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OPERATORS AND THE SPACE OF INTEGRABLE SCALAR FUNCTIONS WITH RESPECT TO A FRÉCHET-VALUED MEASURE

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Abstract

We consider the space $L^1(\nu, X)$ of all real functions that are integrable with respect to a measure ν with values in a real Fréchet space X. We study L-weak compactness in this space. We consider the problem of the relationship between the existence of copies of ℓ^∞ in the space of all linear continuous operators from a complete DF-space Y to a Fréchet lattice E with the Lebesgue property and the coincidence of this space with some ideal of compact operators. We give sufficient conditions on the measure ν and the space X that imply that $L^1(\nu, X)$ has the Dunford-Pettis property. Applications of these results to Fréchet AL-spaces and Köthe sequence spaces are also given.

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1. Introduction

In this paper we study operators with values in, or defined on, spaces of scalar-valued integrable functions with respect to a vector measure with values in a real Fréchet space.

This kind of integration was introduced by Lewis in [19] and developed essentially by Kluvánek and Knowles in [18] for locally convex spaces. Let us recall briefly the basic definitions (see [19] and [18]).

Throughout the paper X will be a real Fréchet space. Denote by $\mathscr{U}_0(X)$ the system of all 0-neighborhoods in X. Given $U \in \mathscr{U}_0(X)$ we denote by p_U the associated

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Minkowski functional, that is,

$$p_U(x) = \sup\{|\langle x', x\rangle|,\ x' \in U^\circ\}, \quad x \in X,$$

where U° denotes the polar set of U. Consider a countably additive measure $\nu : \Sigma \to X$ defined on a σ-algebra Σ of subsets of a non-empty set Ω. For every 0-neighborhood U in X the U-semivariation of ν is the set function $\|\nu\|_U : \Sigma \to [0, \infty)$ defined by

$$\|\nu\|_U(A) := \sup\{|x'\nu|(A),\ x' \in U^\circ\},$$

where $|x'\nu|$ is the variation measure of the signed measure $x'\nu(A) = \langle x', \nu(A)\rangle$, $A \in \Sigma$, $x' \in X'$. Observe that for all $A \in \Sigma$

$$\sup\{p_U(\nu(B)),\ B \in \Sigma_A\} \le \|\nu\|_U(A) \le 2\sup\{p_U(\nu(B)),\ B \in \Sigma_A\}, \tag{1.1}$$

where $\Sigma_A := \{B \in \Sigma,\ B \subseteq A\}$; see [18, Lemma II.2].

Let $L^1(\nu, X)$ be the space of (classes of ν-almost everywhere equal) scalar-valued integrable functions with respect to ν. A real-valued, Σ-measurable function f on Ω is called *ν-integrable* (see [18, 19]) if $f \in L^1(|x'\nu|)$, for all $x' \in X'$, and if for each $A \in \Sigma$ there is a vector $\int_A f\,d\nu \in X$ (necessarily unique) satisfying $\langle x', \int_A f\,d\nu\rangle = \int_A f\,d(x'\nu)$ for all $x' \in X'$. We identify two functions f and g if they are equal ν-almost everywhere, that is, if

$$\|\nu\|_U(\{\omega \in \Omega : f(\omega) \neq g(\omega)\}) = 0,$$

for all $U \in \mathscr{U}_0(X)$. The space $L^1(\nu, X)$ is a Fréchet lattice with the Lebesgue property when it is equipped with the topology of convergence in mean and the order $f \le g$ if and only if $f(\omega) \le g(\omega)$, ν-almost everywhere; see [19, Theorem 2.2] or [18, Corollary II.4.2]. Recall that a locally solid Riesz space (L, τ) is said to have the Lebesgue property (or that τ is a Lebesgue topology) if $u_\alpha \downarrow 0$ in L implies $u_\alpha \xrightarrow{\tau} 0$. The characteristic function χ_Ω is a weak order unit of the Fréchet lattice $L^1(\nu, X)$, since $\inf\{f, \chi_\Omega\} = 0$ implies $f = 0$. Moreover, a system of lattice seminorms for this topology is given by

$$\|f\|_U := \sup\left\{\int_\Omega |f|\,d|x'\nu|,\ x' \in U^\circ\right\}, \quad f \in L^1(\nu, X),\ U \in \mathscr{U}_0(X).$$

The associated integration map I_ν given by $I_\nu(f) := \int_\Omega f\,d\nu$ is linear and continuous from $L^1(\nu, X)$ into X.

In Section 2 we characterize the L-weakly compact sets of $L^1(\nu, X)$ via equi-integrability (Theorem 2.2). L-weak compactness of the range of a positive vector measure with values in a Fréchet lattice is also proved (Theorem 2.4).

In Section 3 we consider the problem of relating the existence of copies of ℓ^∞ in the Fréchet space $L_b(Y, E)$, consisting of all linear continuous operators from a complete DF-space Y to a Fréchet lattice E (with the Lebesgue property) and having the topology of uniform convergence on the bounded sets of Y, with the coincidence of $L_b(Y, E)$ to a certain ideal of compact operators. Our results (Theorems 3.2 and 3.3) extend to the locally convex setting those of Curbera in [9]. Similar problems have been considered in [4, 5] and [6].

In Section 4 we study sufficient conditions on the measure ν and the space X in order that the space $L^1(\nu, X)$ has the Dunford-Pettis property (Theorem 4.1 and Corollary 4.2). Some applications to generalized Fréchet AL-spaces (Corollary 4.3) and Köthe spaces are also given; see Section 4 for the definition of these concepts.

Our notation and terminology is standard. For details concerning the lattice properties we refer the reader to [20, 21] and [23] and for the topological concepts in Riesz spaces to [1] and [2]. Aspects related to locally convex spaces can be seen in [15]. For the general theory of vector measures and integration we refer to the monographs [11] and [18].

2. L-weakly compact sets in $L^1(\nu, X)$

In this section we obtain a characterization of L-weakly compact sets in the space $L^1(\nu, X)$, where X is a Fréchet space and $\nu : \Sigma \to X$ is a countably additive measure defined on a σ-algebra Σ of subsets of a non-empty set Ω. Recall (see [21, Definition 3.6.1]) that a (non-empty) subset A of a Fréchet lattice E is said to be L-weakly compact if $x_n \to 0$ in the topology of E for every disjoint sequence $(x_n)_n$ contained in the solid hull $S(A)$ of A, where $S(A) := \{v \in E, |v| \leq |u| \text{ for some } u \in A\}$.

