ON EXTENSIONS OF VALUATIONS TO SIMPLE TRANSCENDENTAL EXTENSIONS

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0. Introduction

Let \( v_0 \) be a valuation of a field \( K_0 \) with residue field \( k_0 \) and value group \( Z \), the group of rational integers. Let \( K_0(x) \) be a simple transcendental extension of \( K_0 \). In 1936, Maclane [3] gave a method to determine all real valuations \( V \) of \( K_0(x) \) which are extensions of \( v_0 \). But his method does not seem to give an explicit construction of these valuations. In the present paper, assuming \( K_0 \) to be a complete field with respect to \( v_0 \), we explicitly determine all extensions of \( v_0 \) to \( K_0(x) \) which have \( Z \) as the value group and a simple transcendental extension of \( k_0 \) as the residue field. If \( V \) is any extension of \( v_0 \) to \( K_0(x) \) having \( Z \) as the value group and a transcendental extension of \( k_0 \) as the residue field, then using the Ruled Residue theorem [4, 2, 5], we give a method which explicitly determines \( V \) on a subfield of \( K_0(x) \) properly containing \( K_0 \).

In Section 1, we prove some results needed for the main results. These results, however, turn out to be of independent interest.

1. Certain extensions of any real valuation to a simple transcendental extension

In this section, \( v_0 \) is a real valuation of a field \( K_0 \) (not necessarily discrete or complete) with residue field \( k_0 \) and \( K_0(x) \) is a simple transcendental extension of \( K_0 \). We shall denote by \( V_0 \) the valuation of \( K_0(x) \) defined on \( K_0[x] \) by

\[
V_0\left( \sum_{i=0}^{n} c_i x^i \right) = \min_i v_0(c_i).
\]

Let \( P(x) \) be a monic polynomial with coefficients in the valuation ring \( \mathfrak{o} \) of \( v_0 \) such that the corresponding polynomial \( \bar{P}(x) \) with coefficients in the residue field \( k_0 \) of \( v_0 \) is irreducible over \( k_0 \). Let \( \theta \) be any positive real number. By successive division by powers of \( P(x) \), any non-zero polynomial \( f(x) \) in \( \mathfrak{o}[x] \) can be uniquely represented as

\[
f(x) = \sum_{i=0}^{m} f_i P(x)^i
\]
where the polynomial $f(x)$ in $\mathfrak{o}[x]$ is either zero or has degree less than that of $P(x)$. (The above representation of $f(x)$ will be referred to as the canonical representation of $f(x)$). We define $V_{P(x)}$ on $\mathfrak{o}[x]$ by

$$V_{P(x)}(f(x)) = \min_i (V_0(f_i(x)) + i\theta).$$

We shall soon prove that $V_{P(x)}$ is a valuation of $\mathfrak{o}[x]$. Its unique extension to $K_0(x)$ will also be denoted by $V_{P(x)}$.

**Lemma 1.** If a non-zero polynomial $F(x)$ in $\mathfrak{o}[x]$ is written, by the division algorithm, as

$$F(x) = P(x)q(x) + r(x),$$

then

$$V_0(r(x)) \geq V_0(F(x))$$

and consequently

$$V_0(q(x)) = V_0(P(x)q(x)) \geq V_0(F(x)).$$

**Proof.** This follows at once if we write $F(x) = \alpha F_1(x)$ with $\alpha$ in $\mathfrak{o}$ such that $V_0(F_1(x)) = 0$ and then write by the division algorithm $F_1(x)$ as $P(x)q_1(x) + r_1(x)$.

The following remark follows immediately from above.

**Remark 1.** With notation as in Lemma 1, $V_0(r(x)) > V_0(F(x))$ if and only if $\bar{P}(x)$ divides $\bar{F}_1(x)$ over $k_0$, where $F_1(x)$ is any constant multiple of $F(x)$ with $V_0(F_1(x)) = 0$.

