SAMPLE PROPERTIES OF WEAKLY STATIONARY PROCESSES

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1. Introduction. Let $X(t) = X(t, \omega)$, $-\infty < t < \infty$, be a stationary stochastic process with

\[(1.1)\]

\[EX(t) = 0, \quad E|X(t)|^2 < \infty, \quad -\infty < t < \infty\]

and the continuous covariance function

\[(1.2)\]

\[\rho(u) = \int_\infty^{-\infty} e^{ixu} dF(x),\]

where $F(x)$ is the spectral distribution function. $X(t)$ then admits the harmonic representation

\[(1.3)\]

\[X(t) = \int_\infty^{-\infty} e^{it\lambda} d\xi(\lambda),\]

where $\xi(\lambda)$ is a stochastic process with orthogonal increments and the property that

\[(1.4)\]

\[Ed\xi(\lambda) = 0, \quad E|d\xi(\lambda)|^2 = dF(\lambda).\]

Two stochastic processes $X(t)$ and $X_1(t)$ are said to be equivalent to each other, if

\[P(X(t) = X_1(t)) = 1, \quad \text{for each } t.\]

When $X(t)$ is equivalent to a process continuous almost surely or differentiable almost surely, $X(t)$ is called sample continuous or sample differentiable respectively.

One of the authors has shown the following theorem [3].

THEOREM A. Suppose that for a given weakly stationary process $X(t)$ there is a function $g(x)$ which is even, non-negative and non-decreasing for $x > 0$ and is such that

\[(1.5)\]

\[\sum_{n=1}^{\infty} \frac{1}{g(n)} < \infty,\]

Received May 28, 1969. This work partly supported by NSF Grant 9396.
(1.6) \[ \int_{-\infty}^{\infty} g(x) dF(x) < \infty. \]

Then \(X(t)\) is sample continuous.

The condition (1.6) with \(g(x) = |x|(|\log^+ |x|)^{\beta}\), \(\beta > 1\), implies the condition

(1.7) \[ \varphi(h) = O(h/|\log |h||^r) \] for \(r > 2\),
as \(h \to 0\), where \(\varphi(h) = 2\rho(0) - \rho(h) - \rho(-h)\) ([3] (3.8) and (3.9)). This generalizes the Cramér-Leadbetter's result on sample continuity of a weakly stationary process ([1], p. 125).

In 2, we shall give the conditions which assure the sample differentiability of a process. We can adopt the method for the proof similar to what we did proving Theorem A, namely we make use of the approximate Fourier series [3] [6] associated with a given weakly stationary process. In 3, we shall show that the same reasoning still applies to get the "sample Hölder property".

In the paper of one of the authors [2], Theorem A was motivated by a theorem on the absolute convergence of the Fourier series of a given process truncated at \(-T\) and \(T\). But it involved some erroneous argument although the theorem itself is right, and the different method using the approximate Fourier series was employed to prove Theorem A in [3]. In 4, it is shown that the original way of proving is effective if some modifications are made with a slight additional condition on \(g(x)\).

Finally we mention that the conditions on the existence of \(g(x)\) in Theorem A are also necessary for all the weakly stationary processes with a given spectral distribution \(F(x)\) to be sample continuous. This has been shown by I. Kubo [4] and will be given in a separate forthcoming paper.

2. Sample differentiability of a weakly stationary process.

M. Loève [5] studied the sample differentiability of a weakly stationary process and proved among others the following theorem.

**Theorem B.** If the covariance function \(\rho(u)\) of a weakly stationary process \(X(t)\) with (1.1) and (1.2), is \((2n + 2)\)-times differentiable, then \(X(t)\) is sample \(n\)-times differentiable.

Cramér and Leadbetter [1] generalized this result to obtain Theorem C below.
Write

\[ \Delta_u^k \rho(-ku) = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} \rho((k-j)u), \]

where \( k \) is a non-negative integer.

**Theorem C.** If the covariance function \( \rho(u) \) of a weakly stationary process \( X(t) \) with (1.1) and (1.2) satisfies

\[ \Delta_u^{2n+1} \rho(-ku) = O(|u|^{2n+1}/|\log|u||^q), \quad \text{as } u \to 0 \quad \text{for } q > 3, \]

then \( X(t) \) is sample \( n \)-times differentiable.

This is a slight completion of the Cramér-Leadbetter's result. They actually have shown Theorem C for the case \( n = 0, 1 \).

The aim of this section is to generalize Theorem C further.

In association with a given weakly stationary process \( X(t) \) with (1.1), (1.2) and the representation (1.3), we define a sequence of uncorrelated random variables

\[ \xi_n = \xi_n(T) = \int \frac{2n+1}{2n} d\xi(\lambda), \quad n = 0, \pm 1, \ldots, \]

where \( T \) is any positive number. We also define

\[ \tilde{X}(t, T) = \tilde{X}(t) = \sum_{n=-\infty}^{\infty} e^{2nit/T} \xi_n. \]

Actually \( \xi_n \)'s are uncorrelated because of the orthogonality of the increments of \( \xi(\lambda) \) and (2.4) is well-defined, the series being interpreted to converge in \( L^2 \)-norm.