By using the disjoint sequence theorem of Aliprantis and Burkinshaw [1, Theorem 21.7] we can prove the following result; see also [21, Proposition 3.6.2].

THEOREM 2.1. *Let E be a Fréchet lattice and K be a bounded subset of E. Then the following assertions are equivalent.*

(1) *K is L-weakly compact.*

(2) *$x'_n(x) \to 0$ uniformly on K for every (positive) disjoint equi-continuous sequence $(x'_n)_n$ in E'.*

Moreover, if E has the Lebesgue property, then the above conditions are equivalent to

(3) *K is almost order bounded, that is, for every solid set $U \in \mathcal{U}_0(E)$ there exists $x \in E^+$ such that $K \subset [-x, x] + U$.*

REMARK 2.1. In the course of this paper we will need the relationship between the concept of an L-weakly compact set and other notions of compactness. The position of the class of solid, bounded, L-weakly compact sets in a Fréchet lattice with the Lebesgue property, among other classes of compact sets, is given in the following items.

(1) L-weakly compact sets are relatively weakly compact, [1, Theorem 21.8]. The converse holds for generalized AL-spaces E. Indeed, E is $|\sigma|(E, E')$-complete, [14, Theorem 1] and so $E = (E')_n^\sim$; see [1, Theorem 22.2]. Since E' is Dedekind complete [1, Theorem 5.7], it follows from Corollary 20.12 of [1], applied to $L = E'$, that if a subset A of E is relatively weakly compact, then its convex solid hull is also relatively weakly compact. So, if $(x_n)_n \subseteq S(A)$ is pairwise disjoint, then also $(|x_n|)_n \subseteq S(A)$ is pairwise disjoint. By [1, Theorem 21.2] we see that $\langle |x'|, |x_n| \rangle \to 0$ for each $x' \in E'$, that is, $x_n \to 0$ with respect to the absolute weak topology $|\sigma|(E, E')$ and hence, $x_n \to 0$ in E by [14, Theorem 1]. Hence, A is L-weakly compact.
(2) Solid, relatively compact sets are L-weakly compact, [1, Theorem 21.15]. The converse is true for discrete Fréchet lattices; see Theorems 21.12 and 21.15 of [1].

Suppose X is a Fréchet space and let $\nu : \Sigma \to X$ be a vector measure. A positive measure $\lambda : \Sigma \to [0, \infty)$ is said to be a control measure for ν if $\lambda(A) \to 0$, $A \in \Sigma$ if and only if $\|\nu\|_U(A) \to 0$, for every $U \in \mathscr{U}_0(X)$. Let us observe that a control measure for a Fréchet valued measure always exists; see [18, II.1. Corollary 2 of Theorem 1].

THEOREM 2.2. *Let X be a Fréchet space, $\nu : \Sigma \to X$ be a countably additive measure and K be a bounded subset of $L^1(\nu, X)$. Then the following assertions are equivalent.*

(1) *K is L-weakly compact in $L^1(\nu, X)$.*
(2) $\lim_n[\sup\{\|f\chi_{A_n}\|_U,\ f \in K\}] = 0$, *for every $U \in \mathscr{U}_0(X)$, and every sequence $(A_n)_n \downarrow \varnothing$ in Σ.*
(3) $\lim_{\lambda(A)\to 0}[\sup\{\|f\chi_A\|_U,\ f \in K\} = 0]$, *for every $U \in \mathscr{U}_0(X)$, and every control measure λ of ν.*
(4) $\lim_{\|\nu\|_U(A)\to 0}[\sup\{\|f\chi_A\|_U,\ f \in K\}] = 0$, *for every $U \in \mathscr{U}_0(X)$.*

PROOF. (1) $\Rightarrow$ (2) Take $U \in \mathscr{U}_0(X)$ and $\varepsilon > 0$. Consider the solid neighborhood of 0 in $L^1(\nu, X)$ given by

$$V_\varepsilon = \left\{ f \in L^1(\nu, X) : \|f\|_U \le \frac{\varepsilon}{2} \right\}.$$

Since K is L-weakly compact and $L^1(\nu, X)$ has the Lebesgue property, there exists $g_\varepsilon > 0$ in $L^1(\nu, X)$ such that $K \subset [-g_\varepsilon, g_\varepsilon] + V_\varepsilon$; see Theorem 2.1. Hence, for every

$f \in K$ we can write $f = u + v$, for some $v \in V_\varepsilon$ and u with $|u| \leq g_\varepsilon$. Now, if $(A_n)_n \downarrow \varnothing$ in Σ and $f \in K$, then

$$\|f\chi_{A_n}\|_U = \|u\chi_{A_n} + v\chi_{A_n}\|_U \leq \||u|\chi_{A_n}\|_U + \tfrac{1}{2}\varepsilon \leq \|g_\varepsilon\chi_{A_n}\|_U + \tfrac{1}{2}\varepsilon. \tag{2.1}$$

But, $g_\varepsilon\chi_{A_n} \to 0$ in $L^1(\nu, X)$, since $g_\varepsilon\chi_{A_n} \downarrow 0$ pointwise and $L^1(\nu, X)$ has the Lebesgue property, and so for some N_ε we have that $\|g_\varepsilon\chi_{A_n}\|_U \leq \varepsilon/2$ for all $n \geq N_\varepsilon$. It is then clear from (2.1) that (2) follows.

(2) $\Rightarrow$ (3) For every $f \in L^1(\nu, X)$ denote by $\nu_f : \Sigma \to X$ the measure $A \mapsto \int_A f\,d\nu$. Take $U \in \mathscr{U}_0(X)$, and consider the following family of countably additive signed measures $\mathscr{M} := \{x'\nu_f : f \in K,\ x' \in U^\circ\}$. This family is uniformly bounded with respect to the total variation norm, since K is bounded in $L^1(\nu, X)$. By hypothesis, it is uniformly countably additive. Observe also that every member of $\mathscr{M}$ is λ-continuous. By [11, Corollary I.2.5], $\mathscr{M}$ is uniformly λ-continuous, that is,

$$\lim_{\lambda(B)\to 0} \sup\{|x'\nu_f(B)|,\ x' \in U^\circ,\ f \in K\} = 0.$$

Hence

$$\lim_{\lambda(B)\to 0} \sup\{p_U(\nu_f(B)),\ f \in K\} = 0$$

and it follows that

$$\lim_{\lambda(A)\to 0} \sup\{\|\nu_f\|_U(A),\ x' \in U^\circ,\ f \in K\} = 0,$$

because $\|\nu_f\|_U(A) \leq 2\sup\{p_U(\nu_f(B)),\ B \in \Sigma_A\}$. The conclusion then follows from the fact that

$$\|\nu_f\|_U(A) = \|f\chi_A\|_U,$$

for all $A \in \Sigma$ and all $f \in K$ [19, Theorem 2.2].