**Lemma 2.** If $a(x)$ and $b(x)$ are two non-zero polynomials in $\mathfrak{o}[x]$, each of degree less than the degree of $P(x)$, then

$$V_{P(x)}(a(x)b(x)) = V_{P(x)}(a(x)) + V_{P(x)}(b(x)).$$

**Proof.** Let $\alpha, \beta$ be elements of $\mathfrak{o}$ such that

$$a(x) = \alpha a_1(x), \ b(x) = \beta b_1(x), \ V_0(a_1(x)) = V_0(b_1(x)) = 0.$$  

On dividing $a_1(x)b_1(x)$ by $P(x)$, we can write

$$a_1(x)b_1(x) = P(x)q_1(x) + r_1(x); \quad (1.1)$$

where either $r_1(x) = 0$ or $\deg r_1(x) < \deg P(x)$. Since $\deg (a_1(x)b_1(x)) < \deg P(x)^2$, therefore $\deg q_1(x) < \deg P(x)$. Also $V_0(r_1(x)) = 0$, i.e. the polynomial $\bar{F}_1(x) \neq 0$ in $k_0[x]$; for other-
wise by (1.1) \( P(x) \) will divide \( \bar{a}_i(x)\bar{b}_i(x) \) and being irreducible must divide at least one of \( \bar{a}_i(x) \) or \( \bar{b}_i(x) \), which is impossible in view of the degrees of \( \bar{a}_i(x) \) and \( \bar{b}_i(x) \). On multiplying (1.1) by \( \alpha\beta \), we have

\[
a(x)b(x) = P(x)\alpha\beta q_1(x) + \alpha\beta r_1(x).
\]

Now by definition of \( V_{P(x)} \), we have

\[
V_{P(x)}(a(x)b(x)) = \min \{ V_0(\alpha\beta q_1(x)) + t, V_0(\alpha\beta r_1(x)) \}
\]

\[
= \min \{ v_0(\alpha\beta) + V_0(q_1(x)) + \theta, v_0(\alpha\beta) \}
\]

\[
= v_0(\alpha\beta) = V_{P(x)}(a(x)) + V_{P(x)}(b(x)),
\]

and the lemma is proved.

**Theorem 1.** \( V_{P(x)} \) is a valuation on \( \mathbb{O}[x] \).

**Proof.** Let \( f(x) \) and \( g(x) \) be non-zero polynomials over \( \mathbb{O} \) with canonical representations

\[
f(x) = \sum_{i=0}^{m} f_i(x)P(x)^i, \quad f_m(x) \neq 0
\]

\[
g(x) = \sum_{j=0}^{n} g_j(x)P(x)^j, \quad g_n(x) \neq 0.
\]

On adding (1.2) and (1.3) we obtain the canonical representation for \( f(x) + g(x) \) and the triangle law, i.e.,

\[
V_{P(x)}(f + g) \geq \min \{ V_{P(x)}(f), V_{P(x)}(g) \}
\]

follows immediately. Also it is easy to prove using Lemma 2 and the triangle law that

\[
V_{P(x)}(fg) \geq V_{P(x)}(f) + V_{P(x)}(g).
\]

Now, it remains to prove that

\[
V_{P(x)}(fg) \leq V_{P(x)}(f) + V_{P(x)}(g).
\]

(1.4)

Let \( t \) and \( u \) be the smallest indices such that

\[
V_{P(x)}(f(x)) = v_0(f_i(x)) + t\theta, \quad V_{P(x)}(g(x)) = v_0(g_u(x)) + u\theta.
\]
Let \( r_i(x) \) and \( q_i(x) \) be the polynomials over \( \mathbb{F} \) determined by the division algorithm from the following equations:

\[
\begin{align*}
    f_0(x)g_0(x) &= q_0(x)P(x) + r_0(x) \\
    f_0(x)g_1(x) + f_1(x)g_0(x) + q_0(x) &= q_1(x)P(x) + r_1(x) \\
    &\vdots \\
    f_m(x)g_n(x) + q_{m+n-1}(x) &= q_{m+n}(x)P(x) + r_{m+n}(x).
\end{align*}
\]

Observe that the degree of each \( q_i(x) \) and \( r_i(x) \) is less than the degree of \( P(x) \). Thus the representation of \( f(x)g(x) \) as

\[
f(x)g(x) = \sum_{i=0}^{m+n} r_i(x)P(x)^i + q_{m+n}(x)P(x)^{m+n+1}
\]

is the canonical representation.