However, we have shown in [3] and [4] that under the conditions either in Theorem A or in Lemma 3 below, the series in (2.4) is absolutely convergent almost surely and hence \( X(t) \) may be identified to be the sum of the series. Also it was shown that in this case \( \tilde{X}_k(t) = \tilde{X}(t, 2^k) \) converges uniformly for every finite interval \( |t| \leq A \) as \( k \to \infty \) almost surely to a weakly stationary process \( \tilde{X}_0(t) \), which is sample continuous, and is equivalent to \( X(t) \).

**Lemma 1.** If

\[ \sum_{n=-\infty}^{\infty} |n|^r |\xi_n| < \infty \]
almost surely, where $r$ is a positive integer, then $X(t)$ is equivalent to a weakly stationary process with the almost sure continuous $r$-th derivative.

**Proof.** Since the series on the right of (2.4) is absolutely and uniformly convergent almost surely, because of (2.5), we may suppose that $X(t)$ itself is represented by the series in (2.4) for every $t$ almost surely, and has the continuous $r$-th derivative almost surely. We shall, however, prove Lemma 1 when $r = 1$. $r$ repetitions of the same argument give us the required.

(2.6) \[
\frac{\hat{X}(t + h) - \hat{X}(t)}{h} = \sum_{n=\infty}^{\infty} e^{2\pi i n^2 T^2} e^{2\pi i n^2 T^2} e^{2\pi i n^2 T^2}.
\]

The series on the right is dominated in absolute value by $(2\pi/T)^{\sum |n|} \xi_n$ almost surely and since each term converges as $h \to 0$, the limit of (2.6) as $h \to 0$ should exist and $\hat{X}'(t)$ is given by $(2\pi/T)^{\sum 2\pi i n^2 T^2} e^{2\pi i n^2 T^2} \xi_n$, which is continuous almost surely. Generally $\hat{X}^{(r)}(t)$ is given by $(2\pi T)^{\sum 2\pi i n^2 T^2} e^{2\pi i n^2 T^2} \xi_n$.

**Lemma 2.** Let $h(x)$ be non-negative and non-decreasing over $[0, \infty)$ and let $F(x)$ be a spectral distribution. Then the inequalities

(2.7) \[
\frac{1}{2} \sum_{n=0}^{\infty} h\left(\frac{|n| - 1}{a}\right) (F(n + 1) - F(n))^{1/2} + \frac{1}{2} h(0)(F(1) - F(0))^{1/2} \leq \sum_{n} h(|n|) (F(a(n + 1)) - F(an))^{1/2} \leq \left(\frac{1}{a} + 1\right)^{1/2} \sum_{n} h\left(\frac{|n| + 1}{a}\right) (F(n + 1) - F(n))^{1/2}
\]

hold for $0 < a < 1$.

**Proof.** Since $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$, we have

$I = \sum_{n} h(|n|) (F(a(n + 1)) - F(an))^{1/2}$

$= \sum_{n} h\left(\left[\frac{k}{a}\right]\right) (F\left(a\left[\frac{k}{a}\right] + a\right) - F\left(a\left[\frac{k}{a}\right]\right))^{1/2} + \sum_{n=\left[\frac{k+1}{a}\right]+1}^{\infty} \sum_{n=\left[\frac{k}{a}\right]+1}^{\infty} h(|n|) (F(a(n + 1)) - F(an))^{1/2}$

$\leq \sum_{n} h\left(\left[\frac{k}{a}\right]\right) (F\left(a\left[\frac{k}{a}\right] + a\right) - F\left(a\left[\frac{k}{a}\right]\right))^{1/2} + \left(\left[\frac{k+1}{a}\right]\right) (F\left(a\left[\frac{k}{a}\right] + a\right) - F(k))^{1/2} + $

$+ \sum_{n=\left[\frac{k+1}{a}\right]+1}^{\infty} h\left(\left[\frac{k+1}{a}\right]\right) (F(a(n + 1)) - F(an))^{1/2}$

$\leq \sum_{n} h\left(\left[\frac{k+1}{a}\right]\right) \left(\frac{1}{a} + 1\right)^{1/2} (F(k + 1) - F(k))^{1/2}$

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The last inequality is obtained by Schwarz inequality. Since $2\sqrt{x} + \sqrt{y} \geq \sqrt{x + y}$ for $x, y \geq 0$, we have similarly

$$
2I \geq \sum_{k} h\left(\frac{k+1}{a}\right)(F(k+1) - F\left(\frac{k+1}{a}\right))^\frac{1}{2} + h\left(\frac{k}{a}\right)(F\left(\frac{k}{a}\right) + a - F(k))^\frac{1}{2} +
$$

$$
+ \left[\sum_{n=\lfloor b/a\rfloor+1}^{\lfloor(b+1)/a\rfloor-1} h\left(-\frac{|k|}{a}\right)(F(a(n+1) - F(an))^\frac{1}{2}\right]
$$

$$
\geq \sum_{k} h\left(-\frac{|k|}{a}\right)(F(1) - F(k))^\frac{1}{2},
$$

with the agreement that $h(u) = h(0)$ for $u \leq 0$.

