



Correlation Functions and Boundary Conditions in Rational Conformal Field Theory and Three-Dimensional Topology

GIOVANNI FELDER,¹ JÜRIG FRÖHLICH,² JÜRGEN FUCHS² and
CHRISTOPH SCHWEIGERT³

¹*Department of Mathematics, ETH-Zentrum, CH-8092 Zürich, Switzerland*

²*Institute for Theoretical Physics, ETH-Hönggerberg, CH-8093 Zürich, Switzerland*

³*LPTHE, Université Paris VI, 4, place Jussieu, F-75252 Paris Cedex 05, France*

(Received: 14 June 2000; accepted in final form: 16 November 2000)

Abstract. We give a general construction of correlation functions in rational conformal field theory on a possibly nonorientable surface with boundary in terms of three-dimensional topological field theory. The construction applies to any modular category in the sense of Turaev. It is proved that these correlation functions obey modular invariance and factorization rules. Structure constants are calculated and expressed in terms of the data of the modular category.

Mathematics Subject Classifications (2000). 81T40, 18D10, 57R56.

Key words. conformal field theory, modular categories, topological field theory.

1. Introduction

In this paper, we study correlation functions in conformal field theory from the point of view of three-dimensional topological field theory.

The problem of constructing correlation functions in rational conformal field theory has two parts. The first part of the problem is to determine the space of conformal blocks. The second part is to use this space to construct correlation functions.

The first part of the problem is by now well understood in mathematical terms. One approach, suggested by Witten's paper [W] on Chern–Simons theory, is given in terms of three-dimensional topological field theory (TFT). Such a theory assigns to every 'extended surface' – an oriented 2-manifold X with marked points carrying labels, and certain additional data related to framing – a finite dimensional complex vector space $\mathcal{H}(X)$, the space of conformal blocks, or of the states of the TFT, and to every 3-manifold M bounded by X , containing a 'ribbon graph', a vector $Z(M)$ in $\mathcal{H}(X)$. The ribbon graph is an embedded graph ending at the marked points, with some additional structure. The spaces $\mathcal{H}(X)$ and the vectors $Z(M)$ are supposed to obey a number of natural axioms relating to homeomorphisms and cutting

and pasting. Turaev showed in [T] how every *modular category* produces a TFT and, in particular, a space of conformal blocks associated to every extended surface.

The purpose of this paper is to give a precise meaning in the same terms to the second part of the problem, the construction of correlation functions out of conformal blocks, and to present a solution. Let us first describe the data of the problem. First, one requires the chiral data of a rational conformal field theory, which for us are the data of a modular category. In particular, there is a set I of distinguished simple objects. These data are needed to define conformal blocks. The conformal field theory itself is given by a system of *correlation functions* obeying certain axioms. Each construction of such correlation functions satisfying the axioms gives a different conformal field theory with the same underlying modular category.

We proceed to describe what the axioms are in the simplest situation (the ‘Cardy case’ [C1]) to which we restrict our attention in this paper. To formulate the axioms we introduce the notion of a *labeled surface*. A conformal field theory is then an assignment of a correlation function $C(X)$ to each labeled surface X . A labeled surface X consists of a (not necessarily orientable) compact two-dimensional manifold with (possibly empty) oriented boundary, a set of marked points on it, and ‘boundary conditions’. The marked points all carry a label from I and certain local data. The boundary conditions are a coloring by I of the boundary arcs between marked boundary points. For example, if X is a disk with m marked points on its boundary, the boundary conditions are a labeling of the m arcs between neighboring points by elements of I .

The correlation function associated to these data is then a linear map $W_{\partial X} \rightarrow \mathcal{H}(\hat{X})$ from a ‘multiplicity space’, $W_{\partial X}$, associated to the boundary ($W_{\partial X} = \mathbb{C}$ if $\partial X = \emptyset$) to the space of conformal blocks, $\mathcal{H}(\hat{X})$, of the *double* \hat{X} of X . The double of a compact surface is an oriented closed surface with an orientation reversing involution σ so that X is obtained from \hat{X} by identifying pairs of points related by σ . For example, the double of a disk is a sphere, and the double of the projective plane is also a sphere, but with a different involution. The torus is the double of the annulus, the Möbius band and the Klein bottle. The double of a closed orientable surface is the disjoint union of two copies of the surface, with opposite orientations. The definition of the double also applies naturally to surfaces with marked points, so that the double of a labeled surface is an extended surface in the sense of Turaev ([T]).

The description of correlation functions in terms of conformal blocks on the double goes back to the early days of conformal field theory ([A1]) and was formulated in the general context of conformal field theory in [FS].

The correlation functions are supposed to behave naturally under homeomorphisms (the modular invariance of correlation functions) and gluing (the factorization properties of correlation functions). There are two types of gluing properties: a (possibly disconnected) surface may be glued by identifying two arcs

on its boundary or by cutting out two disks in the interior and identifying the boundaries of the resulting holes. In both cases these operations induce gluing operations on the double of the surface, which in turn, by the rules of TFT, induce homomorphisms between the corresponding spaces of conformal blocks. The requirement of factorization means that the correlation functions behave naturally under these homomorphisms.

Our main result is the construction of an assignment $X \mapsto C(X) \in \text{Hom}_{\mathbb{C}}(W_{\partial X}, \mathcal{H}(\hat{X}))$ of a correlation function to every labeled surface and a proof that it obeys the required modular and factorization properties. The basic idea is that the correlation function $C(X)$ is the vector $Z(M_X) \in \mathcal{H}(\hat{X})$ associated to a 3-manifold M_X with a ribbon graph, the *connecting 3-manifold of X* , whose boundary is the double \hat{X} of X . The vector $Z(M_X)$ depends linearly on the colorings of the vertices of the ribbon graphs by morphisms of the modular category. This space of colorings is identified with $W_{\partial X}$. The connecting manifold was first considered by Hořava ([H]) in his study of Chern–Simons theory on \mathbb{Z}_2 -orbifolds.

Let us first describe our construction in a simple example, suppressing for the moment the framing. Let X be a disk with n marked points in its interior with labels $i_1, \dots, i_n \in I$ and m points on the boundary, with labels $j_1, \dots, j_m \in I$. Let the boundary condition on the arcs between the k th and $k + 1$ st boundary points be labeled by $a_k \in I$. Then M_X is a 3-ball and the correlation function $C(X)$ is the conformal block on the sphere associated to the ribbon graph depicted in Figure 1. The points

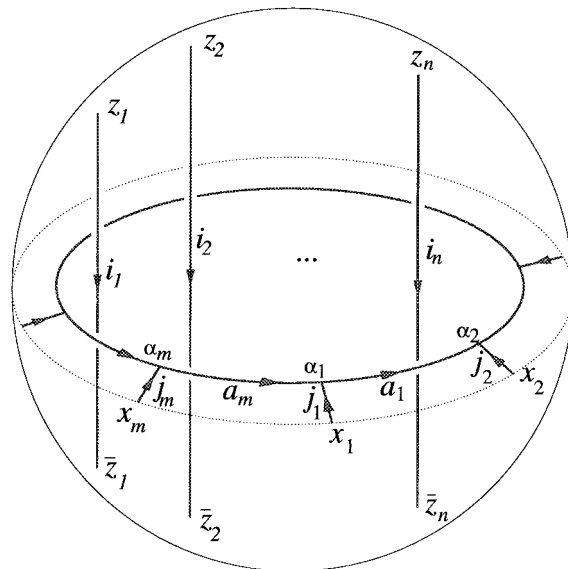


Figure 1.

z_k, \bar{z}_k project to the k th interior point on the disk, and x_l projects to the l th boundary point.