The inequality (1.4) follows at once if we prove that

\[
V_0(r_{t+u}(x)) = V_0(f_t(x)) + V_0(g_u(x)). \tag{1.5}
\]

We first show that

\[
V_0(f_i(x)g_j(x)) > V_0(f_i(x)g_u(x)) \quad \text{if } i + j < t + u \tag{1.6}
\]

and

\[
V_0(f_i(x)g_j(x)) > V_0(f_i(x)g_u(x)) \quad \text{if } i + j = t + u, \quad i \neq t. \tag{1.7}
\]

Both (1.6) and (1.7) follow immediately from the following observation. For \( 0 \leq i \leq m \), \( V_0(f_i) + \theta \geq V_0(f_i) + \theta \) with strict inequality if \( i < t \); and for \( 0 \leq j \leq n \), \( V_0(g_j) + j\theta \geq V_0(g_u) + \theta \) with strict inequality if \( j < u \).

Define a polynomial \( F(x) \) over \( \mathbb{F} \) by

\[
F(x) = \sum_{i+j=t+u} f_i(x)g_j(x) + q_{t+u-1}(x).
\]

Recall that \( q_{t+u}(x) \) and \( r_{t+u}(x) \) are respectively the quotient and remainder when \( F(x) \) is divided by \( P(x) \) i.e.

\[
F(x) = q_{t+u}(x)P(x) + r_{t+u}(x).
\]

In view of (1.6) and Lemma 1, it is clear that

\[
V_0(q_{t+u-1}) > V_0(f_tg_u). \tag{1.8}
\]

In view of (1.7), we have
Let \( \alpha \) and \( \beta \) be elements of \( \mathfrak{o} \) such that \( f_i(x) = \alpha F_i(x) \), \( g_u(x) = \beta g_u(x) \) with \( V_0(f_i(x)) = \nu_0(\alpha) \) and \( V_0(g_u(x)) = \nu_0(\beta) \).

Let \( F_1(x) \) be the polynomial over \( \mathfrak{o} \) defined by \( F(x) = \alpha \beta F_1(x) \). In view of (1.7) and (1.8) it is clear that

\[
F_1(x) = \bar{F}_1(x) \bar{G}_u(x).
\]

Since both \( \bar{F}_1(x) \) and \( \bar{G}_u(x) \) are of degree less than that of \( \bar{P}(x) \), therefore \( \bar{P}(x) \) does not divide \( \bar{F}_1(x) \). It now follows from equation (1.9), Lemma 1 and the remark following the lemma that

\[
V_0(f_i g_u) = V_0(F) = V_0(r_i + u),
\]

which proves (1.5) and hence completes the proof of the fact that \( V_p(x) \) is a valuation of \( \mathfrak{o}[x] \).

Notation. If \( a \) is an element of the valuation ring of a valuation \( V \) of a field \( K \), then \( \bar{a} \) will denote its image in the residue field of \( V \).

The following theorem determines the residue fields of the valuations \( V_0 \) and \( V_p(x) \). The residue field of \( V_0 \) is well known \([1, \S 10.2, \text{Prop. 2}]\). For the sake of completeness we determine it here also.

**Theorem 2.** With \( \nu_0, k_0, V_0, \theta, V_p(x) \) as before and with \( G_0 \) as the value group of \( \nu_0 \), we have:

(i) The residue field of the valuation \( V_0 \) is \( k_0(\bar{x}) \) with \( \bar{x} \) (the image of \( x \) in the residue field of \( V_0 \)) transcendental over \( k_0 \).

(ii) If \( \theta \) is free modulo \( G_0 \), then the residue field of \( V_p(x) \) is \( k_0(\bar{x}) \) with \( \bar{x} \) (the image of \( x \) in the residue field of \( V_p(x) \)) algebraic over \( k_0 \).