**Lemma 3.** If the spectral distribution function $F$ of a given stationary process $X(t)$ satisfies

$$
\sum_{n} |n|^r(F(n+1) - F(n))^{1/2} < \infty
$$

for a non-negative integer $r$, then (2.5) holds almost surely.

**Proof.** In order to show (2.5) it is sufficient to prove

$$
E \sum_{n=\infty} |n|^r |\xi_n| < \infty.
$$

By Lemma 2, we have that, for $0 < T \leq 2\pi$,

$$
E \sum |n|^r |\xi_n| \leq \sum |n|^r \left[ E \left| \int_{2\pi n/T}^{2\pi(n+1)/T} d\xi(\lambda) \right| ^2 \right]^{1/2} =
$$

$$
= \sum |n|^r \left( F\left(\frac{2(n+1)\pi}{T}\right) - F\left(\frac{2n\pi}{T}\right) \right)^{1/2} \leq
$$

$$
\left( \frac{T}{2\pi} + 1 \right)^{1/2} \sum \frac{T}{2\pi} |n|^{1} \left( F(n+1) - F(n) \right)^{1/2} < \infty.
$$

If $T \geq 2\pi$, then (2.9) follows from the first inequality of (2.7).

It is easy to show that $\dot{X}(r)(t)$ is a weakly stationary process, observing

$$
\sum |n|^{2r}(F(2(n+1)\pi/T) - F(2n\pi/T)) < \infty.
$$

Now we shall prove

**Theorem 1.** If a weakly stationary process $X(t)$ with (1.1) and (1.2) satisfies (2.8), then $X(t)$ is equivalent to a weakly stationary process which has the continuous $r$-th derivative almost surely.

**Proof.** First we prove the theorem for $r = 1$. Denote the differential quotients of $X(t)$ and $\dot{X}(t)$ by
respectively. From Lemma 1, the series in (2.4) is absolutely convergent
and \( \hat{X}(t) \) may be supposed to be defined by this series. By Lemma 1 and
Lemma 3, \( \hat{X}(t) \) has the continuous derivative almost surely.

Write \( \xi_{n,k} \) for \( \xi_n \) with \( T = 2^k \), \( \hat{X}_k(t) \) for the corresponding \( \hat{X}(t) \), \( k \) being
a positive integer. Then

\[
\hat{X}_{k+1}(t) - \hat{X}_k(t) = \sum_{n=-\infty}^{\infty} \exp\left( \frac{2\pi i t}{2^{k+1}} \right) \xi_{n,k+1} - \sum_{m=-\infty}^{\infty} \exp\left( \frac{2\pi i t}{2^k} \right) \xi_{m,k} =
\]

\[
= \sum_{m=-\infty}^{\infty} \left[ \exp\left( \frac{2\pi i (2m) t}{2^{k+1}} \right) \xi_{2m,k+1} + \right.
+ \exp\left( \frac{2\pi i (2m+1) t}{2^{k+1}} \right) \xi_{2m+1,k+1} - \exp\left( \frac{2\pi i t}{2^k} \right) \xi_{m,k} \right].
\]

Since \( \xi_{m,k} = \xi_{2m,k+1} + \xi_{2m+1,k+1} \), we may write

\[
(2.13) \quad \hat{X}_{k+1}(t) - \hat{X}_k(t) =
\]

\[
= \sum_{m=-\infty}^{\infty} \left[ \exp\left( \frac{2\pi i (2m+1) t}{2^{k+1}} \right) - \exp\left( \frac{2\pi i (2m) t}{2^{k+1}} \right) \right] \xi_{2m+1,k+1}.
\]

Write \( \hat{D}_k(t,h) \) for the differential quotient of \( \hat{X}_k(t) \).