In the general case, the main property of the connecting manifold is that it comes with a projection $p: M_X \rightarrow X$ whose fibers over interior points are closed intervals, and an inclusion $i: X \rightarrow M_X$, which is a homotopy equivalence such that $p \circ i = \text{id}_X$. The ribbon graph then consists of fibers of p over the marked points in the interior, and a loop running close to $i(\partial X)$ and connected by short lines to the marked boundary points.

After formulating and proving the modular invariance and factorization properties of our correlation functions, we compute ‘structure constants’, namely the correlation functions for elementary building blocks: the disk with three boundary points, the disk with one interior and one boundary point, and the projective plane with one point. General correlation functions are then in principle obtainable from these using the factorization theorems. The structure constants are given in terms of the data (fusing matrices, modular S -matrix) of the modular category.

We also give formulae for the annulus, the Klein bottle and the Möbius strip without marked points. In these cases, the double is a torus; we show that the coefficients of correlation functions with respect to a natural basis of the space of conformal blocks are integers.

The correlation functions are thus given in our approach as states of a TFT. To get actual functions of the position of the marked points and moduli, say for WZW models, one uses the modular category of integrable modules of an affine Lie algebra $([KL, F])$. The dependence on the moduli should then be obtained by integrating the Knizhnik–Zamolodchikov connection. In the case of the sphere and general WZW models such a construction might be possible along the lines of [K].

The paper is organized as follows. We begin by giving a review of three-dimensional topological field theory, following [T], in Section 2. In this section, modular categories, TFT and modular functors are introduced, and subtleties such as the framing anomaly are explained. In Section 3 we present our proposal for correlation functions. We also formulate and prove the factorization and modular properties they obey. These properties imply, in particular, that correlation functions on general surfaces may be expressed in terms of basic correlation functions. We compute these basic correlation functions in Section 4. In Section 5 we compute correlation functions in the cases where the double has genus one, and prove integrality results.

In Appendix A we give the definition of modular categories and in Appendix B we describe how to obtain the real projective space by surgery on the unknot, a result needed to compute correlation functions on the projective plane.

Finally, in Appendix C, we give a short overview of the physical background of our results. In particular, we briefly describe some of the main applications of boundary conformal field theory to physical problems.

Some of these results were announced in [FFFS2].

2. Modular Categories and Three-Dimensional TFT

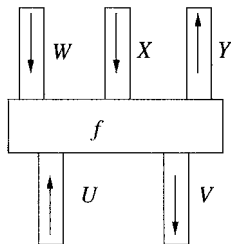
2.1. MODULAR CATEGORIES

A modular category is a strict monoidal Ab-category \mathcal{M} (see Appendix A) with unit object $\mathbf{1}$ and an additional set of data obeying a system of axioms. The data are

- (1) A finite set I of simple objects containing $\mathbf{1}$.
- (2) For each pair of objects V, W , a braiding morphism $c_{V,W} \in \text{Hom}(V \otimes W, W \otimes V)$.
- (3) For each object V , a twist $\theta_V \in \text{Hom}(V, V)$.
- (4) A duality: for each object V there is a unique dual object V^* and morphisms $b_V \in \text{Hom}(\mathbf{1}, V \otimes V^*)$ and $d_V \in \text{Hom}(V^* \otimes V, \mathbf{1})$.

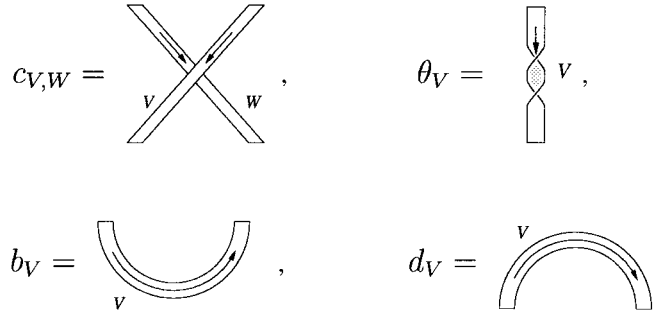
These data obey a number of axioms, which we describe in Appendix A (see [T] for more details). In general modular categories, $\text{Hom}(\mathbf{1}, \mathbf{1})$ is a general ring. Here we assume that $\text{Hom}(\mathbf{1}, \mathbf{1}) = \mathbb{C}$. The general case may be dealt with similarly, as we only use the axioms and results in the general theory of [T]. However, in order to simplify certain normalizations, we do use the fact that \mathbb{C} is an algebraically closed field.

The axioms can be best understood in the language of ribbon graphs, i.e., finite collections of disjoint ribbons, annuli and coupons. *Ribbons* are oriented rectangles $[-1/10, 1/10] \times [0, 1]$ embedded in $\mathbb{R}^2 \times [0, 1]$, so that $[-1/10, 1/10] \times (0, 1)$ is entirely contained in $\mathbb{R}^2 \times (0, 1)$. *Annuli* are oriented annuli $[-1/10, 1/10] \times S^1$, embedded in $\mathbb{R}^2 \times (0, 1)$. Ribbons and annuli are labeled by objects of the category. The *core* $\{0\} \times [0, 1]$ or $\{0\} \times S^1$ of each ribbon or annulus is also given an orientation. The *coupons* are oriented rectangles embedded in $\mathbb{R}^2 \times (0, 1)$ with two preferred opposite sides, the top and the bottom, and are labeled by morphisms of the category. The ends $[-1/10, 1/10] \times \{0\}$, $[-1/10, 1/10] \times \{1\}$ of the ribbons are glued to the top or the bottom of coupons, so that the orientations match to give an oriented (topological) 2-manifold with boundary, or are contained in $\mathbb{R}^2 \times \{0, 1\}$. The coupons are labeled by morphisms from the tensor product of the objects labeling the ribbons glued to the bottom boundary, or their dual objects, to the tensor product of the objects labeling the ribbons glued to the top boundary, or their dual objects. The ordering of the tensor product reflects the ordering (from left to right) of the ribbons meeting at the coupon. The dual object is taken when the orientation of the core points in the upwards (bottom to top) direction. For example, if the upper side of the coupon



is the top boundary, and the orientation is the standard orientation of the plane, then $f \in \text{Hom}(U^* \otimes V, W \otimes X \otimes Y^*)$.