(iii) If \( \theta \) is torsion modulo \( G_0 \), with \( s \) as the smallest positive integer such that \( s\theta(=\nu_0(\alpha)) \) is in \( G_0 \), then the residue field of \( V_p(x) \) is \( k_0[\bar{x}](t) \) where \( t = \) (the residue class of \( P(x)^{1/\alpha} \) (in the residue field of \( V_p(x) \)), is transcendental over \( k_0[\bar{x}] \) and \( \bar{x} \) is algebraic over \( k_0 \).

**Proof.** In all the three cases, we denote by \( \Delta \) the residue field of the valuation under
consideration and by \( \bar{\xi} = (f(x)/g(x))^{-1} \) an arbitrary non-zero element of \( \Delta \), with \( f(x) \) and \( g(x) \) in \( o[x] \).

(i) It is easy to verify that the image \( \bar{x} \) of \( x \) in the residue field of \( V_0 \) is transcendental over \( k_0 \) in this case. Let \( \beta \) be an element of \( o \) such that

\[
V_0(f(x)) = V_0(g(x)) = v_0(\beta).
\]

Then

\[
V_0(f(x)/\beta) = 0, \quad V_0(g(x)/\beta) = 0.
\]

So \( \xi = \xi_1 \xi_2^{-1} \) where \( \xi_1 = (f(x)/\beta)^{-1} \) and \( \xi_2 = (g(x)/\beta)^{-1} \) are in \( k_0[\bar{x}] \). This proves that \( \Delta = k_0(\bar{x}) \).

(ii) Let

\[
f(x) = \sum_i f_i(x)P(x)^i
\]

\[
g(x) = \sum_j g_j(x)P(x)^j
\]

be the canonical expression for \( f(x) \) and \( g(x) \) respectively. For non-zero polynomials \( f_i(x) \) and \( g_j(x) \), define polynomials \( F_i(x) \) and \( G_j(x) \) with coefficients in \( o \) by

\[
f(x) = \beta_i F_i(x), \quad g(x) = \gamma_j G_j(x)
\]

where \( v_0(\beta_i) = V_0(f_i(x)) \) and \( v_0(\gamma_j) = V_0(g_j(x)) \). Thus

\[
f(x) = \sum_i \beta_i F_i(x)P(x)^i, \quad (1.10)
\]

\[
g(x) = \sum_j \gamma_j G_j(x)P(x)^j. \quad (1.11)
\]

Since \( V_{P(x)}(f(x)) = V_{P(x)}(g(x)) \), i.e., \( \min_i(\nu_0(\beta_i) + i\theta) = \min_j(\nu_0(\gamma_j) + j\theta) \), therefore there exist subscripts \( h \) and \( k \) such that

\[
\nu_0(\beta_h) + h\theta = \nu_0(\gamma_k) + k\theta.
\]

In this case, \( \theta \) being free modulo \( G_0 \), the above equality is possible only if \( h = k \) and \( \nu_0(\beta_h) = \nu_0(\gamma_h) \), also \( \nu_0(\beta_i) + i\theta > \nu_0(\beta_h) + h\theta \) if \( i \neq h \) and \( \nu_0(\gamma_j) + j\theta > \nu_0(\gamma_k) + k\theta \) if \( j \neq k \). So if we write \( \xi_1 = f(x)/\beta_h P(x)^h \) and \( \xi_2 = g(x)/\beta_h P(x)^h \), we have \( \xi = \xi_1/\xi_2 = F_h(\bar{x})/(G_h(\bar{x}))^{-1} \) is in \( k_0(\bar{x}) \); here \( \bar{G}_h(\bar{x}) \neq 0 \), because the degree of the polynomial \( \bar{G}_h(\bar{x}) \) is less than the degree of \( \bar{P}(\bar{x}) \) which is the minimal polynomial of \( \bar{x} \) over \( k_0 \). This proves that \( \Delta = k_0(\bar{x}) \) which is an algebraic extension of \( k_0 \).

