedness this latter condition is in fact equivalent to the apparently stronger condition that \( V_{y, z} = G \) for all \( y, z \) in \( I \cup \Lambda \); if \( S \) is replete then \( V_{y, x} \) and \( V_{x, z} \) are both non-empty by connectedness and so
\[
V_{y, z} = V_{y, x} V_{x, x} V_{x, z} = V_{y, x} G V_{x, z} = G.
\]
A semigroup \( S \) with set of idempotents \( E \) is called \textit{idempotent-generated} if \( \langle E \rangle = S \). We now have the following obvious corollary to Theorem 1.

**Corollary.** The completely 0-simple semigroup \( M_0[G; I, \Lambda; P] \) is idempotent-generated if and only if it is replete.

2. **The completely simple case.** The case where \( S \) has no zero and is completely simple is easier, since the matrix \( P \) has no zero entries and connectedness is automatic. The results corresponding to Theorem 1 and its corollary do not require separate statement. One easy consequence of Theorem 1 is worth recording. A subsemigroup \( U \) of a semigroup \( S \) is called \textit{unitary} if, for all \( u \) in \( U \) and all \( s \) in \( S \),
\[
us \in U \Rightarrow s \in U, \quad su \in U \Rightarrow s \in U.
\]

**Theorem 2.** In a completely simple semigroup \( S \) with set \( E \) of idempotents, the subsemigroup \( \langle E \rangle \) generated by the idempotents is unitary.

**Proof.** Let \( S = M[G; I, \Lambda; P] \) and suppose that \( u = (a, i, \lambda) \in \langle E \rangle \), \( s = (b, j, \mu) \in S \) and \( us = (ap_{\lambda, j} b, i, \mu) \in \langle E \rangle \). Then \( a \in V_{i, \lambda} \) and \( ap_{\lambda, j} b \in V_{i, \mu} \), from which it follows that
\[
b = p_{\lambda, j}^{-1} a^{-1} ap_{\lambda, j} b \in V_{i, \lambda} V_{\lambda, i} V_{i, \mu} = V_{i, \mu}.
\]
Thus \( s \in \langle E \rangle \). Similarly \( su \in \langle E \rangle \Rightarrow s \in \langle E \rangle \), and so \( \langle E \rangle \) is unitary.

We may remark that a closely analogous result exists for the completely 0-simple case. If \( S \) is a semigroup with zero element 0 then a subsemigroup \( U \) containing 0 is called
0-unitary if, for all \( u \) in \( U \setminus \{0\} \) and all \( s \) in \( S \setminus \{0\} \),

\[
us \in U \setminus \{0\} \Rightarrow s \in U \setminus \{0\}, \quad su \in U \setminus \{0\} \Rightarrow s \in U \setminus \{0\}.
\]

Then the following theorem can be proved. The details of the proof differ only slightly from those of the last proof and so may safely be omitted.

**Theorem 3.** In a completely 0-simple semigroup with set \( E \) of idempotents, the subsemigroup \( \langle E \rangle \) generated by the idempotents is 0-unitary.

Returning now to the completely simple case, we consider the simplifications that occur when we assume that the sandwich matrix \( P \) is normal. As remarked by Clifford [2], every completely simple semigroup is isomorphic to a Rees matrix semigroup \( \mathcal{M}[G; I, \Lambda; P] \) in which \( P = (p_{ki}) \) is normal, in the sense that there exist \( k \) in \( I \) and \( v \) in \( \Lambda \) such that \( p_{k\lambda} = e \) (the identity element of \( G \)) for all \( \lambda \) in \( \Lambda \) and \( p_{vi} = e \) for all \( i \) in \( I \). To put it another way, \( P \) is normal if it contains at least one row and at least one column consisting entirely of \( e \)'s.

Let us now suppose that \( S = \mathcal{M}[G; I, \Lambda; P] \) and that \( P \) is normal, with \( p_{k\lambda} = e \) for all \( \lambda \) and \( p_{vi} = e \) for all \( i \).

**Lemma 2.** With these assumptions, \( V_{x,y} = V_{z,t} \) for all \( x, y, z, t \) in \( I \cup \Lambda \).

**Proof.** The first step is to show that \( e \in V_{x,y} \) for all \( x, y \) in \( I \cup \Lambda \). This is straightforward if we consider separately the four cases (i) \( x, y \in I \), (ii) \( x \in I, y \in \Lambda \), (iii) \( x \in \Lambda, y \in I \), (iv) \( x, y \in \Lambda \). In case (i) we have a path \((x, \nu, y)\) from \( x \) to \( y \) with value \( e \) and so \( e \in V_{x,y} \). In case (ii) the path \((x, \nu, k, y)\) has value \( e \). Cases (iii) and (iv) are similar.

The desired result now follows easily, since for all \( x, y, z, t \) in \( I \cup \Lambda \),

\[
V_{x,y} = e V_{x,y} e \subseteq V_{x,x} V_{x,y} V_{y,t} = V_{z,t},
\]

and, similarly, \( V_{z,t} \subseteq V_{x,y} \).

There is thus a fixed subset \( V \) of \( G \) equal to \( V_{x,y} \) for every choice of \( x, y \) in \( I \cup \Lambda \). An alternative description of \( V \) is as follows:

**Lemma 3.** \( V = \langle \{ p_{k\lambda}: \lambda \in \Lambda, i \in I \} \rangle \), the subgroup of \( G \) generated by the elements \( p_{k\lambda} \).

**Proof.** Since \( V = V_{x,y} \) for arbitrarily chosen elements \( x, y \) in \( I \cup \Lambda \), it is immediate that each element of \( V \), being the value of a path from \( x \) to \( y \), is a product of the entries of \( P \) and their inverses. Conversely, to show that \( V \) contains every such product we need only observe (a) that each \( p_{k\lambda} \in V_{\lambda,i} = V \), (b) that each \( p_{k\lambda}^{-1} \in V_{i,\lambda} = V \), and (c) that if \( a \in V = V_{x,y} \) and \( b \in V = V_{y,z} \) then \( ab \in V_{x,y} V_{y,z} = V_{x,z} = V \).

The final easy consequence of Theorem 1 and Lemma 3 is the following theorem, which can of course be verified more directly. Part of this result is implicit in the proof of Theorem 1 in Benzaken and Mayr [1].

**Theorem 4.** Let \( S = \mathcal{M}[G; I, \Lambda; P] \) be a completely simple semigroup in which \( P \) is normal. Then \( \langle E \rangle = V \times I \times \Lambda \), where \( V \) is the subgroup of \( G \) generated by the entries of \( P \). The semigroup \( S \) is idempotent-generated if and only if \( V = G \).
That this is untrue without normalisation is evident from the following elementary example. Let \( S = \mathcal{M}[G; I, \Lambda; P] \), where \( I = \{1, 2\} \), \( \Lambda = \{3, 4\} \), \( G = \mathbb{Z}_2 = \{e, a\} \), \( p_{31} = p_{32} = p_{41} = p_{42} = a \). Then the subgroup generated by the entries of \( P \) is \( G \), but

\[
\langle E \rangle = E = \{(a, 1, 3), (a, 1, 4), (a, 2, 3), (a, 2, 4)\}.
\]

In fact \( V_{1,3} = V_{1,4} = V_{2,3} = V_{2,4} = \{a\} \), in accord with Theorem 1.

REFERENCES


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