Together with the relation for \( \hat{X}_k(t + h) \) similar to (2.13) and noting
that, for \( |t| \leq A, A \) being a positive number,

\[
|e^{iy(t+h)} - e^{iy t} - e^{iz(t+h)} + e^{iz t}| \leq
\]

\[
\leq |(e^{iy t} - e^{iz t}) (e^{iy h} - 1)| + |e^{iz t} (e^{iy h} - e^{iz h})| \leq
\]

\[
\leq 4 \left| \sin \frac{y - z}{2} t \sin \frac{y h}{2} + 2 \sin \frac{y - z}{2} h \right| \leq |h| \left| y - z \right| \cdot (1 + A |y|),
\]

we obtain

\[
(2.15) \quad |\hat{D}_{k+1}(t,h) - \hat{D}_k(t,h)| \leq
\]

\[
= \sum_{m=-\infty}^{\infty} \left[ \exp\left( \frac{\pi i (2m+1) (t + h)}{2^k} \right) - \exp\left( \frac{\pi i (2m) (t + h)}{2^k} \right) \right] -
\]

\[
- \exp\left( \frac{\pi i (2m+1) t}{2^k} \right) + \exp\left( \frac{\pi i (2m) t}{2^k} \right) \right] \xi_{2m+1,k+1} \leq
\]

\[
\leq \sum_{m=-\infty}^{\infty} \left( \frac{\pi}{2^k} + \frac{A \pi^2 |2m|}{2^k} \right) |\xi_{2m+1,k+1}|. \]
Therefore we can see by Lemma 2 that for any \( \varepsilon_k > 0 \)

\[
Q_k = \mathbb{P}(\sup_{|t| \leq A, h>0} |\hat{D}_{k+1}(t, h) - \hat{D}_k(t, h)| > \varepsilon_k) \leq \\
\leq \frac{1}{\varepsilon_k} \frac{\pi}{2^k} \sum_{m}(1 + \frac{2A|m|}{2^k}) E|\xi_{2m+1,k+1}| \leq \\
\leq \frac{1}{\varepsilon_k} \frac{\pi}{2^k} \sum_{m}(1 + \frac{2A|m|}{2^k}) (F(\frac{(2m+1)\pi}{2^k}) - F(\frac{(2m)\pi}{2^k}))^{1/2} \leq \\
\leq \frac{1}{\varepsilon_k} \frac{\pi}{2^k} \sum_{m}(1 + \frac{A|m|}{2^k}) (F(\frac{(n+1)\pi}{2^k}) - F(\frac{n\pi}{2^k}))^{1/2} \leq \\
\leq \frac{1}{\varepsilon_k} \frac{(2^k + \pi)^{1/2}}{2^k \pi^{1/2}} \sum_{n=1}(1 + A(|n| + 1)) (F(n + 1) - F(n))^{1/2} \leq \\
\leq \frac{1}{\varepsilon_k} C_1 \frac{C_2}{2^{k/2}} \sum_{n=1}(F(n + 1) - F(n))^{1/2} + C_3 \frac{C_2}{2^{k/2}} \sum_{n=1}(F(n + 1) - F(n))^{1/2},
\]

where \( C_1 \) and \( C_2 \) are constants independent of \( k \). In what follows \( C_j, j = 3, 4, \ldots \), mean some constants independent of \( k \). From (2.8), it follows that

\[
Q_k \leq \frac{1}{\varepsilon_k} \frac{C_3}{2^{k/2}}.
\]

If \( \varepsilon_k \) is chosen to be \( 2^{-k/4} \), then \( \sum \varepsilon_k < \infty \) and \( \sum \frac{1}{2^{k/2} \varepsilon_k} < \infty \), so that \( \sum Q_k < \infty \). Then Borel-Cantelli lemma gives us that, with probability one

\[
\sup_{h>0, |t| \leq A} |\hat{D}_{k+1}(t, h) - \hat{D}_k(t, h)| < \varepsilon_k
\]

except for a finite number of \( k \). Hence almost surely \( \hat{D}_k(t, h) \) converges as \( k \to \infty \) uniformly for \( |t| \leq A \) and \( h \).

Now from (2.18) we have, for \( k \) larger than some \( k_0 \),

\[
\sup_{|t| \leq A} |\hat{D}_k(t, h) - \hat{D}_m(t, h)| \leq \eta_k \quad \text{(uniformly in \( h \))}
\]

almost surely, where \( \eta_k = \sum_{j=k}^{\infty} \varepsilon_j \). From the italicized statement before Lemma 1, we have, letting \( m \to \infty \)

\[
\hat{D}_k(t, h) - \frac{1}{h} [X_\alpha(t + h) - X_\alpha(t)] \leq \eta_k \quad \text{(uniformly in \( h \))}
\]

almost surely. Let \( h \to 0 \). Then from Lemma 1 and 2 with \( r = 1 \), \( \hat{D}_k(t, h) \) converges almost surely and hence \( X_\alpha(t) \) is differentiable almost surely, and is equivalent to \( X(t) \).
Finally (2.19) implies that the derivative $X'_k(t)$ of $X_k(t)$ converges uniformly to the derivative $X'_0(t)$ of $X_0(t)$. Since Lemmas 1 and 2 give us that $X'_k(t)$ is continuous almost surely, $X'_0(t)$ is also sample continuous for every $|t| \leq A$. This proves the theorem for the case $r = 1$.