(3) $\Rightarrow$ (4) If λ is any control measure for ν, then $\lim_{\|\nu\|_U(A)\to 0}\lambda(A) = 0$, for every $U \in \mathscr{U}_0(X)$ and so (4) follows from (3).

(4) $\Rightarrow$ (2) This is immediate since $(A_n)_n \downarrow \varnothing$ in Σ implies that $\|\nu\|_U(A_n) \to 0$, for all $U \in \mathscr{U}_0(X)$; see [18, II.1. Lemma 3].

(3) $\Rightarrow$ (1) Let $(f_n)_n$ be a disjoint sequence in the solid hull of K. By definition of $S(K)$ there exist $g_n \in K$ such that $|f_n| \leq |g_n|$, for all $n = 1, 2, \ldots$. Consider the disjoint measurable sets $A_n := \{\omega \in \Omega : |f_n(\omega)| > 0\}$, $n = 1, 2, \ldots$, and observe that $|f_n|\chi_{A_n} \leq |g_n|\chi_{A_n}$, for all $n = 1, 2, \ldots$. Let $U \in \mathscr{U}_0(X)$ and $\varepsilon > 0$. By the hypothesis (3) there exists $\delta > 0$ such that

$$\sup\{\|g\chi_A\|_U,\ g \in K\} \leq \varepsilon,$$

for all $A \in \Sigma$ with $\lambda(A) < \delta$. To finish the proof observe that

$$\|f_n\|_U = \||f_n|\chi_{A_n}\|_U \leq \||g_n|\chi_{A_n}\|_U \leq \sup\{\|g\chi_{A_n}\|_U,\ g \in K\} \leq \varepsilon,$$

for n large enough, because $\lim_n \lambda(A_n) = 0$. □

For X a Banach space the following result can be found in [10, Claim 1, p. 3803].

COROLLARY 2.3. *Let X be a Fréchet space with the Schur property and let $\nu : \Sigma \to X$ be a countably additive measure. Then in $L^1(\nu, X)$ relatively weakly compact sets coincide with L-weakly compact sets.*

PROOF. As we have already pointed out, every L-weakly compact set is relatively weakly compact. Suppose that there exists a set K in $L^1(\nu, X)$ which is relatively weakly compact but is not L-weakly compact. By the condition (2) of Theorem 2.2, there exist $U \in \mathscr{U}_0(X)$, a sequence $(A_n)_n \downarrow \varnothing$ in Σ, and a sequence $(f_n)_n \subset K$ such that

$$\|f_n\chi_{A_n}\|_U \geq \delta, \tag{2.2}$$

for some $\delta > 0$ and all $n = 1, 2, \ldots$. Since K is relatively weakly compact, by [15, Corollary 9.8.3] there exists a subsequence, that we still denote by $(f_n)_n$, which converges weakly in $L^1(\nu, X)$. Since $\|f\chi_A\|_U \leq \|f\|_U$, for all $U \in \mathscr{U}_0(X)$ and $A \in \Sigma$, the linear map $\Phi_A : f \mapsto f\chi_A$ is continuous from $L^1(\nu, X)$ into $L^1(\nu, X)$ for each $A \in \Sigma$. Hence, the composition map $I_\nu \circ \Phi_A : f \mapsto \int_A f\,d\nu$ is continuous from $L^1(\nu, X)$ into X. In particular, $I_\nu \circ \Phi_A$ is also continuous for the weak topologies on $L^1(\nu, X)$ and X. Accordingly, the sequence of integrals $(\int_A f_n d\nu)_n$ converges weakly in X for every $A \in \Sigma$. Since X is a Schur space, the convergence also holds in the topology of X. Let ν_n be the vector measure $A \mapsto \int_A f_n d\nu$, $A \in \Sigma$. These measures ν_n are countably additive and absolutely continuous with respect to any control measure λ of ν. Since $(\nu_n(A))_n$ converges in X for every $A \in \Sigma$, the Vitali-Hahn-Saks theorem implies that

$$\lim_{\lambda(A)\to 0} \sup_n p_V(\nu_n(A)) = 0,$$

for every $V \in \mathscr{U}_0(X)$. Then, we have

$$\lim_{\lambda(A)\to 0} \sup_n \|f_n\chi_{A_n}\|_V = 0,$$

for every $V \in \mathscr{U}_0(X)$. But this is a contradiction of (2.2). □

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We finish this section with an application of Theorem 2.2 to the properties displayed by the range of a vector-valued measure.

It is a classical result of Bartle, Dunford and Schwartz [3, Theorem 2.9] that the range of a vector measure with values in a Banach space is relatively weakly compact. This result was extended (in particular for Fréchet-valued measures) by Tweddle in [22] (see also [17, Theorem 2], [18, Theorem IV.6.1]). When the vector measure takes its values in a Fréchet lattice more can be said.

THEOREM 2.4. *Let E be a Fréchet lattice and let $\nu : \Sigma \to E$ be a positive countably additive measure. Then the solid hull of the range of ν is L-weakly compact.*

PROOF. By using the condition (2) of Theorem 2.1 it is enough to show that $x_n'(x) \to 0$ uniformly with respect to $x \in S(\nu(\Sigma))$, for every positive, disjoint and equi-continuous sequence $(x_n')_n$ in E'. Now, for all $n = 1, 2, \dots$ we have that

$$\sup\{|x_n'(x)|,\ x \in S(\nu(\Sigma))\} \leq \sup\{x_n'\nu(A),\ A \in \Sigma\} \leq x_n'\nu(\Omega) = x_n' I_\nu(\chi_\Omega).$$