(iii) Let \( s \) be the smallest positive integer such that \( s\theta \in G_0 \), (say) \( s\theta = \nu_0(\alpha) \) with \( \alpha \) in \( o \).
We first prove that the residue class \((P(x)^{\gamma}/\alpha)^{-1} = t\) (say) is transcendental over \(k_0\). Suppose \(t\) is algebraic over \(k_0\). Let \(y^{m} + a_1y^{m-1} + \cdots + a_m\) be a polynomial over \(k_0\) satisfied by \(t\). Therefore

\[ V_{P(x)}((P(x)^{\gamma}/\alpha)^{m} + a_1(P(x)^{\gamma}/\alpha)^{m-1} + \cdots + a_m) > 0 \]

i.e., if we write

\[ F(x) = P(x)^{\gamma} + a_1xP(x)^{(m-1)} + \cdots + a_mx^m \quad (1.12) \]

then

\[ V_{P(x)}(F(x)) > v_0(\alpha^m) = ms\theta, \]

which is impossible because (1.12) is a canonical expression for \(F(x)\) and by definition of \(V_{P(x)}\), we must have \(V_{P(x)}(F(x)) \leq sm\theta\). This contradiction proves that \(t\) is transcendental over \(k_0\); in fact \(t\) is transcendental over \(k_0(\bar{x})\) because \(\bar{x}\), satisfying the polynomial \(\bar{P}(y)\), is algebraic over \(k_0\).

Let expressions for \(f(x)\) and \(g(x)\) be as in (1.10) and (1.11). Let \(\beta\) be an element of \(\mathfrak{o}\) and \(h\) an integer such that

\[ v_0(\beta) + h\theta = \min_i (v_0(\beta_i) + i\theta) = \min_j (v_0(\gamma_j) + j\theta). \]

Write \(\xi_1 = f(x)/\beta P(x)^h\), \(\xi_2 = g(x)/\beta P(x)^h\). Then

\[ \xi_1 = \sum_i F_i(\bar{x})(\beta_i P(x)^i/\beta P(x)^h)^{-} \]

where the sum \(\sum_i\) is carried over all those \(i\) for which \(v_0(\beta_i) + i\theta = v_0(\beta) + h\theta\) (the rest of the terms are zero in the residue field). For each \(i\) in \(\sum_i\), \(v_0(\beta_i) + i\theta = v_0(\beta) + h\theta\), i.e., \((i-h)\theta = v_0(\beta) - v_0(\beta_i)\). So \((i-h)\) is an integral multiple of \(s\), say \((i-h) = ms\). Therefore the residue class of \(\beta_i P(x)^i/\beta P(x)^h = (P(x)^{\gamma}/\alpha)^{m} \cdot (\beta \alpha^m/\beta)\) is an integral power of \(t = (P(x)^{\gamma}/\alpha)^{-}\) multiplied by an element of \(k_0\). Thus \(\xi_1\) is in the field \(k_0[\bar{x}](t)\). Similarly \(\xi_2\) and hence \(\xi\) are in the same field. This proves (iii).

**Remark 2.** As in [1, § 10.2, Prop. 2], it is easy to prove that if \(V\) is a real valuation of \(K_0(x)\) extending the valuation \(v_0\) of \(K_0\) with \(V(x) = 0\) and if \(\bar{x}\) is transcendental over \(k_0\) then \(V = V_0\).

2. **Construction of extensions of \(v_0\) with residue field \(k_0(t)\)**

In what follows, \(K_0\) is a complete valuation field with respect to a valuation \(v_0\) having the value group \(\mathbb{Z}\), the valuation ring \(\mathfrak{o}\) and the residue field \(k_0\). As before \(x\) is an indeterminate, We shall consider only those extensions \(V\) of \(v_0\) to \(K_0(x)\) for which \(V(x) \geq 0\).
Theorem 3. Let $V$ be an extension of $v_0$ to $K_0(x)$ with value group $\mathbb{Z}$ and residue field $\Delta$ such that $\Delta \cap (\text{the algebraic closure of } k_0) = k_0(\bar{x})$, then $V = V_{P(x)}$ for some monic polynomial $P(x)$ over $\mathfrak{o}$ where $\bar{P}(y)$ is the minimal polynomial of $\bar{x}$ over $k_0$.

