Canad. Math. Bull. Vol. **23** (4), 1980

A THEOREM ON THE DENSENESS OF ORBITS IN METRIC SPACES

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ABSTRACT. Let τ be a transformation from a compact metric space X into itself. Conditions are presented which ensure that there exists an orbit $\{\tau^n(x)\}_{n=1}^{\infty}$ which is dense in X. An example is given.

1. **Introduction.** Let X be a compact metric space and τ a mapping from X into X. If X is an interval of the real line, sufficient conditions are known [1] which ensure the existence of a point $x \in X$ such that $\{\tau^n(x)\}_{n=0}^{\infty}$ is dense in X, where $\tau^n = \tau \circ \tau^{n-1}$ is the nth iterate of τ. The aim of this note is to extend this result to a general compact metric space. In [2, 3, 6, 7] the existence of invariant measures is established for expanding maps. In [2, 3, 6] the underlying space is restricted to a compact, connected smooth manifold. These existence results do not say anything about the supports of the invariant measures. The existence of dense orbits in a subset $\bar{X} \subset X$ would be helpful in locating the support of the invariant measure. In [5], conditions are presented which guarantee the existence of orbits that do not approach any periodic pattern, but in themselves they do not establish denseness even in a subset of the underlying space.

2. **Denseness of orbit.** Let (X, ρ) be a compact metric space, and let $\{X_1, X_2, \ldots, X_n\}$ be a finite partition of X such that each X_i is closed and compact. Let $H = \bigcup_{i \neq j} (X_i \cap X_j)$ denote the boundary of the partition. Let $\tau : X \to X$ be a mapping such that $\tau|_{X_i}$ is continuous. Define $B = \bigcup_{n=0}^{\infty} \tau^{-n}(H)$.

THEOREM. *Let (X, ρ) and τ be as above. Assume $B \neq X$ and (1) $\tau|_{X_i}$ is expanding for $i = 1, \ldots, n$, i.e., there exist $d_i > 1$ such that $\rho(\tau(x), \tau(y)) > d_i \rho(x, y)$ for every $x, y \in X_i$.*

(2) for each pair (i, j), $i, j \in (1, \ldots, n)$, there exists an integer $n_{ij} \ni \tau^{n_{ij}}(X_i) \supset X_j$

(3) for each i there exist integers $n_i, l_i \in (1, \ldots, n) \ni \tau(X_i) = \bigcup_{j=l_i}^{n_i} X_j$. Then there is a point $x \in X$ such that the orbit $\{\tau^n(x)\}_{n=0}^{\infty}$ starting at x is dense in X.

Proof. Associate with each $x \notin B$ the coding $\langle x \rangle$ of x as follows:

$$\langle x \rangle = .\, i_1 i_2 i_3 \ldots\ldots\ldots\ldots,$$

Received by the editors March 2, 1979 and in revised form August 8, 1979.

* The research of this author was supported by a grant from the National Research Council of Canada: Grant No. A-9072.

where $x \in X_{i_1}$, $\tau(x) \in X_{i_2}$, $\tau^2(x) \in X_{i_3}, \ldots\ldots$. This coding is well-defined since we cannot have $\tau^i(x) \in X_k \cap X_l$. Further, we claim:

(i) if two codings are equal, i.e., $\langle x\rangle = \langle y\rangle$, then $x = y$, and

(ii) if $.\, i_1 i_2 i_3 \ldots\ldots$ is a coding and τ satisfies $\tau(X_{i_j}) \supset X_{i_{j+1}}$, $j = 1, 2, 3, \ldots$ (we refer to such a coding as allowable), then there exists a unique $x \in X$ such that $\langle x\rangle = .\, i_1 i_2 i_3 \ldots$.

To prove (i), we note that our hypotheses implies $\rho(\tau^n(x), \tau^n(y)) \geq d^n \rho(x, y)$, where $d = \min(d_1, d_2, \ldots, d_n)$. Since $d > 1$ and ρ is bounded on $X \times X$, this gives a contradiction unless $x = y$.

To prove (ii), we define

$$J_k = \{x : x \in X_{i_1}, \tau(x) \in X_{i_2}, \ldots, \tau^{k-1}(x) \in X_{i_k}\}$$

We note that $\{J_k\}_{k=1}^{\infty}$ is a collection of compact sets with the finite intersection property. Thus $\bigcap_{k>1} J_k$ is non-empty and by (i) has no more than one element.

We can now associate allowable codings in a $1-1$ and onto fashion with $x \in X - B$. To prove the existence of a dense orbit, form the set of finite allowable sequences

$$\{i_k,\, i_k i_l,\, i_k i_l i_m, \ldots\}$$

where, for example, the symbol $i_k i_l$ represents all sequences $i_k i_l$ such that $\tau(X_{i_k}) \supset X_{i_l}$. These are countable. Let $S_1, S_2, S_3, \ldots$ be an enumeration of them. Now form the coding $.\, S_1 T_1 S_2 T_2 S_3 T_3 \ldots$, where the T_i are inserted to make this an allowable sequence. This is permitted by hypothesis (2). Let the coding represent the point $x \in X$. Let $y \in X$ and $\varepsilon > 0$ be given. We want to show that there exists an integer n such that $|\tau^n(x) - y| < \varepsilon$. Let the symbol S_p correspond to the coding of $\{y, \tau(y), \ldots, \tau^m(y)\}$, i.e., if $y \in X_{i_1}, \ldots, \tau^m(y) \in X_{i_m}$, then $S_p = i_1 \ldots i_m$. We then have:

$$\begin{array}{ll} \tau^p(x) \in X_{i_1}, & y \in X_{i_1} \\ \tau^{p+1}(x) \in X_{i_2}, & \tau(y) \in X_{i_2} \\ \cdot & \\ \cdot & \\ \cdot & \\ \tau^{p+m}(x) \in X_{i_m}, & \tau^m(y) \in X_{i_m} \end{array}$$

By the expanding property of τ,

$$|\tau^p(x) - y| < d^{-m}\, |\tau^{p+m}(x) - \tau^m(y)| < \varepsilon,$$

if m is chosen sufficiently large. This proves the denseness of the orbit $\{\tau^n(x)\}_{n=0}^{\infty}$ in X.

We note that the theorem does not require τ to be non-singular. If it is and X is an n-dimensional space, then $B \neq X$ is implied by the condition $m(H) = 0$, where m is the Lebesgue measure.

3. An example. Let $\tau:[0, 1]\times[0, 1]\to[0, 1]\times[0, 1]$ be defined by $\tau(x, y)=(f(x), f(y))$, where

$$f(t)=\begin{cases} 2t & 0\le t\le\frac{1}{2} \\ 2-2t, & \frac{1}{2}\le t\le 1\end{cases}$$

Let $X_1=[0,\frac{1}{2}]\times[0,\frac{1}{2}]$, $X_2=[0,\frac{1}{2}]\times[\frac{1}{2}, 1]$, $X_3=[\frac{1}{2}, 1]\times[0,\frac{1}{2}]$, and $X_4=[\frac{1}{2}, 1]\times[\frac{1}{2}, 1]$. It is easy to show that τ and the partition $\{X_1, X_2, X_3, X_4\}$ satisfy the conditions of the theorem. Hence there exists an orbit $\{\tau^n(x)\}_{n=0}^{\infty}$ which is dense in $[0, 1]\times[0, 1]$. The one dimensional version of this result is known [4].

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