Indeed, the second inequality follows from the positivity of x_n' (given) and I_ν (easily verified). To verify the first inequality, note that if $x \in S(\nu(\Sigma))$, then $|x| \leq |y|$ for some $y \in \nu(\Sigma)$. Since ν is a positive measure $y = \nu(A)$ for some $A \in \Sigma$ and so $|y| = y$, that is, $|x| \leq \nu(A)$ for some $A \in \Sigma$. Since $x_n' > 0$ we have $x_n'(|x|) \leq x_n'(\nu(A))$. But, $|x_n'| = x_n'$ and so, by [1, p. 21] we have $|x_n'(x)| \leq x_n'(|x|)$. Hence $|x_n'(x)| \leq x_n'(\nu(A))$ for some $A \in \Sigma$ whenever $x \in S(\nu(\Sigma))$, which establishes the first inequality. Since the order interval $[-\chi_\Omega, \chi_\Omega]$ is L-weakly compact in $L^1(\nu, E)$, we conclude the proof by showing that the equi-continuous sequence of positive functionals $(x_n' I_\nu)_n$ is disjoint. If $f > 0$ in $L^1(\nu, E)$, then $u := I_\nu(f) \geq 0$ in E. Therefore,

$$\begin{aligned}
\inf\{x_n' I_\nu, x_m' I_\nu\}(f) &= \inf\{x_n' I_\nu(g) + x_m' I_\nu(f - g),\ 0 \leq g \leq f\} \\
&\leq \inf\{x_n'(x) + x_m'(u - x),\ x \in E,\ 0 \leq x \leq u\} \\
&= \inf\{x_n', x_m'\}(u) = 0.
\end{aligned}$$

□

REMARK 2.2. It is a well known fact that the unit ball of ℓ^2 is the range of a vector measure [18, VII.4. Examples 1 and 2]. By considering the basis vectors $(e_n)_n$ it is clear that the unit ball of ℓ^2 is a solid set which is not L-weakly compact. This tells us that the statement of Theorem 2.4 is not true in general. Nevertheless, it still holds under a weaker hypothesis on the measure. A vector measure $\mu : \Sigma \to E$ is said to be dominated by a positive measure $\nu : \Sigma \to E$, if $|\mu(A)| \leq \nu(A)$, for all $A \in \Sigma$. In this case, the solid hull of the range of μ is obviously contained in the solid hull of

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the range of ν. Thus, the solid hull of the range of μ is L-weakly compact, since any subset of a L-weakly compact set is also L-weakly compact. Observe that any vector measure μ with a Jordan decomposition ($\mu = \mu_1 - \mu_2$, with μ_1 and μ_2 positive vector measures) or any measure μ with a so called Hahn decomposition (that is, there exists $A \in \Sigma$ such that $\mu(B) \geq 0$ if $B \subseteq A$ and $\mu(B) \leq 0$ if $B \subseteq \Omega \setminus A$) is dominated by a positive measure.

3. Operators with values in $L^1(\nu, X)$

Let Y be a complete DF-space (see [15, Section 12.4] for the definition) and let X be a Fréchet space. Recall that $L_b(Y, X)$ denotes the Fréchet space (see [15, 12.4 Theorem 2]) of all linear continuous operators from Y to X, equipped with the topology of uniform convergence on the bounded sets of Y. This topology is defined by the seminorms

$$p_{U,H}(T) := \sup\{p_U(Ty),\ y \in H\}, \qquad T \in L_b(Y, X),$$

where U is any 0-neighborhood in X and H is any bounded set in Y. In this section we extend to the locally convex case the results obtained by Curbera [9, Theorems 9 and 10] in the Banach case, about the existence of copies of ℓ^∞ in $L_b(Y, E)$ and its relationship to the coincidence of this space with some ideal of compact operators. Similar results have been proved in [5], [4] and [6] for pairs (Y, X), where X and Y are either Fréchet or complete DF-spaces. To do this, we associate to each continuous linear operator $T : Y \to L^1(\nu, X)$ a vector measure taking values in the space of all linear continuous operators from Y to X, and we characterize those operators whose associated measure is countably additive in the topology of uniform convergence on bounded sets of Y.

Consider the operator-valued set function $\nu_T : \Sigma \to L(Y, X)$, associated to the continuous linear operator $T : Y \to L^1(\nu, X)$ and the given vector measure $\nu : \Sigma \to X$, which is defined by

$$\nu_T(A) : y \mapsto \int_A Ty\, d\nu \in X \qquad (y \in Y),$$

that is, $\nu_T(A) = I_\nu \circ \Phi_A \circ T$ for each $A \in \Sigma$. It is then clear that ν_T is $L(Y, X)$-valued and finitely additive. Moreover, using (1.1) it can be shown that for every bounded set H of Y and each 0-neighborhood U of X, we have the following estimates for the $\|\cdot\|_{U,H}$-semivariation of ν_T;

$$\text{(3.1)} \qquad \tfrac{1}{2}\sup\{\|Ty \cdot \chi_A\|_U,\ y \in H\} \leq \|\nu_T\|_{U,H}(A) \leq 2\sup\{\|Ty \cdot \chi_A\|_U,\ y \in H\},$$

for all $A \in \Sigma$. Moreover, it is easy to show that ν_T is countably additive in $L_s(Y, X)$, the space $L(Y, X)$ equipped with the topology of pointwise convergence. In general, ν_T is not countably additive in $L_b(Y, X)$; see [9, Example p. 322].

The following result can be found in [9, Theorem 4] for the case when both X and Y are Banach spaces.

THEOREM 3.1. *Let Y be a complete* DF*-space, X be a Fréchet space, $\nu : \Sigma \to X$ be a countably additive measure and $T : Y \to L^1(\nu, X)$ be a continuous linear operator. The following conditions are equivalent.*

(1) *The operator T is L-weakly compact, that is, T maps bounded sets of Y into L-weakly compact sets of the Fréchet lattice $L^1(\nu, X)$.*

(2) *The measure ν_T is strongly additive in $L_b(Y, X)$, that is, $\nu_T(A_n) \to 0$ in $L_b(Y, X)$ whenever $(A_n)_n$ is a disjoint sequence in Σ.*

(3) *The measure ν_T is countably additive in $L_b(Y, X)$.*

PROOF. (1) $\Rightarrow$ (2) Suppose that ν_T is not strongly additive. By the Fréchet space version of [11, Corollary I.1.18] there exist a bounded set H in Y, a 0-neighborhood U in X, a pairwise disjoint sequence of measurable sets $(A_n)_n$ and an $\varepsilon > 0$ such that $\|\nu_T\|_{U,H}(A_n) \geq \varepsilon > 0$, for all $n = 1, 2, \ldots$. By using the bounds given for the semivariations of the measure ν_T in (3.1) we can choose a $y_n \in H$ such that $\|Ty_n \cdot \chi_{A_n}\|_U \geq \varepsilon/2$, for each $n = 1, 2, \ldots$. But this contradicts (1), since $(Ty_n \cdot \chi_{A_n})_n$ is then a disjoint sequence in the solid hull of $T(H)$ that does not converge to 0.