Using this correspondence the basic morphisms are represented as follows:



Choosing an orientation of a ribbon in an oriented 3-manifold is the same as choosing a preferred side, which in our drawings will usually face the reader. The other side is drawn as shaded.

The tensor product of morphisms is represented by the juxtaposition of the factors, and the composition fg is obtained by drawing f on top of g and gluing the ends of the ribbons. Then the axioms are such that the morphism corresponding to a ribbon graph depends only on its isotopy class. In particular, for every ribbon graph Γ in $S^3 = \mathbb{R}^3 \cup \infty$, we get an isotopy invariant $|\Gamma| \in \text{Hom}(\mathbf{1}, \mathbf{1}) = \mathbb{C}$.

In drawing the graphs representing morphisms we will often make use of the ‘blackboard framing’ notation. Instead of drawing ribbons, we will only draw their cores, with the understanding that the ribbons are contained in the plane of the page (or of the screen) and inherit the orientation from the standard orientation of the plane.

As a consequence of the axioms, one then proves that the objects in I are pairwise non-isomorphic and are a system of representatives of all isomorphism classes of simple objects. Also one shows that there is a canonical isomorphism $V \rightarrow (V^*)^*$ for all objects V given in terms of braiding, twist and duality. We will tacitly identify $(V^*)^*$ with V via this isomorphism below.

Of particular importance is the (quantum) *trace* of an endomorphism $f \in \text{Hom}(V, V)$. It is defined by the formula $\text{tr}(f) = d_V c_{V, V^*} (\theta_V f \otimes \text{id}_{V^*}) b_V \in \mathbb{C}$. It obeys $\text{tr}(fg) = \text{tr}(gf)$ (whenever both sides are defined) and $\text{tr}(\text{id}_1) = 1$.

Out of the trace one then defines the *quantum dimensions* of simple objects: $\dim(i) = \text{tr}(\text{id}_i)$, and the *modular matrix* with matrix elements $s_{i,j} = \text{tr}(c_{j,i} c_{i,j})$. These numbers depend only on the isomorphism classes of the simple objects i, j . One shows

* We usually denote simple objects by lower case letters i, j, \dots , and general objects by capital letters U, V, \dots

that, as a consequence of the axioms, $\dim(i) = \dim(i^*) \neq 0$, $\dim(\mathbf{1}) = 1$, and that the modular matrix obeys $s_{i,j} = s_{j,i} = s_{i^*,j^*}$, $s_{i,\mathbf{1}} = \dim(i)$.

On the simple modules i , the isomorphism θ_i acts as a scalar $v_i \neq 0$ times the identity.

A rank of a modular category is a number $\mathcal{D} \in \mathbb{C}$ such that $\mathcal{D}^2 = \sum_{i \in I} \dim(i)^2$. In the following, we shall assume that a rank has been fixed. Related to the rank is the charge κ of the modular category. It is defined by the formula $\kappa = \mathcal{D}^{-1} \sum_{j \in I} v_j^{-1} \dim(j)^2$. The charge appears in the description of the framing anomaly.

Out of $s_{i,j}$, v_i , \mathcal{D} one constructs a projective representation of the mapping class group of the torus, see 2.5.2 below.

EXAMPLES. (1) For any positive integer N , let \mathcal{M}_N be the category with object set \mathbb{Z} and morphism spaces $\text{Hom}(j, k) = \mathbb{C}$ if $j \equiv k \pmod{2N}$ and $\text{Hom}(j, k) = \{0\}$ otherwise. The composition of morphisms is the multiplication of complex numbers. The tensor structure is defined on objects as $j \otimes k = j + k$ and on morphisms as $f \otimes g = (-1)^{js} fg$, if $f \in \text{Hom}(j, j + 2Nk)$, $g \in \text{Hom}(r, r + 2Ns)$. The unit object is $\mathbf{1} = 0$. This category can be made into a modular category with $I = \{0, \dots, 2N - 1\}$ as follows. Let $\zeta = \exp(i\pi/2N)$. We set $j^* = -j$, $d_j = b_j = 1$, $\theta_j = \zeta^{-j^2}$ and $c_{j,k} = \zeta^{-jk}$. Then

$$v_j = \zeta^{-j^2}, \quad s_{j,k} = \zeta^{-2jk}, \quad \mathcal{D} = \sqrt{2N}, \quad \kappa = e^{i\pi/4}.$$

In particular, $\dim(j) = 1$ for all objects j . An equivalent category appears in conformal field theory and is related to the free boson field with values in a circle of radius $1/\sqrt{2N}$.

(2) The ‘purified’ category of representations of the quantum group $U_q(\mathfrak{sl}_2)$ with $q = \exp(\pi i/(\ell + 2))$, which is related to the $SU(2)$ WZW model at ℓ , has $\ell + 1$ simple objects $0, \dots, \ell$ up to isomorphism. In this case we have

$$v_j = e^{-\pi i \frac{j(j+2)}{2(\ell+2)}}, \quad s_{j,k} = \frac{\sin\left(\frac{\pi(j+1)(k+1)}{\ell+2}\right)}{\sin\left(\frac{\pi}{\ell+2}\right)}, \quad \mathcal{D} = \frac{\sqrt{\ell+2}}{\sin\left(\frac{\pi}{\ell+2}\right)}, \quad \kappa = e^{\frac{3\pi i \ell}{4(\ell+2)}}.$$

2.2. SPACES OF MORPHISMS

Let U, V, W be objects of a modular category. Then we have linear isomorphisms

$$\begin{aligned} \text{Hom}(U \otimes V, W) &\rightarrow \text{Hom}(U, W \otimes V^*), \\ \text{Hom}(U \otimes V, W) &\rightarrow \text{Hom}(V, U^* \otimes W), \end{aligned}$$

given by

$$\phi \mapsto (\phi \otimes \text{id}_{V^*}) \circ (\text{id}_U \otimes b_V), \quad \phi \mapsto (\text{id}_{U^*} \otimes \phi) \circ (b_{U^*} \otimes \text{id}_V), \tag{1}$$

respectively. In particular, we have an isomorphism

$$\text{Hom}(i_1 \otimes \dots \otimes i_n, j_1 \otimes \dots \otimes j_m) \rightarrow \text{Hom}(\mathbf{1}, i_n^* \otimes \dots \otimes i_1^* \otimes j_1 \otimes \dots \otimes j_m),$$

for any simple objects i_1, \dots, j_m .