Proof. Let $\pi$ be a uniformizer of $v_0$ in $K_0$ and $\phi(y)$ be the minimal polynomial of $\bar{x}$ over $k_0$ of degree $n$. We claim that there exists a monic polynomial $P(x)$ over $\mathfrak{o}$ with $\bar{P}(x) = \phi(x)$ such that the residue class $(P(x)/\pi^n)^{-}$ (in the residue field of $V$) is transcendental over $k_0$, $r$ being given by $V(P(x)) = r$. Let $P_1(x)$ be any monic polynomial with coefficients in $\mathfrak{o}$ such that $\bar{P}_1(x) = \phi(x)$. Since $\bar{P}_1(\bar{x}) = \phi(\bar{x}) = 0$, therefore $s_1 = V(P_1(x)) > 0$. If $(P_1(x)/\pi^n)^{-}$ is transcendental over $k_0$ then our claim is proved. If it is algebraic over $k_0$ then by hypothesis there exists a polynomial $f_1(x)$ in $\mathfrak{o}[x]$ of degree $\leq n - 1$ such that

$$(P_1(x)/\pi^n)^{-} = f_1(\bar{x}).$$

So

$$V(P_1(x) - \pi^n f_1(x)) > s_1.$$ 

Write $P_2(x) = P_1(x) - \pi^n f_1(x)$ and define an integer $s_2 > s_1$ by $V(P_2(x)) = s_2$. If $(P_2(x)/\pi^n)^{-}$ is transcendental over $k_0$, we stop here otherwise we continue the process. We show that the process cannot continue indeﬁnitely. Suppose it does. So we obtain a sequence of polynomial $f_i(x)$ in $\mathfrak{o}[x]$ each of degree $\leq n - 1$ and a strictly increasing sequence of positive integers $s_1 < s_2 < \cdots$ such that

$$V\left(P_1(x) - \sum_{j=1}^{i} f_j(x)\pi^j\right) = s_{i+1}.$$ 

Since $K_0$ is complete and since each $f_i(x)$ is of degree $\leq n - 1$, therefore $\sum_{j=1}^{n} f_j(x)\pi^j$ is a polynomial over $\mathfrak{o}$ of degree $\leq n - 1$, which we shall denote by $F(x)$. By choice $V(P_1(x) - F(x)) > s_i$ for all $i$, so $P_1(x) - F(x)$ must be the zero polynomial. Which is impossible because $P_1(x) - F(x)$ is a monic polynomial of degree $n$ over $\mathfrak{o}$. This contradiction proves the claim.

Let $P(x)$ be a monic polynomial over $\mathfrak{o}$ such that $\bar{P}(x) = \phi(x)$, $V(P(x)) = r$ and $(P(x)/\pi^n)^{-}$ is transcendental over $k_0$ and hence over $k_0(\bar{x})$. We now prove that the valuation $V$ is nothing but $V_{\theta P(x)}$ with $\theta = V_{P(x)}(P(x)) = r$. Since the minimal polynomial satisfied by $\bar{x}$ over $k_0$ has degree $n$, therefore for any polynomial $a(x)$ over $\mathfrak{o}$ of degree $\leq n - 1$, $V(a(x)) = V_0(a(x))$ holds. Let $f(x)$ be any non-zero element of $\mathfrak{o}[x]$ and let

$$f(x) = \sum_{i=1}^{m} f_i(x)P(x)^i$$

be the canonical representation of $f(x)$. If $f_i(x) \neq 0$ then $\deg f_i(x) \leq n - 1$. So

$$V(f_i(x)) = V_0(f_i(x)) = a_i, \quad \text{(say)}.$$
By definition of \( V_{f(x)} \), we have

\[
V_{f(x)}(f(x)) = \min_i (a_i + ir)
\]

where the minimum is carried over those \( i \) for which \( f(x) \neq 0 \). We shall prove that

\[
V(f(x)) = \min_i (a_i + ir) = V_{f(x)}(f(x)). \tag{2.2}
\]