(2) $\Rightarrow$ (1) Suppose that T is not L-weakly compact. Then there exists a bounded set H in Y such that $T(H) \subset L^1(\nu, X)$ is not L-weakly compact. Then we can take a positive and disjoint sequence $(f_n)_n$ in $L^1(\nu, X)$ such that $f_n \leq |Ty_n|$ for certain $y_n \in H$ $(n = 1, 2, \ldots)$ but $(f_n)_n$ does not converge to 0. By passing to a subsequence, there exists $U \in \mathscr{U}_0(X)$ such that $\|f_n\|_U \geq 1$, for all $n = 1, 2, \ldots$. Consider the disjoint sequence $(A_n)_n$ of measurable sets $A_n := \{\omega \in \Omega : f_n(\omega) > 0\}$. Then (3.1) implies that

$$\begin{aligned} 1 &\leq \|f_n\|_U \leq \||Ty_n| \cdot \chi_{A_n}\|_U = \||Ty_n \cdot \chi_{A_n}|\|_U = \|Ty_n \cdot \chi_{A_n}\|_U \\ &\leq \sup\{\|Ty\chi_{A_n}\|_U,\ y \in H\} \leq 2\|\nu_T\|_{U,H}(A_n), \end{aligned}$$

for all $n = 1, 2, \ldots$. Once again, by [11, Corollary I.1.18], ν_T is not strongly additive.

(2) $\Rightarrow$ (3) Since the measure $\nu_T : \Sigma \to L_b(Y, X)$ is countably additive in a weaker Hausdorff topology (that is, in $L_s(Y, X)$) it is routine to check that the strong additivity of ν_T in $L_b(Y, X)$ implies its countable additivity.

(3) $\Rightarrow$ (2) This is obvious. □

For the case when Y is a Banach space and E is a Banach lattice with order continuous norm and a weak order unit, the following result can be found in [9, Theorem 9].

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THEOREM 3.2. *Let E be a Fréchet lattice with the Lebesgue property and with a weak order unit and let Y be a complete* DF*-space. If $L_b(Y, E)$ does not contain an isomorphic copy of ℓ^∞, then every continuous linear operator T from Y to E is L-weakly compact.*

PROOF. By the representation theorem [12, Proposition 2.4 (vi)] there exists a measurable space (Ω, Σ) and a countably additive vector measure $\nu : \Sigma \to E$ such that E is lattice isomorphic to $L^1(\nu, E)$. Thus, we can consider the operator T as mapping Y into $L^1(\nu, E)$. Then the associated measure ν_T takes its values in $L(Y, E)$ and has bounded range in $L_b(Y, E)$. By a theorem of Diestel and Faires [8, Corollary 4.1.44 and Theorem 4.7.16] the measure ν_T is strongly additive. Accordingly, the operator T is L-weakly compact by Theorem 3.1. □

Recall that a Fréchet lattice is said to be discrete if there exists a complete disjoint system of atoms. (See [1, p. 17 and Example 9, p. 31].) In this setting, we know that L-weakly compact sets are relatively compact; see Remark 2.1. The theorem to follow is an extension of part of [9, Theorem 10] (a similar result to [16, Theorem 6], without restrictions in the first space). For its proof we will need the following lemma which, in the Banach space case, is contained in the proof of [9, Theorem 10]. We include it for the sake of completeness.

LEMMA 3.1. *Let E be a Fréchet lattice with the Lebesgue property. Let $A(E)$ be a maximal disjoint system of positive atoms in E. For every $x \in E$ the set $A(x) := \{z \in A(E),\ \inf\{z, |x|\} \neq 0\}$ is countable.*

PROOF. Since E has the Lebesgue property it is Dedekind complete, [1, Theorem 10.3]. Hence the order projection P_z associated with the element $z \in E$ exists, [1, Theorem 2.11], and satisfies

$$(3.2) \qquad P_z(v) = \sup\{\inf\{v, n|z|\} : n \in \mathbb{N}\}, \qquad v \in E^+;$$

see [1, p. 13]. Fix any $x \in E$. For every $\varepsilon > 0$ and every continuous lattice seminorm q, the set $\{z \in A(E),\ q(P_z(|x|)) \geq \varepsilon\}$ is finite. If this is not the case, we can find an infinite sequence of atoms $(z_n)_n$ from $A(E)$ such that $q(P_{z_n}(|x|)) \geq \varepsilon$, for all $n = 1, 2, \ldots$. Consider the increasing sequence $u_k := P_{z_1+\cdots+z_k}(|x|)$, $k = 1, 2, \ldots$. This sequence is order bounded by $|x|$. Since E has the Lebesgue property, $(u_k)_k$ must be convergent. But, it follows from (3.2) that

$$q(u_k - u_{k-1}) = q(P_{z_k}(|x|))) \geq \varepsilon \quad \text{for all } k = 2, 3, \ldots$$

which is a contradiction.

Now, consider an increasing sequence $(q_n)_n$ of lattice seminorms generating the topology of E. Then it can be shown that

$$A(x) = \bigcup_{n,m=1}^{\infty} \left\{ z \in A(E), q_n(P_z(|x|)) \geq \frac{1}{m} \right\}$$

and this set is countable. □

THEOREM 3.3. *Let E be a discrete Fréchet lattice with the Lebesgue property and Y be a complete* DF-*space. If $L_b(Y, E)$ does not contain an isomorphic copy of ℓ^∞, then every continuous linear operator from Y to E is compact.*

PROOF. If Y is a DF-space and X a Fréchet space, then the compact operators from Y to X coincide with Montel operators from Y to X. (See the remark (1) after [5, Corollary 19].) Recall that a continuous linear map from Y to X is called Montel if it transforms bounded sets into relatively compact sets.