For $n + m = 3$, it is then sufficient to consider the space

$$H^{i,j,k} = \text{Hom}(\mathbf{1}, i \otimes j \otimes k).$$

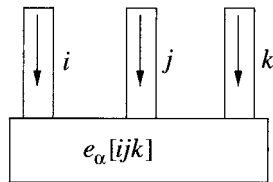
We have a nondegenerate pairing $\langle \cdot, \cdot \rangle: H^{k^*j^*i^*} \otimes H^{i,j,k} \rightarrow \mathbb{C}$, given by

$$\langle \phi, \psi \rangle = d_k \circ (\text{id} \otimes d_j \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes d_i \otimes \text{id} \otimes \text{id}) \circ (\phi \otimes \psi).$$

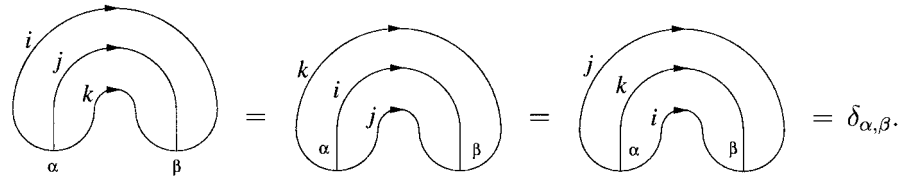
It is useful to fix bases $(e_\alpha[ijk], \alpha = 1, \dots, N^{i,j,k})$ of the spaces $H^{i,j,k}$ for i, j, k simple objects, so that

$$\langle e_\alpha[k^*j^*i^*], e_\beta[ijk] \rangle = \delta_{\alpha,\beta}. \tag{2}$$

Graphically, a basis element $e_\alpha[ijk]$ is represented by a coupon



or in the simplified blackboard framing notation by a trivalent vertex with a label α drawn where the bottom of the coupon should be. The relation (2) may then be written as



Using the isomorphisms above, the bilinear pairing may also be formulated as a pairing between $H_{i,j}^k = \text{Hom}(i \otimes j, k)$ and $H_k^{i,j} = \text{Hom}(k, i \otimes j)$ given by the trace: $\langle \phi, \psi \rangle = \text{tr}(\phi\psi)$. The dimensions (Verlinde numbers) of $\text{Hom}(i \otimes j, k)$ will be denoted by $N_{i,j}^k$. Similar notations are used for the other spaces. Thus $N^{i,j,k} = \dim(H^{i,j,k}) = N_{i^*j^*}^{k^*}$ and so on.

The first computations with these bases are the following two identities. The first identity is

$$\begin{array}{c} \alpha \\ \downarrow i \\ \circlearrowleft \\ \uparrow i \\ \beta \end{array} = \delta_{\alpha,\beta} \frac{1}{\dim(i)} \begin{array}{c} \downarrow i \end{array}, \tag{3}$$

or $e_\alpha[ik^*j^*]e_\beta[jki^*] = \delta_{\alpha,\beta} \dim(i)^{-1} \text{id}_i$, if $e_\alpha[ik^*j^*]$, $e_\beta[jki^*]$ are identified via (1) with basis elements of $H_{j,k}^i$, $H_i^{j,k}$, respectively. The second identity is

$$\begin{array}{c} \downarrow i \\ \downarrow j \end{array} = \sum_{k \in I} \sum_{\alpha=1}^{N_{j,k^*,i}} \dim(k) \begin{array}{c} i \quad j \\ \searrow \quad \swarrow \\ \alpha \\ \downarrow k \\ \swarrow \quad \searrow \\ \alpha \\ i \quad j \end{array}, \tag{4}$$

whose meaning is

$$\text{id}_i \otimes \text{id}_j = \sum_{k \in I, \alpha} \dim(k) e_\alpha[jk^*i] e_\alpha[i^*kj^*],$$

where $e_\alpha[jk^*i]$ is identified with a basis element of $H_k^{i,j}$ and $e_\alpha[i^*kj^*]$ with a basis element of $H_{i,j}^k$.

The first identity can be obtained by noticing that both sides of the equations are elements of the one-dimensional vector space $\text{Hom}(i, i) = \mathbb{C} \text{id}_i$, so they are proportional. The constant of proportionality is computed by taking the trace on both sides.

The second identity follows from the domination axiom (Appendix A(xi)) which implies that the left-hand side can be expressed as a linear combination of the morphisms on the right-hand side, possibly with different basis elements e_α, e_β in the factors. To compute the coefficients $c_{\alpha,\beta}(k)$ in

$$\text{id}_i \otimes \text{id}_j = \sum_{k \in I, \alpha, \beta} c_{\alpha,\beta}(k) e_\alpha[jk^*i] e_\beta[i^*kj^*],$$

we compose both sides of the equation with $e_\gamma[i^*lj^*] \in H_{i,j}^l$. Since $\text{Hom}(k, l) = 0$ for $l \neq k$, the only term contributing to the sum over k is the one with $k = l$. Using the first identity, we get $e_\gamma = \sum_{\alpha,\beta} c_{\alpha,\beta}(l) \delta_{\alpha\gamma} \dim(l)^{-1} e_\beta$, from which the result $c_{\alpha,\beta}(l) = \dim(l) \delta_{\alpha,\beta}$ follows.

2.3. FROBENIUS–SCHUR INDICATORS

A *self-dual object* in a modular category is an object isomorphic to its dual object. To each simple self-dual object one associates a scalar squaring to one, called the

Frobenius–Schur indicator. Its role in conformal field theory was first emphasized in [B]. It is a generalization of a classical notion in group representation theory: if V is an irreducible representation of a group G then there is an action of \mathbb{Z}_2 on the space of invariants $(V \otimes V)^G$ by permutation of factors. This space is at most one-dimensional. If it is nontrivial, the generator of \mathbb{Z}_2 acts by multiplication by the Frobenius–Schur indicator.

In the case of a general modular category, the Frobenius–Schur indicator is defined as follows. Suppose V is a simple self-dual object and $\phi: V^* \mapsto V$ is an isomorphism. Then the Frobenius–Schur indicator is the factor of proportionality $v(V)$ in the identity

$$(\theta_{V^*} \otimes \text{id}_V) c_{V, V^*} b_V = v(V) (\phi^{-1} \otimes \phi) b_V,$$

between two nonzero elements of the one-dimensional space $\text{Hom}(\mathbf{1}, V^* \otimes V)$. Since V is simple, ϕ is unique up to a nonzero scalar, so $v(V)$ is independent of the choice of isomorphism ϕ .