Repeating similar arguments, the general case is shown.

**Theorem 2.** If, for a given weakly stationary process $X(t)$ with (1.1) and (1.2), there is a function $g(x)$, $-\infty < x < \infty$, which is non-negative, even and non-decreasing for $x \geq 0$ and satisfies

(2.20) \[ \sum_{n=1}^{\infty} \frac{n^{2r}}{g(n)} < \infty, \]

(2.21) \[ \int g(x) \, dF(x) < \infty, \]

then $X(t)$ is equivalent to a weakly stationary process which has the continuous $r$-th derivative almost surely.

**Proof.** By Schwarz inequality,

\[
\left\{ \sum_{n=0}^{\infty} |n|^r(F(n+1) - F(n))^{1/2} \right\}^2 \leq \sum_{n=0}^{\infty} \frac{n^{2r}}{g(n)} \sum_{n=0}^{\infty} g(n)(F(n+1) - F(n))
\]

\[ \leq \sum_{n=0}^{\infty} \frac{n^{2r}}{g(n)} \int_{0}^{\infty} g(x) \, dF(x) < \infty. \]

Similarly, we have $\sum_{n=-\infty}^{\infty} |n|^r(F(n+1) - F(n))^{1/2} < \infty$. By Theorem 1, the proof is completed.

**Example 1.** If, for some $\varepsilon > 0$ and $B > 0$

(2.22) \[ \int_{|x|<B} |x|^{r+1} \log |x| \cdot \log_{(2)} |x| \cdot \ldots \cdot \log_{(n)} |x| \cdot (\log_{(n+1)} |x|)^{1+\varepsilon} dF(x) < \infty \]

holds, then $X(t)$ is sample $r$-times differentiable, where $\log_{(1)} x = \log x$ and $\log_{(n+1)} x = \log(\log_{(n)} x)$ for $n \geq 1$.

**Example 2.** Suppose that $F(x)$ is absolutely continuous with the density $f(x)$. If

(2.23) \[ |f(x)| \leq [(|x|^{r+1} \log |x| \cdot \log_{(2)} |x| \cdot \ldots \cdot \log_{(n)} |x| \cdot (\log_{(n+1)} |x|)^{1+\varepsilon}]^{-2}. \]

holds for sufficiently large $|x|$ with some $\varepsilon > 0$, then $X(t)$ is sample $r$-times differentiable.
EXAMPLE 3. Besides the same assumption in Example 2, further suppose that \( f(x) \) is non-decreasing as \( x \to \pm \infty \). If

\[
\int |x|^r f^{1/2}(x)dx < \infty
\]

holds, then \( X(t) \) is sample \( r \)-times differentiable.

3. Sample Hölder continuity.

Let \( \Psi(h) \) be a non-decreasing function defined over an interval \((0,1] \) such that \( \Psi(h) \) decreases to zero as \( h \) does. If a function \( f(x) \) on \((a,b)\) satisfies

\[
|f(t + h) - f(t)| \leq M \Psi(h)
\]

for \( t, t + h \in (a,b), |h| < 1 \) with some \( M \), then it is said to be \( \Psi \)-Hölder continuous.

We are going to give sufficient conditions which assure the sample \( \Psi \)-Hölder continuity of a weakly stationary process. The method similar to the one applied to the proofs of Theorems 1 and 2 is also applicable.

LEMMA 4. Let \( \Psi(h) \) be a non-decreasing function over \((0,1]\) such that \( \Psi(h)/h \) is non-increasing. Then for \( 0 < h \leq 1 \)

\[
|\sin xh| \leq \frac{\Psi(h)}{\Psi(x^{-1})}, \quad \text{for } x \geq 1,
\]

\[
|\sin xh| \leq \frac{1}{x + 1}, \quad \text{for } x \geq 0.
\]

Proof. If \( 0 < xh < 1 \), then

\[
|\sin xh| \leq xh = \frac{\Psi(h)}{\Psi(x^{-1})} \cdot \frac{\Psi(x^{-1})}{x^{-1}} \cdot \frac{h}{\Psi(h)} \leq \frac{\Psi(h)}{\Psi(x^{-1})}.
\]

If \( xh \geq 1 \), then since \( \Psi(h) \) is non-decreasing

\[
|\sin xh| \leq 1 \leq \frac{\Psi(h)}{\Psi(x^{-1})}.
\]

Similarly, we can prove (3.3) observing \( xh \leq h(x + 1) \).