If E has a weak order unit, bearing in mind (by Remark 2.1 (2)) that L-weakly compact sets are relatively compact, it follows that every operator from Y to E is Montel, by the previous paragraph and Theorem 3.2. Now consider the general case (that is, no weak order unit) and suppose that there exists a continuous linear operator $T : Y \to E$ which is not Montel. Then there exist a bounded sequence $(x_n)_n$ in Y, and $U \in \mathscr{U}_0(E)$ such that

$$p_U(Tx_n - Tx_m) \geq 1, \text{ for all } n \neq m. \tag{3.3}$$

By Lemma 3.1, the set $H := \bigcup_{n\geq 1} A(Tx_n)$ is countable. The band F generated by H coincides with the subspace generated by H, since all of the elements of H are atoms. If we consider on F its relative topology, it is a discrete Fréchet lattice with the Lebesgue property. Moreover it has a weak order unit, since it is separable [1, Example 7 p. 123]. On the other hand, $Tx_n \in F$, for all $n = 1, 2, \ldots$ since $\inf\{|Tx_n|, z\} = 0$, for all $z \notin A(Tx_n), n = 1, 2, \ldots$. Denote by $P_F : E \to F$ the order projection band onto F. Then $P_F T \in L(Y, F)$ and is not compact by (3.3) as $p_U(P_F Tx_n - P_F Tx_m) = p_U(Tx_n - Tx_m)$. Moreover $L_b(Y, F)$ does not contain an isomorphic copy of ℓ^∞ as it is a closed subspace of $L_b(Y, E)$. But, the previous case shows that $P_F T$ is Montel and so we have a contradiction. □

REMARK 3.1. The converse of the above theorem is not true in general. (See the remark after the corollary below.) Nevertheless, it is true if the discrete Fréchet lattice E is non-Montel, in addition to having the Lebesgue property. Suppose that $L_b(Y, E) = M_b(Y, E)$ has a copy of ℓ^∞. According to [5, Corollary 19 (a)], Y contains a complemented copy of ℓ^1 or E contains a copy of ℓ^∞. The latter case is impossible, because E has the Lebesgue property [1, Theorem 10.7]. If we take a bounded sequence $(x_n)_n$ in E without convergent subsequences (which is possible as E is non-Montel), then the operator $T(\alpha) := \sum_n \alpha_n x_n$, for $\alpha \in \ell^1$, defines a non-Montel operator from ℓ^1 to E. It is then possible to construct a non-Montel operator from Y to E and a contradiction follows.

Since no infinite dimensional Banach space is Montel, we point out that Remark 3.1 and Theorem 3.3 together are an extension of (all of) [9, Theorem 10].

We complete this section with an application to Köthe spaces. Compare our result with [6, Theorem 3] and [5, Propositions 29 and 30]. Consider an index set I, not assumed to be countable. Recall that an increasing sequence $A = (a_k)_k$ of positive families $a_k = (a_{ki})_{i\in I}$ is called a Köthe matrix if for each $i \in I$ there exists a $k \geq 1$ such that $a_{ki} > 0$. For $1 \leq p < \infty$ we define

$$\lambda_p(I, A) := \left\{ x = (x_i)_{i\in I} : \|x\|_k^p := \left(\sum_{i\in I} |x_i|^p a_{ki} \right)^{1/p} < \infty, k = 1, 2, \ldots \right\}$$

equipped with the topology generated by the seminorms $\|\cdot\|_k^p$, $k = 1, 2, \ldots$. Then $\lambda_p(I, A)$, called a Köthe space, is a discrete Fréchet lattice with the Lebesgue property which has a weak order unit if and only if the index set I is countable. To check that $\lambda_p(I, A)$ has the Lebesgue property use the p-additivity property of each seminorm $\|\cdot\|_k^p$, $k = 1, 2, \ldots$ as in [1, Theorem 10.10].

COROLLARY 3.4. *Let $\lambda_p(I, A)$ be a Köthe space and Y be a complete* DF-*space. If $L_b(Y, \lambda_p(I, A))$ does not contain an isomorphic copy of ℓ^∞, then every continuous linear operator from Y to $\lambda_p(I, A)$ is compact* (*or Montel*).

REMARK 3.2. Equivalent conditions under which $L_b(Y, \lambda_p(I, A))$ has a copy of ℓ^∞, for a Fréchet or a complete DF-space Y and $2 \leq p < \infty$ have been studied in [4, Corollary 21].

4. Operators defined on $L^1(\nu, X)$

Let X be a Fréchet space and $\nu : \Sigma \to X$ be a vector measure of bounded variation. That is, for every 0-neighborhood U on X, we have

$$|\nu|_U(\Omega) := \sup_\pi \sum_{A\in\pi} p_U(\nu(A)) < \infty,$$

where the supremum is taken over all partitions π of Ω. For technical reasons we will require that

(4.1) $$D_\nu := \{x' \in X' : \nu \text{ is } |x'\nu|\text{-continuous}\} \neq \varnothing,$$

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where to say that ν is λ-continuous with respect to a positive measure λ on Σ means that every λ-null set is a ν-null set. Measures of the form $|x'\nu|$ with $x' \in D_\nu$ (when they exist) are called Rybakov control measures for ν. Conditions on the space X for which (4.1) holds have been studied in [13], where it is shown that if X admits a continuous norm, then every X-valued vector measure has a Rybakov control measure. In this section we study sufficient conditions on ν and X in order that the space $L^1(\nu, X)$ has the Dunford-Pettis property. Recall that a Fréchet space is said to have the Dunford-Pettis property if every weakly compact operator defined on it maps relatively weakly compact sets into relatively compact sets. We also apply these results to the class of Fréchet AL-spaces with a continuous norm.

We first establish some preparatory results concerning the representation of operators from $L^1(\nu, X)$ to Y. The next lemma is the vector version of [13, Lemma 3.1 (B)].

LEMMA 4.1. *Let X and Y be Fréchet spaces, $\nu : \Sigma \to X$ be a countably additive vector measure and $T : L^1(\nu, X) \to Y$ be a continuous linear map. Then the map $\mu_T : \Sigma \to Y$, given by $\mu_T(A) := T(\chi_A)$, defines a countably additive vector measure (henceforth called the representing measure for T), the inclusion $L^1(\nu, X) \subset L^1(\mu_T, Y)$ holds in the sense of vector spaces, and*

$$T(f) = \int_\Omega f \, d\mu_T \quad (f \in L^1(\nu, X)). \tag{4.2}$$

PROOF. The formula $\mu_T(A) := T(\chi_A)$ defines a finitely additive measure $\mu_T : \Sigma \to Y$. Actually, this measure is countably additive by the dominated convergence theorem for vector measures [18, Theorem II.4.2] and the continuity of T. To prove the inclusion $L^1(\nu, X) \subset L^1(\mu_T, Y)$ it suffices to show that every non-negative function $f \in L^1(\nu, X)$ belongs to $L^1(\mu_T, Y)$. To see this, choose a sequence of Σ-measurable simple functions $0 \le \varphi_1 \le \varphi_2 \le \cdots \le f$ which increases pointwise to f on Ω. By the dominated convergence theorem for vector measures we have $\chi_A\varphi_n \to \chi_A f$ in $L^1(\nu, X)$, for each $A \in \Sigma$. The continuity of T implies that $T(\chi_A\varphi_n) \to T(\chi_A f)$ in Y as $n \to \infty$. That is, the sequence $T(\chi_A\varphi_n) = \int_A \varphi_n d\mu_T$, $n = 1, 2, \ldots$ is convergent in Y for each $A \in \Sigma$. Now, since ν-null sets are μ_T-null sets, it follows from [19, Lemma 2.3] that $f \in L^1(\mu_T, Y)$. By the dominated convergence theorem for μ_T it is also follows (now knowing that $f \in L^1(\mu_T, Y)$) that $\int_\Omega \varphi_n d\mu_T \to \int_\Omega f \, d\mu_T$, and this establishes that $\int_\Omega f \, d\mu_T = T(f)$. ☐