LEMMA 2.1. *Let V be a simple self-dual object. Then*

- (i) $v(\mathbf{1}) = 1$.
- (ii) $v(V)^2 = 1$.
- (iii) *If V is isomorphic to W then W is simple and self-dual and $v(V) = v(W)$. In particular $v(V^*) = v(V)$.*

Proof: (i) follows from $\theta_{\mathbf{1}} = \text{id}_{\mathbf{1}}$, $c_{\mathbf{1}, \mathbf{1}} = \text{id}_{\mathbf{1}} \otimes \text{id}_{\mathbf{1}}$. To prove (ii), let us act on the equation defining v with the morphism $(\theta_V \otimes \text{id}_{V^*}) c_{V^*, V}$. By the naturality of the braiding, the left-hand side becomes $c_{V^*, V} c_{V, V^*} (\theta_V \otimes \theta_{V^*}) b_V$, which, by the twist axiom (Appendix A, (iv)), is equal to $\theta_{V \otimes V^*} b_V = b_V \theta_{\mathbf{1}} = b_V$. The right-hand side is

$$\begin{aligned} & (\theta_V \otimes \text{id}_{V^*}) c_{V^*, V} v(V) (\phi^{-1} \otimes \phi) b_V \\ &= v(V) (\phi \otimes \phi^{-1}) (\theta_V \otimes \text{id}_{V^*}) c_{V, V^*} b_V \\ &= v(V)^2 b_V. \end{aligned}$$

Thus $v(V)^2 = 1$.

To prove (iii), let us introduce the dual morphism $f^*: W^* \mapsto V^*$ of a morphism

$$f: V \mapsto W : f^* = (d_W \otimes \text{id})(\text{id} \otimes f \otimes \text{id})(\text{id} \otimes b_V).$$

It is easy to see (by drawing the corresponding graphs) that $\text{id}_V^* = \text{id}_{V^*}$ and that $(fg)^* = g^* f^*$, whenever the composition of the morphisms f, g is defined. In particular, f^* is an isomorphism if and only if f is an isomorphism. Moreover, $b_W = f \otimes (f^*)^{-1} b_V$ if $f: V \mapsto W$ is an isomorphism. Using the naturality of twist and braiding, we act by $(f^*)^{-1} \otimes f$ on the equation defining $v(V)$ and get

$$\begin{aligned} (\theta_{W^*} \otimes \text{id}_W) c_{W, W^*} b_W &= v(V) (f^{*-1} \phi^{-1} \otimes f \phi) b_V \\ &= v(V) ((f \phi f^*)^{-1} \otimes f \phi f^*) b_W, \end{aligned}$$

Since $f\phi f^*: W^* \mapsto W$ is an isomorphism, it follows that W is self-dual (it is clear that W is simple) and that $v(W) = v(V)$. □

2.4. FUSING MATRICES (6J-SYMBOLS)

A modular category is in principle determined by a set of numerical data. Additionally to the modular matrix and the scalars v_i , one has to specify the 6j-symbols. Let i, j, k, l, m, n be simple objects (in the applications they are either elements of I or duals of elements of I). Then one shows that the linear homomorphisms

$$\Phi : \bigoplus_{l \in I} H_l^{i,j} \otimes H_m^{l,k} \rightarrow \text{Hom}(m, i \otimes j \otimes k),$$

given by $\phi \otimes \psi \mapsto (\phi \otimes \text{id}_k) \circ \psi$, and

$$\Phi' : \bigoplus_{n \in I} H_n^{i,k} \otimes H_m^{j,n} \rightarrow \text{Hom}(m, i \otimes j \otimes k),$$

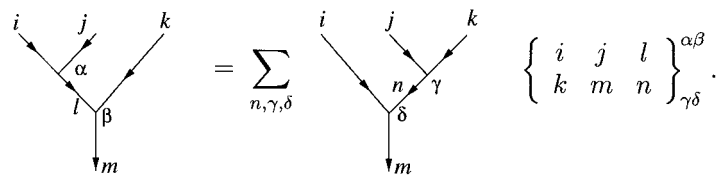
given by $\phi \otimes \psi \mapsto (\text{id}_i \otimes \phi) \circ \psi$, are isomorphisms. Therefore we have an isomorphism $(\Phi')^{-1} \circ \Phi$. Its components are the 6j-symbols:

$$\left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\} : H_l^{i,j} \otimes H_m^{l,k} \rightarrow H_n^{j,k} \otimes H_m^{i,n}.$$

The matrix elements of the 6j-symbols with respect to the above bases are defined by

$$\begin{aligned} & \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\} e_\alpha[l^*ij] \otimes e_\beta[m^*lk] \\ &= \sum_{\gamma, \delta} \left\{ \begin{matrix} i & j & k \\ l & m & n \end{matrix} \right\}_{\gamma\delta}^{\alpha\beta} e_\gamma[n^*jk] \otimes e_\delta[m^*in]. \end{aligned}$$

Graphically, these matrix elements are defined by



We also introduce inverse 6j-symbols as the components of the inverse map $\Phi^{-1} \circ \Phi'$:

$$\begin{array}{c} i \\ \swarrow \\ \delta \\ \downarrow \\ m \end{array} \begin{array}{c} j \\ \swarrow \\ n \\ \downarrow \\ m \end{array} \begin{array}{c} k \\ \swarrow \\ \gamma \\ \downarrow \\ m \end{array} = \sum_{l, \alpha, \beta} \begin{array}{c} i \\ \swarrow \\ \alpha \\ \downarrow \\ m \end{array} \begin{array}{c} j \\ \swarrow \\ l \\ \downarrow \\ m \end{array} \begin{array}{c} k \\ \swarrow \\ \beta \\ \downarrow \\ m \end{array} \overline{\left\{ \begin{array}{ccc} i & j & l \\ k & m & n \end{array} \right\}}_{\alpha\beta, \gamma\delta} .$$

2.5. THE THREE-DIMENSIONAL TOPOLOGICAL FIELD THEORY

To every modular category one can associate a three-dimensional topological field theory. It is a formalization and generalization of the Chern–Simons path integral of [W]. The TFT associates a finite-dimensional vector space $\mathcal{H}(X)$ (the space of conformal blocks) to each surface X with marked points and some additional structure, and an element of $\mathcal{H}(X)$ to each three-dimensional manifold with a graph of Wilson lines bounding X .

To deal properly with the ‘framing anomaly’, we need to endow surfaces with additional structures and use ribbons instead of Wilson lines. We start by introducing the definitions, following [T].

An *extended surface* is an oriented closed 2-manifold X with a finite set of disjoint oriented embedded arcs labeled by simple objects, and a Lagrangian subspace $\lambda(X)$ of the first homology group $H_1(X, \mathbb{R})$. A homeomorphism of extended surfaces $f: X \mapsto Y$ is an orientation preserving homeomorphism mapping arcs to arcs with the same label and the same orientation. A homeomorphism $f: X \rightarrow Y$ of extended surfaces will be called *strong* if it also maps $\lambda(X)$ to $\lambda(Y)$ *. The *opposite* $-X$ of an extended surface is the surface X with opposite orientation and the same arcs, so that if an arc of X is labeled by a simple object i then it has opposite orientation and it is labeled by i^* in $-X$.