LEMMA 5. If \( \Psi(h) \) is non-decreasing and \( \Psi(h)/h \) is non-increasing over \((0,1]\), and

\[
\sum_{n \neq 0} \Psi^{-1}\left(\frac{1}{|n|}\right) |\xi_n| < \infty
\]
almost surely, then $\hat{X}(t) = X(t, T)$, $T > \pi$, is sample $\Psi$-Hölder continuous where $\xi_n$ is defined by (2.3) and $\hat{X}(t)$ is defined by (2.4).*

Proof. Using Lemma 4, we have

\[ |X(t + h) - X(t)| \leq \sum_{n \geq 0} |\sin nh\pi/T| |\xi_n| \leq \Psi\left(\frac{h\pi}{T}\right) \sum_{n \geq 0} \Psi^{-1}\left(\frac{1}{|n|}\right) |\xi_n| \leq \Psi(|h|) \sum_{n \geq 0} \Psi^{-1}\left(\frac{1}{|n|}\right) |\xi_n|. \]

**Lemma 6.** If the spectral distribution function $F(x)$ satisfies

\[ \sum_{n \geq 0} \Psi^{-1}\left(\frac{1}{|n|}\right) (F(n + 1) - F(n))^{1/2} < \infty. \]

Then (3.4) holds almost surely, where $\Psi(h)$ is the function in Lemma 4. Hence $X(t)$ is $\Psi$-Hölder continuous almost surely.

The proof is carried out as in that of Lemma 3.

**Theorem 3.** If a given weakly stationary process $X(t)$ satisfies (3.6), then $X(t)$ is equivalent to a weakly stationary process which is $\Psi$-Hölder continuous almost surely, where $\Psi(h)$ is the function in Lemma 4.

The proof is very similar to that for Theorem 1. Write

\[ D_{\psi}(t, h) = \frac{X(t + h) - X(t)}{\Psi(|h|)} , \quad \hat{D}_{\psi}(t, h) = \frac{\hat{X}_k(t + h) - \hat{X}_k(t)}{\Psi(|h|)} , \]

where $\hat{X}_k(t)$ is, as before, defined by (2.4) with $T = 2^k$. Using the same notations as in the proof of Theorem 1, we have, analogously to (2.15), by Lemma 4 and (2.14),

\[ |\hat{D}_{\psi}(t, h) - \hat{D}_{\psi}(t, h)| \leq \frac{1}{\Psi(|h|)} \sum_{m = -\infty}^{\infty} \left( 4 \sin \frac{\pi t}{2^k + 1} \sin \frac{2\pi h}{2^k + 1} + 2 \left| \sin \frac{\pi h}{2^k + 1} \right| |\xi_{2m+1,k+1}| \right) \]

for $|t| \leq A$. Therefore we obtain by Lemma 2

\[ Q_k' \equiv P\left( \sup_{0 < |h| \leq 1, |t| \leq A} |\hat{D}_{\psi}(t, h) - \hat{D}_{\psi}(t, h)| > \varepsilon_k \right) \leq \frac{1}{\varepsilon_k} \sum_{m = -\infty}^{\infty} \left( \Psi^{-1}(1) + 2\Psi^{-1}\left(\frac{2^k}{2^k + 1}\right) \right) \left( F\left(\frac{(2m+1)\pi}{2^k}\right) - F\left(\frac{2m\pi}{2^k}\right) \right)^{1/2} \]

\[ * \Psi^{-\eta}(x) = (\Psi(x))^{-\eta}. \]
\[ \leq \frac{1}{\epsilon_k} \frac{\pi}{2^k} \sum_{n=-\infty}^{\infty} \left( \Psi^{-1}(1) + 2A \Psi^{-1}\left( \frac{2^{k+1}}{|n| + 2^{k+1}} \right) \right) \left( F\left( \frac{n+1}{2^k} \right) - F\left( \frac{n}{2^k} \right) \right) \leq \frac{1}{\epsilon_k} \frac{\pi}{2^k} \sqrt{\frac{2^k}{\pi} + 1} \sum_{n=-\infty}^{\infty} \left( \Psi^{-1}(1) + 2A \Psi^{-1}\left( \frac{2}{|n| + 3} \right) \right) (F(n+1) - F(n))^{1/2}. \]

Since \( \Psi^{-1}\left( \frac{2}{|n| + 3} \right) \leq \frac{|n| + 3}{|n|} \Psi^{-1}\left( \frac{1}{|n|} \right) \) and \( \Psi\left( \frac{1}{|n|} \right) \leq \Psi(1) \) for \( n \neq 0 \), we get

\[(3.7) \quad Q'_k \leq \frac{1}{\epsilon_k} \frac{C_k}{2^{k/2}} \left[ (F(1) - F(0))^{1/2} + \sum_{n=0}^{\infty} \Psi^{-1}\left( \frac{1}{|n|} \right) (F(n+1) - F(n))^{1/2} \right] \leq \frac{1}{\epsilon_k} \frac{C_k}{2^{k/2}}. \]

Choosing \( \epsilon_k \) as in the proof of Theorem 1, we see from (3.7) that \( \hat{X}_k(t) \) converges uniformly to a weakly stationary process \( X_0(t) \) and

\[ \left| \hat{X}_k(t, h) - \frac{X_0(t+h) - X_0(t)}{\Psi(|h|)} \right| \leq \epsilon_k \quad \text{for} \quad k \geq k_0. \]

By Lemma 5, \( \sup_{0<|h| \leq 1, |t| \leq A} |\hat{D}_{\nu,k}(t, h)| < \infty \) almost surely, we conclude that \( X_0(t) \) is \( \Psi \)-Hölder continuous for \( |t| \leq A \) for any \( A > 0 \), which completes the proof.