Since \( V(f(x))P(x)^k = a_i + ir \), it follows from (2.1) that

\[
V(f(x)) \geq \min_i (a_i + ir). \tag{2.3}
\]

We now prove (2.2). Let \( h \) (be the smallest subscript such that \( \min_i (a_i + ir) = a_h + hr \). For a non-zero polynomial \( f(x) \) define \( F_i(x) \) in \( o[x] \) by \( f(x) = \pi x F_i(x) \), so that \( V_0(F_i(x)) = V(F_i(x)) = 0 \). Suppose strict inequality holds in (2.3). Then there exist positive integers \( h = h_0 < h_1 < \cdots < h_l \leq m \), such that

\[
V\left( \sum_{i=0}^{l} \pi^{a_i} F_h(x)P(x)^h_i \right) > a_h + hr \tag{2.4}
\]

and

\[
V(\pi^{a_i} F_h(x)P(x)^h) = a_h + hr \tag{2.5}
\]

for \( 0 \leq i \leq l \). It follows from (2.5) that

\[
(a_h - a_{h_i}) = (h_i - h_0) = n_i r, \quad \text{(say)} \tag{2.6}
\]

for \( 1 \leq i \leq l \). Observe that \( n_1 < n_2 < \cdots < n_l \). It follows at once from (2.4) and (2.6) that

\[
V\left( F_h(x) + \sum_{i=1}^{l} (P(x)/\pi^r)^{n_i} F_h(x) \right) > 0
\]

which shows that \( (P(x)/\pi^r)^{-} \) is algebraic over \( k_0[\bar{x}] \). This contradiction proves that equality holds in (2.3) and hence \( V = V_{f(x)} \).

Remark 3. If \( V \) is as in the above theorem then we have shown in Theorem 2, part (iii) of Section 1 that the residue field of \( V \) is a simple transcendental extension of \( k_0(\bar{x}) \).

**Theorem 4.** Let \( V \) be an extension of \( v_0 \) to \( K_0(\bar{x}) \) with value group \( \mathbb{Z} \) and residue field a simple transcendental extension of \( k_0 \) then either \( V = V_0 \) or \( V = V_{f(x)} \) for some monic linear polynomial \( P(x) \) over \( o \).
Proof. If $\bar{x}$ is transcendental over $k_0$ then by the remark in the end of Section 1, $V = V_0$. Suppose now that $\bar{x}$ is algebraic over $k_0$, therefore $\bar{x}$ is in $k_0$. So the minimal polynomial of $\bar{x}$ over $k_0$ is a linear polynomial. The desired assertion now follows immediately from Theorem 3.

3. Method of construction of valuations with residue field transcendental over $k_0$

Notation and assumptions are as in the previous section. Now we assume that $k_0$ is a perfect field. Let $V$ be an extension of $v_0$ to $K_0(x)$ with value group $\mathbb{Z}$ and residue field $\Delta$ transcendental over $k_0$. By the Ruled Residue theorem [5], there exists a finite extension $k_1$ of $k_0$ such that $\Delta = k_1(t)$ with $t$ transcendental over $k_0$. If $k_1 = k_0$, then by Theorem 4, either $V = V_0$ or $V = V_{P(x)}$ where $P(x)$ is a linear polynomial over $o$. Suppose now that $k_1$ is a proper extension of $k_0$. Since $k_0$ is perfect, therefore there exists $y$ in $K_0(x)$ such that $V(y) = 0$ and $k_1 = k_0(y)$. Then $y$ does not belong to $K_0$, so $K_0(x)$ is a finite extension of $K_0(y)$. Let $V_1$ denote the restriction of $V$ to $K_0(y)$. The hypotheses of Theorem 3 are clearly satisfied for the valuation $V_1$ of $K_0(y)$, so by this theorem $V_1 = V_{P(y)}$ for some monic polynomial $P(y)$ with coefficients in $o$. Thus the valuation $V$ is completely determined on $K_0(y)$.

REFERENCES