For the notions of strongly λ-measurable and λ-integrable functions $f : \Omega \to X$ (with X a Fréchet space) with respect to a finite positive measure λ we refer to [7], for example. The function f is called λ-Pettis integrable if $\langle x', f\rangle \in L^1(\lambda)$, for all $x' \in X'$, and for each $A \in \Sigma$ there is a vector $\int_A f \, d\lambda \in X$ such that $\langle x', \int_A f \, d\lambda\rangle = \int_A \langle x', f\rangle d\lambda$

for all $x' \in X'$. For X and Y Banach spaces the following result can be found [10, p. 3804]; see the proof of Claim 2 given there.

LEMMA 4.2. *Let X and Y be Fréchet spaces, let $\nu : \Sigma \to X$ be a countably additive vector measure of bounded variation, and let $T : L^1(\nu, X) \to Y$ be a weakly compact operator. Then, for every control measure $\lambda : \Sigma \to [0, +\infty)$ of ν, the representing measure μ_T has a Radon-Nikodým derivative with respect to λ, that is, there exists a strongly λ-measurable function $g : \Omega \to Y$ which is λ-integrable such that*

$$\mu_T(A) = \int_A g d\lambda, \quad \textit{for all } A \in \Sigma.$$

Moreover, for every function $f \in L^1(\nu, X)$, the function fg is strongly λ-measurable and λ-Pettis integrable and its Pettis integral $\int_\Omega fg d\lambda$ satisfies

$$T(f) = \int_\Omega fg d\lambda. \tag{4.3}$$

PROOF. By using the continuity of T and the fact that ν has bounded variation we can see that the measure μ_T has bounded variation. Moreover, μ_T is λ-continuous and has locally relatively weakly compact (hence, s-dentable by [7, Theorem 1.1]) average range, meaning that for every $A \in \Sigma^+$ there exists $B \in \Sigma^+$, $B \subseteq A$, such that

$$\mathscr{R}_B(\mu_T) := \left\{ \frac{\mu_T(C)}{\lambda(C)}, \ C \in \Sigma^+, \ \text{and } C \subseteq B \right\}$$

is relatively weakly compact, where $\Sigma^+ = \{A \in \Sigma, \ \lambda(A) > 0\}$. To see this, observe that $\mu_T(C)/\lambda(C) = T(\chi_C/\lambda(C))$ for all $C \in \Sigma^+$, so that $\mathscr{R}_B(\mu_T) = T(\mathscr{R}_B(\mu))$, for all $B \in \Sigma^+$, where $\mu : \Sigma \to L^1(\nu, X)$ is the countably additive λ-continuous vector measure of bounded variation given by $\mu(A) := \chi_A$. With this observation, and recalling that T is weakly compact, it is enough to show that μ has locally bounded average range in $L^1(\nu, X)$. But, this follows from [7, Lemma 3.1]. Now, by [7, Theorem 2.1] there exists a strongly λ-measurable and λ-integrable function $g : \Omega \to Y$ (called Bochner integrable in the Banach space case) such that

$$\mu_T(A) = \int_A g d\lambda, \qquad A \in \Sigma.$$

Moreover,

$$\langle y', \mu_T(A) \rangle = \int_A \langle y', g \rangle d\lambda, \qquad A \in \Sigma, \ y' \in Y'. \tag{4.4}$$

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Now, from Lemma 4.1 and (4.4), we can prove that $\langle y', fg\rangle \in L^1(\lambda)$ for all $f \in L^1(\nu, X)$ and all $y' \in Y$. Moreover,

$$\left\langle y', \int_A f d\mu_T \right\rangle = \int_A f d(y'\mu_T) = \int_A f\langle y', g\rangle d\lambda = \int_A \langle y', fg\rangle d\lambda.$$

Hence $fg : \Omega \to Y$ is λ-Pettis integrable and its Pettis integral is $\int_A fgd\lambda = \int_A fd\mu_T$, for all $A \in \Sigma$. This together with (4.2) gives (4.3).

Finally, to see that fg is strongly λ-measurable note that g takes its values in a separable subspace of Y and hence, so does fg. Clearly $\langle y', fg\rangle$ is Σ-measurable for each $y' \in Y'$ and so $fg : \Omega \to Y$ is scalarly measurable. Then the Pettis measurability theorem (which is also valid in Fréchet spaces) implies that fg is strongly λ-measurable. □

For Banach spaces the following result occurs in [10, Claim 2, p. 3804].

THEOREM 4.1. *Let X and Y be Fréchet spaces and let $\nu : \Sigma \to X$ be a vector measure of bounded variation for which $D_\nu \neq \varnothing$. If $T : L^1(\nu, X) \to Y$ is a weakly compact operator, then T maps L-weakly compact sets into relatively compact sets.*

PROOF. Let λ be a Rybakov control measure for ν. By Lemma 4.2, there is a strongly λ-measurable and λ-integrable function $g : \Omega \to Y$, such that

$$T(f) = \int_\Omega fgd\lambda, \quad f \in L^1(\nu, X).$$

Let K be an L-weakly compact (solid) set in $L^1(\nu, X)$. To see that $T(K)$ is a relatively compact subset of Y it is enough to show (see [2, Theorem 9.1] and [15, Theorem 3.5.1]) that for every $V \in \mathcal{U}_0(Y)$ there exists a relatively compact set $K_V \subset Y$ such that $T(K) \subset K_V + V$. So, fix a 0-neighborhood V in Y.