**Theorem 4.** If for a given weakly stationary process \( X(t) \), there is an even, non-negative, non-decreasing function \( g(x) \) such that

\[(3.8) \quad \sum_{n=1}^{\infty} \Psi^{-1}\left( \frac{1}{n} \right) \cdot g^{-1}(n) < \infty, \]

\[(3.9) \quad \int g(x) dF(x) < \infty. \]

Then \( X(t) \) is equivalent to a weakly stationary process which is \( \Psi \)-Hölder continuous almost surely, where \( \Psi(h) \) is the function in Lemma 4.

**Proof.** By (3.8) and (3.9), we have

\[ \left[ \sum_{n=1}^{\infty} \Psi^{-1}\left( \frac{1}{n} \right) (F(n+1) - F(n))^{1/2} \right]^2 \leq \sum_{n=1}^{\infty} \Psi^{-2}\left( \frac{1}{n} \right) g^{-1}(n) \cdot \sum_{n=1}^{\infty} g(n)(F(n+1) - F(n)) \]

\[ \leq \sum_{n=1}^{\infty} \Psi^{-2}\left( \frac{1}{n} \right) g^{-1}(n) \cdot \int g(x) dF(x) < \infty. \]
Similarly, we can see that \( \sum_{n=1}^{\infty} \Psi^{-1}\left(\frac{1}{|n|}\right) (F(n + 1) - F(n))^{1/2} < \infty \). Hence the assertion follows from Theorem 3.

**Example 4.** Suppose that \( F(x) \) is absolutely continuous with the density \( f(x) \) and that \( f(x) \) is non-increasing as \( x \to \pm \infty \). If

\[
(3.10) \quad \int_{|x|>1} \Psi^{-1}\left(\frac{1}{|x|}\right) f^{1/2}(x) \, dx < \infty,
\]

then \( X(t) \) is sample \( \Psi \)-Hölder continuous.

**Example 5.** If a separable stationary process \( X(t) \) satisfies

\[
(3.11) \quad \int_{|x|>B} \Psi^{-2}\left(\frac{1}{|x|}\right) |x| \cdot \log |x| \cdot \log(2|x|) \cdot \cdots \cdot \log(\alpha|x|) \cdot (\log(\alpha+1)|x|)^{1+\varepsilon} \, dF(x) < \infty,
\]

for sufficiently large \( B > 0 \) with \( \varepsilon > 0 \), then

\[
\limsup_{h \to 0} \frac{|X(t + h) - X(t)|}{\Psi(h)} = 0 \quad \text{a.s.}
\]

Especially if, \( F(x) \) is absolutely continuous with the density \( f(x) \) which satisfies

\[
|f(x)| \leq \Psi^2\left(\frac{1}{|x|}\right) |x| \cdot \log |x| \cdot \cdots \cdot \log(\alpha|x|) \cdot (\log(\alpha+1)|x|)^{1+\varepsilon} \Psi^{-2},
\]

then (3.11) holds.

**4. Absolute convergence of the Fourier series of a weakly stationary process.**

Let \( X(t) \) be a weakly stationary process described in 1. Let \( T \) be any positive number. Define

\[
(4.1) \quad Y(t) = X(t), \quad t \geq 0,
\]

\[
X(-t), \quad t \leq 0.
\]

We consider the Fourier series of \( Y(t) \) over \((-T,T)\),

\[
(4.2) \quad A_n = \frac{1}{T} \int_{-T}^{T} Y(t) \cos \frac{n\pi t}{T} \, dt,
\]

\[
(4.3) \quad \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi t}{T}.
\]

As in [2].
THEOREM 5. Let $g(x)$ be even, non-negative and non-decreasing for $x > 0$, such that $g(x)/x^2$ is non-increasing for large $x$ and

(4.6) \[ \sum_{n=1}^{\infty} \frac{1}{g(n)} < \infty. \]

If

(4.7) \[ \int_{-\infty}^{\infty} g(x) dF(x) < \infty, \]

then $\sum_{n=0}^{\infty} |A_n|$ converges almost surely.