A *cobordism of extended surfaces* is a triple $(M, \partial_- M, \partial_+ M)$ such that

- (1) M is a three-dimensional manifold with boundary containing a ribbon graph**. A ribbon graph consists of ribbons, annuli and coupons as in 2.1, but the ribbons and annuli are labeled by simple objects only. Ribbon ends are glued to coupons or are contained in the boundary ∂M .
- (2) $\partial_{\pm} M$ are disjoint disconnected subsets of the boundary ∂M so that $\partial M = \partial_+ M \cup (-\partial_- M)$, endowed with Lagrangian subspaces of their first homology groups. The marked arcs at which the ribbons in M end are given the label

* As we rarely use strong homeomorphisms we depart here slightly from the notation of [T]: there a homeomorphism is called weak homeomorphism and a strong homeomorphism is called homeomorphism.

** Instead of ribbon graphs, one often considers framed graphs, whose edges (assumed to be smoothly embedded in a smooth manifold) come with a normal vector field. A ribbon graph can be made into a framed graph by taking a vector field normal to the ribbons. The present approach [T] works also in the topological category.

of the ribbons whose core is oriented inwards, and the dual label otherwise. The Lagrangian subspaces and the oriented labeled arcs give $\partial_{\pm}M$ the structure of extended surfaces.

We say that $(M, \partial_-M, \partial_+M)$ is a cobordism from ∂_-M to ∂_+M .

The TFT* (Z, \mathcal{H}) associated to a modular category \mathcal{M} over \mathbb{C} consists of the following data.

- (i) For each extended surface X there is a finite-dimensional complex vector space $\mathcal{H}(X)$, the space of states (or of conformal blocks), such that $\mathcal{H}(\emptyset) = \mathbb{C}$ and $\mathcal{H}(X \sqcup Y) = \mathcal{H}(X) \otimes \mathcal{H}(Y)$.
- (ii) To each homeomorphism of extended surfaces $f: X \mapsto Y$ there is an isomorphism $f_{\sharp}: \mathcal{H}(X) \mapsto \mathcal{H}(Y)$.
- (iii) If $(M, \partial_-M, \partial_+M)$ is a cobordism of extended surfaces, then the TFT associates to it a homomorphism

$$Z(M, \partial_-M, \partial_+M) : \mathcal{H}(\partial_-M) \rightarrow \mathcal{H}(\partial_+M)$$

depending linearly on the labels of the coupons.

These data obey the following axioms.

- (1) (Naturality) Let $(M, \partial_-M, \partial_+M)$, $(N, \partial_-N, \partial_+N)$ be cobordisms of extended surfaces. Let $f: M \rightarrow N$ be a degree one homeomorphism mapping the ribbon graph in M onto the ribbon graph in N , restricting to homeomorphisms $f_{\pm}: \partial_{\pm}M \mapsto \partial_{\pm}N$ preserving the Lagrangian subspaces. Then

$$(f_+)_{\sharp} \circ Z(M, \partial_-M, \partial_+M) = Z(N, \partial_-N, \partial_+N) \circ (f_-)_{\sharp}$$

- (2) (Multiplicativity) If M_1, M_2 are two cobordisms of extended surfaces, then under the identification

$$\mathcal{H}(\partial_{\pm}M_1 \sqcup \partial_{\pm}M_2) = \mathcal{H}(\partial_{\pm}M_1) \otimes \mathcal{H}(\partial_{\pm}M_2)$$

we have $Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2)$.

- (3) (Functoriality) Suppose a cobordism M is obtained from the disjoint union of M_1 and M_2 by gluing ∂_+M_1 to ∂_-M_2 along a degree one homeomorphism $f: \partial_+M_1 \rightarrow \partial_-M_2$ preserving marked arcs with their orientation. Then

$$\begin{aligned} Z(M, \partial_-M_1, \partial_+M_2) \\ = \kappa^m Z(M_2, \partial_-M_2, \partial_+M_2) \circ f_{\sharp} \circ Z(M_1, \partial_-M_1, \partial_+M_1), \end{aligned}$$

for some integer m .

- (4) (Normalization) Let X be an extended surface. Let the *cylinder over X* be the 3-manifold $X \times [-1, 1]$, with the ribbon graph consisting of the ribbons $z \times [-1, 1]$, where z runs over the marked arcs of X . Their orientation is such

* The notation in [T] for this TFT is (τ^e, T^e) .

that they induce the orientation of the arcs on $X \times \{1\}$. Their core is oriented from 1 to -1 . Then

$$Z(X \times [-1, 1], X \times \{-1\}, X \times \{1\}) = \text{id}_{\mathcal{H}(X)}. \quad (5)$$

The homomorphism $Z(M, \partial_- M, \partial_+ M)$ is called the *invariant* of the cobordism of extended surfaces $(M, \partial_- M, \partial_+ M)$. By the naturality axiom it is invariant under degree one homeomorphisms that restrict to the identity on the boundary.

Moreover, the invariant does not change if we remove an edge with label **1** or we replace the label of an edge by its dual and reverse the orientation of its core.

The TFT gives the system of vector spaces $\mathcal{H}(X)$ the structure of a *modular functor*.

Next we list some of the properties of the modular functor \mathcal{H} which we shall use.

2.5.1. Duality

The space $\mathcal{H}(-X)$ associated to the opposite of the extended surface X is canonically isomorphic to the dual space to $\mathcal{H}(X)$. The isomorphism is induced by the pairing

$$Z(X \times [-1, 1], X \sqcup (-X), \emptyset) : \mathcal{H}(X) \otimes \mathcal{H}(-X) \rightarrow \mathbb{C}. \quad (6)$$

Here X is identified with $X \times \{1\}$ and $-X$ with $X \times \{-1\}$.

2.5.2. Mapping Class Group

The action $f \mapsto f_{\sharp}$ of homeomorphisms may be expressed in terms of the TFT. Namely, let $f: X \rightarrow Y$ be a homeomorphism of extended surfaces. Then the 3-manifold obtained by gluing the cylinder over X to the cylinder over Y defines a cobordism (M_f, X, Y) . The normalization and functoriality axioms then imply that $f_{\sharp} = Z(M_f, X, Y)$. Moreover, it can be shown, using the naturality axiom, that if f, g are homotopic in the class of homeomorphism of extended surfaces, then $f_{\sharp} = g_{\sharp}$. In particular, if $X = Y$, $f \mapsto f_{\sharp}$ defines a projective representation of the mapping class group of X .

EXAMPLE. Let X be a torus with no marked arcs. View X as the boundary of a solid torus $H = D^2 \times S^1$ and take $\lambda = \lambda(X)$ to be the kernel of the map induced by the inclusion $X \hookrightarrow H$. Then a basis of $\mathcal{H}(X)$ is given by

$$\chi_j(X) = Z((H, j), \emptyset, X),$$

$j \in I$ where (H, j) is H with a ribbon graph consisting of an untwisted annulus $[-\varepsilon, \varepsilon] \times S^1$ with label j . Let $S(z, w) = (w^{-1}, z)$, $T(z, w) = (zw, z)$ be the standard generators of the mapping class group $SL(2, \mathbb{Z})$ of $X = S^1 \times S^1$. Then S_{\sharp}, T_{\sharp} are represented in this basis by the matrices $S = (\mathcal{D}^{-1} s_{i,j})$ and $\hat{T} = (v_i^{-1} \delta_{i,j})$ respectively. The map $f \mapsto f_{\sharp}$ is a projective representation of the mapping class group: the

matrices S and \hat{T} obey the relations $S^4 = 1, (\hat{T}S)^3 = \kappa S^2$. Moreover S^2 is the matrix (δ_{ij^*}) . Note that if we choose a third root of κ , a genuine representation may be obtained by replacing \hat{T} by $T = (\kappa^{-1/3} v_i^{-1} \delta_{ij})$.