First of all, observe that $\{\int_\Omega |f| d\lambda,\ f \in K\}$ is a bounded set, since λ is a Rybakov control measure for ν and K is an L-weakly compact set (hence bounded by Remark 2.1(1)) in $L^1(\nu, X)$. Then we can put $\rho := \sup\{\int_\Omega |f| d\lambda,\ f \in K\}$.

From the continuity of T there exists a 0-neighborhood U in X such that $p_V(Tf) \le \|f\|_U$, for all $f \in L^1(\nu, X)$.

By Theorem 2.2, we have that $\lim_{\lambda(A)\to 0} \sup\{\|f\chi_A\|_U,\ f \in K\} = 0$ and so there exists $\delta > 0$ such that $\sup\{\|f\chi_A\|_U,\ f \in K\} < \frac{1}{2}$, for all $A \in \Sigma$ with $\lambda(A) < \delta$.

Since the function g is strongly λ-measurable, by Egoroff's theorem there exist a Σ-simple function $\varphi : \Omega \to Y$ and a set $B \in \Sigma$ with $\lambda(B) < \delta$ such that

$$p_V(g(\omega) - \varphi(\omega)) \le \frac{1}{2\rho}, \qquad \omega \in \Omega \setminus B. \tag{4.5}$$

For f in K we have

$$T(f) = T(f\chi_B) + \int_{\Omega\setminus B} f\varphi d\lambda + \int_{\Omega\setminus B} f(g-\varphi)d\lambda.$$

Then

$$\begin{aligned} p_V\left(T(f) - \int_{\Omega\setminus B} f\varphi d\lambda\right) &\leq p_V(T(f\chi_B)) + p_V\left(\int_{\Omega\setminus B} f(g-\varphi)d\lambda\right) \\ &\leq \|f\chi_B\|_U + \int_{\Omega\setminus B} |f| p_V(g-\varphi)d\lambda \\ &\leq \frac{1}{2} + \rho\frac{1}{2\rho} = 1. \end{aligned}$$

Hence $T(K) \subset \{\int_{\Omega\setminus B} f\varphi d\lambda,\ f \in K\} + V$. Finally, the set

$$K_V := \left\{\int_{\Omega\setminus B} f\varphi d\lambda,\ f \in K\right\}$$

is relatively compact in Y because φ is a Σ-simple function and K is bounded. □

For the case when X is a Banach space the following result occurs in [10, Theorem 4].

COROLLARY 4.2. *Let X be a Fréchet space with the Schur property and let $\nu : \Sigma \to X$ be a vector measure of bounded variation for which $D_\nu \neq \varnothing$. Then the space $L^1(\nu, X)$ has the Dunford-Pettis property.*

PROOF. The result follows from Corollary 2.3 and Theorem 4.1. □

We conclude this section by showing that an important class of Fréchet lattices, the so called generalized AL-spaces, have the Dunford-Pettis property. Such spaces have been studied intensively in [14, Section 2]. A Fréchet lattice E is a generalized AL-space if its topology can be defined by a family of lattice seminorms p that are additive on the positive cone, that is, with $p(x+y) = p(x) + p(y)$, for $x, y \in E^+$.

We recall at this point (see Section 3) that a Köthe space $\lambda_1(I, A)$ is a generalized AL-space with a weak order unit if and only if the index set I is countable. Moreover it has a continuous norm if one of the steps, say $a_k = (a_{ki})_{i\in I}$, is strictly positive.

COROLLARY 4.3. *Let E be a generalized* AL-*Fréchet space with a weak order unit and a continuous norm. Then E has the Dunford-Pettis property.*

PROOF. Since every AL-space has the Lebesgue property (see Section 3), according to the representation theorem [12, Proposition 2.4 (vi)], E is lattice isomorphic to $L^1(\nu, E)$, for a certain countably additive measure $\nu : \Sigma \to E$. Hence, every operator defined on E can be considered as being defined on $L^1(\nu, E)$. Moreover, an examination of the proof of [12, Proposition 2.4 (vi)] (see also p. 364 there) shows that $\nu(\Sigma) \subseteq E^+$. So, if p_U is a lattice seminorm for E which is additive on E^+, then it is routine to verify that $\sum_{A\in\pi} p_U(\nu(A)) = p_U(\nu(\Omega))$ for every partition π of Ω. Accordingly, $|\nu|_U(\Omega) \le p_U(\nu(\Omega)) < \infty$ which shows that ν has bounded variation. Since E has a continuous norm, $D_\nu \neq \varnothing$, that is, the measure ν has a Rybakov control measure [13, Theorem 2.2]. Note that the relatively weakly compact sets coincide with L-weakly compact sets in $L^1(\nu, E)$, provided that $L^1(\nu, E)$ is an AL-space; see Remark 2.1(1). The proof then follows by applying Theorem 4.1. So, it remains to establish that $L^1(\nu, E)$ is a generalized AL-space whenever E is a generalized AL-Fréchet space and $\nu : \Sigma \to E$ is a positive measure (necessarily having bounded variation). The following lemma establishes this fact. □

LEMMA 4.3. *Let E be a generalized* AL-*Fréchet space and $\nu : \Sigma \to E$ be a positive measure. Then $L^1(\nu, E)$ is a generalized* AL-*space.*

PROOF. Denote by $\mathscr{U}_0(E)$ the system of all solid 0-neighborhoods in E. Since E is a generalized AL-Fréchet space its topology is $|\sigma|(E, E')$; see [14, Theorem 1]. Now, by applying [2, Theorem 11.11(1)] we have that for every $U \in \mathscr{U}_0(E)$ there exists $x'_U > 0$ in E' such that $U^\circ \subseteq [-x'_U, x'_U]$, that is, $|x'| \le x'_U$, for all $x' \in U^\circ$. Observe that

$$|x'\nu|(A) \le x'_U \nu(A), \qquad x' \in U^\circ,\ A \in \Sigma \tag{4.6}$$

Consider the following system of lattice seminorms ($x'_U\nu$ is a positive measure) on $L^1(\nu, E)$:

$$|f|_U := \int_\Omega |f| d(x'_U \nu), \qquad f \in L^1(\nu, E),$$

where $U \in \mathscr{U}_0(E)$. Now, it is clear from (4.6) that this system of seminorms define the topology of $L^1(\nu, E)$. Moreover, $|f+g|_U = |f|_U + |g|_U$, for all positive functions $f, g \in L^1(\nu, E)$ and every $U \in \mathscr{U}_0(E)$. This shows that $L^1(\nu, E)$ is a generalized AL-space. □

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