Proof. We may suppose that $g(x)/x^2$ is non-decreasing over $(0, \infty)$. In fact, if $g(x)/x^2$ is non-increasing for $x \geq B$, then we may define $g(x)$ as it is for $(x \leq B)$, and $g(B)(x/B)^2$ for $(x \geq B)$. By (4.5),

(4.8) \[ \sum_{n=2}^{\infty} g(n) E|A_n|^2 = 8 \sum_{n=2}^{\infty} g(n) \frac{\sin^2\left[ \frac{1}{2} (|\lambda| T - n\pi) \right]}{|\lambda| T + n\pi} \frac{e^{2\pi}}{|\lambda| T + \pi} dF(\lambda) = \]

(4.9) \[ = 8 \int_{|\lambda| > \pi/T} \left( \sum_{n \geq |\lambda|/T} \right) dF(\lambda) + \sum_{|\lambda| \geq \pi/T} \right) dF(\lambda) \]

(4.10) \[ + 8 \int_{|\lambda| > 2\pi/T} \left( \sum_{|\lambda|/T - 1 \geq n} \right) dF(\lambda) + \sum_{|\lambda| \leq \pi/T} \right) dF(\lambda) = \]

(4.11) \[ = I_1 + I_2 + I_3 + I_4, \]

say.

Noting that

(4.12) \[ g(AX) \leq A^3 g(x), \]

for $A > 1$ $x \geq 1$ which follows from the assumption that $g(x)/x^2$ is non-increasing for $x \geq 1$, we see that
\[
I_1 \leq 8 \int_{|\lambda| > \pi/T} \sum_{n \geq \frac{|\lambda T/\pi|-1}{2}} \frac{g(n) \lambda^2 T^2}{\pi^2 (n - \frac{|\lambda T}{\pi} - 1)^2 (|\lambda T + \pi n|^2)} \, dF(\lambda) \\
\leq 8C_1 \int_{|\lambda| > \pi/T} \frac{\lambda T}{\pi} \, dF(\lambda) < C_2 T^2 \int_{|\lambda| > \pi/T} g(\lambda) dF(\lambda) < \infty,
\]

where \( C_1 \) and \( C_2 \) are constants. Here we have used that

\[
g(n) \lambda^2 T^2 (|\lambda T + \pi n|^2) \leq g(\lambda T/\pi).
\]

\[
I_2 \leq C_3 \int_{|\lambda| > \pi/T} \left[ g\left(\frac{\lambda T}{\pi} + 1\right) + g\left(\frac{\lambda T}{\pi}\right) \right] \, dF(\lambda) \leq C_4 T^2 \int_{|\lambda| > \pi/T} g(\lambda) dF(\lambda) < \infty,
\]

\( C_3, C_4 \) being constants.

\[
I_3 \leq \int_{|\lambda| > \pi/T} \sum_{n \leq \frac{|\lambda T/\pi|-1}{2}} \frac{\lambda^2 T^2 g(|\lambda T/\pi| - 1)}{(|\lambda T - \pi n|^2)^2 (|\lambda T + 2|^2)} \, dF(\lambda) \\
\leq C_5 \sum_{m=1}^{\infty} \frac{1}{m^2} \int_{|\lambda| > \pi/T} \frac{\lambda T}{\pi} \, dF(\lambda) \leq C_6 T^3 \int_{|\lambda| > \pi/T} g(\lambda) dF(\lambda) < \infty,
\]

where \( C_5 \) and \( C_6 \) are constants.

Since \( \lambda^2 T^2 (|\lambda T - \pi n|^2) \leq (n - 1)^{-2} \) for \( |\lambda T| \leq \pi \),

\[
I_4 \leq 8 \sum_{n=2}^{\infty} \frac{g(n)}{\pi^2 n^2 (n - 1)^2} \int_{|\lambda| \leq \pi/T} dF(\lambda).
\]

Since \( g(n) \leq n^2 g(1) \) from (4.8),

\[
I_4 < \infty.
\]

Hence we have obtained that

\[
\sum_{n=2}^{\infty} g(n) E|A_n|^2 < \infty.
\]

From this, our conclusion follows immediately, for

\[
E \sum_{n=2}^{\infty} |A_n| = E \sum_{n=2}^{\infty} g^{1/2}(n) g^{1/2}(n)|A_n| \\
\leq \left[ \sum_{n=2}^{\infty} \frac{1}{g(n)} \right]^{1/2} E \left[ \sum_{n=2}^{\infty} g(n) |A_n|^2 \right]^{1/2} \\
\leq \left[ \sum_{n=2}^{\infty} \frac{1}{g(n)} \right]^{1/2} E \left[ \sum_{n=2}^{\infty} g(n) |A_n|^2 \right]^{1/2}
\]
which is finite by (4.6) and (4.8), and \( E \sum |A_n| < \infty \) implies the almost sure convergence of \( \sum |A_n| \).

As an implication of the conclusion of Theorem 5 is that \( X(t) \) is sample continuous in \((0,T)\) for every \( T > 0 \) which, of course, implies that \( X(t) \) is sample continuous in \((0,\infty)\). However, for this statement we need the unnecessary condition that \( g(x)x^2 \) is non-decreasing.

**References**