2.5.3. *Gluing Homomorphisms*

If X is an extended surface with arcs γ, γ' labeled by i, i^* , let X' be an extended surface obtained as follows: Let $\phi, \phi': D^2 \rightarrow X$ be orientation preserving disjoint embeddings of the unit disk $D^2 \subset \mathbb{C}$ such that their restriction to $[-1, 1]$ are parametrizations of the oriented arcs γ, γ' . Then X' is obtained from X by removing the interiors of the disks $\phi(D^2), \phi'(D^2)$ and gluing their boundaries by identifying $\phi(z)$ with $\phi'(-\bar{z})$, for $z \in S^1$. The arcs of X' are the remaining arcs of X and the Lagrangian subspace $\lambda(X')$ consists of images in X' of homology classes of cycles in $X - (\phi(D^2) \cup \phi'(D^2))$ which are mapped by the inclusion to cycles in $\lambda(X)$.

Then one has a *gluing homomorphism* $g_{X,X'}: \mathcal{H}(X) \mapsto \mathcal{H}(X')$.

The gluing homomorphism is obtained from the TFT: Let M be the 3-manifold obtained from $X \times [-1, 1]$, the cylinder over X , by gluing $\phi(D^2) \times \{1\}$ to $\phi'(D^2) \times \{1\}$ via the identification $(\phi(z), 1) = (\phi'(-\bar{z}), 1), z \in D^2$. Let the ribbon graph in M be obtained from the ribbon graph in $X \times [-1, 1]$ by replacing the part of the ribbon through the glued disks by a narrower one, so as to fit inside M . Then M has boundary $-X \sqcup X'$ and defines a cobordism of extended surfaces from X to X' . Then $g_{X,X'} = Z(M, X, X')$. It is then known that the gluing homomorphism has the following completeness property.

If X_j is the extended surface X as above, but with γ labeled by j and γ' labeled by j^* , then the sum of gluing homomorphisms $\bigoplus_{j \in I} \mathcal{H}(X_j) \rightarrow \mathcal{H}(X')$ is an isomorphism.

2.5.4. *Description of $\mathcal{H}(X)$ as a Vector Space*

Let X be a 2-sphere with n marked arcs labeled by simple objects j_1, \dots, j_n . Let M be a 3-ball with boundary X and a ribbon graph consisting of one vertex connected to the marked arcs by n ribbons. Then $Z(M, \emptyset, X)$ depends linearly on the label of the vertex and is thus a linear map

$$Z(M, \emptyset, X) : \text{Hom}(\mathbf{1}, j_1 \otimes \dots \otimes j_n) \rightarrow \mathcal{H}(X)$$

By construction of the TFT out of the modular category, this map is an isomorphism. Combining this result with the completeness of the gluing map, one deduces, by attaching handles to the sphere, that $\mathcal{H}(X) \simeq \bigoplus_{k_1, \dots, k_g \in I} \text{Hom}(\mathbf{1}, j_1 \otimes \dots \otimes j_n \otimes \bigotimes_{s=1}^g (k_s \otimes k_s^*))$ for a surface of genus g with n marked points. Under this identification, the invariant of a cobordism changes in a covariant way if we replace the labels of the edges by equivalent simple objects.

2.6. TRACE FORMULA

One important variant of the functoriality axiom is a formula [T] for the invariant of a closed 3-manifold of the form $M = X \times S^1$, for a closed oriented surface X , with

some ribbon graph Γ . We may obtain M by gluing the two components of the boundary of $N = X \times [0, 1]$. Then the ribbon graph intersects the boundary along arcs and $\partial_+ N = X \times \{1\}$ becomes an extended surface. $\partial_- N = X \times \{0\}$ is then canonically strongly homeomorphic to the same extended surface, and the *trace formula* holds:

$$Z(M, \emptyset, \emptyset) = \text{Tr}_{\mathcal{H}(\partial_+ N)}(Z(N, \partial_- N, \partial_+ N)). \tag{7}$$

2.7. FRAMING ANOMALY

We give here the formula for the integer m appearing in the functoriality axiom, following Section IV.7 of [T]. It is given in terms of Maslov indices, which we proceed to define.

Let H be a symplectic real vector space with symplectic form ω , and $\lambda_1, \lambda_2, \lambda_3 \subset H$ be Lagrangian subspaces. Then on the subspace $(\lambda_1 + \lambda_2) \cap \lambda_3$ we have a quadratic form $Q(x) = \omega(x_2, x)$, where $x = x_1 + x_2$ with $x_1 \in \lambda_1, x_2 \in \lambda_2$ ($Q(x)$ does not depend on the choice of the decomposition of x). The *Maslov index* $\mu(\lambda_1, \lambda_2, \lambda_3)$ is by definition the signature of Q . It is a function which is antisymmetric under permutations of its three arguments, and in particular vanishes if any two arguments coincide.

If X is an oriented closed 2-manifold, the intersection form on $H_1(X, \mathbb{R})$ is symplectic. Moreover, if M is a 3-manifold with boundary and $\partial M = \partial_+ M \sqcup \partial_- M$ is a decomposition of the boundary into closed disjoint subsets, then we have a map N_* from the set of the Lagrangian subspaces of $H_1(\partial_- M, \mathbb{R})$ to the set of Lagrangian subspaces of $H_1(\partial_+ M, \mathbb{R})$: $x \in N_*(\lambda)$ if and only if there exists an $x' \in \lambda$ so that $x - x'$ is homologous to zero as a cycle in M . Similarly we have a map N^* sending Lagrangian subspaces of $H_1(\partial_+ M, \mathbb{R})$ to the Lagrangian subspaces of $H_1(\partial_- M, \mathbb{R})$. Then the integer m appearing in the functoriality property is

$$m = \mu(f_* N_* \lambda(\partial_- M_1), f_* \lambda(\partial_+ M_1), N^* \lambda(\partial_+ M_2)) + \mu(f_* \lambda(\partial_+ M), \lambda(\partial_- M_2), N^* \lambda(\partial_+ M_2)).$$

The following property is useful for surfaces with orientation reversing involutions, such as doubles.

LEMMA 2.2. *Let H be a real symplectic vector space with symplectic form ω . Suppose $\sigma \in \text{End}_{\mathbb{R}}(H)$ is an involution such that $\sigma^* \omega = -\omega$. If $\lambda_1, \lambda_2, \lambda_3$ are Lagrangian subspaces invariant under σ , then*

$$\mu(\lambda_1, \lambda_2, \lambda_3) = 0.$$

Proof. Let x be an element of the invariant space $(\lambda_1 + \lambda_2) \cap \lambda_3$. If $x = x_1 + x_2$ with $x_i \in \lambda_i$, then $\sigma(x) = \sigma(x_1) + \sigma(x_2)$ is a decomposition of $\sigma(x)$ into a sum of elements of λ_1, λ_2 . Thus

$$Q(\sigma(x)) = \omega(\sigma(x_2), \sigma(x)) = -\omega(x_2, x) = -Q(x).$$

On the other hand, the signature is an invariant, so the signature of Q is equal to the signature of $Q \circ \sigma = -Q$. Thus the signature vanishes. \square

3. Boundary Conditions and Correlation Functions

3.1. THE DOUBLE OF A SURFACE

Suppose that X is a two-dimensional compact manifold with boundary, possibly nonorientable. Then X may be identified with \hat{X}/\mathbb{Z}_2 for a closed oriented manifold \hat{X} , the *double* of X , with an orientation reversing action of the generator of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The double is constructed by taking the total space of the orientation bundle $p: \text{Or}(X) \rightarrow X$ (the \mathbb{Z}_2 -bundle over X whose fiber at x consists of the two orientations of the tangent plane at x) and identifying the two points of each fiber over the boundary: $\hat{X} = \text{Or}(X)/\sim$ with $x \sim x'$ iff $p(x) = p(x') \in \partial X$. The double comes with a projection $p: \hat{X} \rightarrow X$ and an orientation reversing involution $\sigma: \hat{X} \rightarrow \hat{X}$ exchanging the two sheets and defining the action of \mathbb{Z}_2 .

Here are some examples. If X is closed and orientable, then \hat{X} consists of two copies of X with opposite orientations. If X is orientable with nonempty boundary, \hat{X} is obtained by taking two copies of X with opposite orientations and gluing the two copies along the boundary. So if X is a disk, then \hat{X} can be viewed as the unit sphere S^2 in \mathbb{R}^3 with \mathbb{Z}_2 action generated by the reflection at the x - y plane. If X is the real projective plane $\mathbb{R}P^2$ then \hat{X} is S^2 with \mathbb{Z}_2 action given by the antipodal map $x \mapsto -x$ of $S^2 \subset \mathbb{R}^3$. The annulus $X = S^1 \times [-1, 1]$ is \hat{X}/\mathbb{Z}_2 with $\hat{X} = S^1 \times S^1$ and involution $(\theta_1, \theta_2) \mapsto (\theta_1, -\theta_2)$, $\theta_i \in \mathbb{R}/2\pi\mathbb{Z} = S^1$. Taking $\hat{X} = S^1 \times S^1$ with involution $(\theta_1, \theta_2) \mapsto (-\theta_1, \theta_2 + \pi)$ gives the Klein bottle. The involution $(\theta_1, \theta_2) \mapsto (\theta_2, \theta_1)$ gives the Möbius strip.

3.2. THE CASE OF CLOSED ORIENTABLE SURFACES

Let us first consider the case of closed orientable surfaces. Suppose X is closed and orientable, and choose an orientation of X . Then the double of X is $\hat{X} = X \sqcup (-X)$, the disjoint union of two copies of X with opposite orientations. The involution exchanges the two copies. Let X be endowed with n distinct points z_1, \dots, z_n on it, labeled by simple objects i_1, \dots, i_n . To these data one associates a correlation function $C(X) \in \mathcal{H}(\hat{X})$.

To be more precise, we should take care of the framing: so z_1, \dots, z_n should be taken as disjoint arcs rather than points. Also $\mathcal{H}(\hat{X})$ is only unambiguously defined if \hat{X} is given a Lagrangian subspace λ in its first homology group with real coefficients. As will be clear below, a convenient choice is to take λ to consist of $a \oplus (-a) \in H_1(\hat{X}, \mathbb{R}) = H_1(X, \mathbb{R}) \oplus H_1(X, \mathbb{R})$, where a runs over $H_1(X, \mathbb{R})$. We call this Lagrangian subspace *canonical Lagrangian subspace* and denote it by $\lambda_-(\hat{X})$.

The natural candidate for $C(X)$ is then the element of $\mathcal{H}(\hat{X})$ associated to the 3-manifold $X \times [-1, 1]$, with ribbon graph consisting of $z_i \times [-1, 1]$, where z_i runs

over the marked arcs on X :

$$C(X) = Z(X \times [-1, 1], \emptyset, \hat{X}).$$

(see 2.5 for the orientations of the ribbons in $X \times [-1, 1]$). Note that the canonical Lagrangian subspace is the kernel of the map induced by inclusion of the boundary.

Let us check that this ansatz obeys the modular and factorization properties one expects for correlation functions.

Let $f: X \mapsto X$ be a degree one homeomorphism and let $\hat{f}: \hat{X} \rightarrow \hat{X}$ be equal to f on each of the two copies of X . The lift \hat{f} of f is the unique degree one homeomorphism of \hat{X} commuting with the involution and projecting to f .

THEOREM 3.1 (Modular invariance). *Let X be a closed oriented surface with labeled arcs z_1, \dots, z_n . Let $f: X \mapsto X$ be a degree one homeomorphism preserving the marked arcs. Let \hat{f} be its lift to \hat{X} . Then $f_{\#} C(X) = C(X)$.*

Proof. Let $F: X \times [-1, 1] \mapsto X \times [-1, 1]$ be defined by $F(x, t) = (f(x), t)$. F is a homeomorphism that restricts to \hat{f} on the boundary. Moreover, \hat{f}_* clearly preserves the Lagrangian subspace. Thus the naturality axiom applies and we get

$$\hat{f}_{\#} Z(X \times [-1, 1], \emptyset, \hat{X}) = Z(X \times [-1, 1], \emptyset, \hat{X}),$$

proving the claim. □

It is useful to express the correlation function for more general Lagrangian subspaces. Let us say that a Lagrangian subspace λ of $H_1(\hat{X}, \mathbb{R})$ is *symmetric* if $\sigma_* \lambda = \lambda$. The Lagrangian subspace $\lambda_-(\hat{X})$ has this property. Let us define, for any symmetric Lagrangian subspace λ ,

$$C_{\lambda}(X) = (\text{id}_{\lambda, \lambda_-})_{\#} C(X).$$

Here $\text{id}_{\lambda_1, \lambda_2}$ denotes the identity map between extended surfaces which differ only in their distinguished Lagrangian subspaces. Then we get the more general modularity property: $\hat{f}_{\#} C_{\lambda}(X) = C_{\lambda}(X)$. This formula follows from the functoriality formula, except that we have to check that the framing anomaly term is trivial. The reason for this is that all Lagrangian subspaces appearing in the calculation of the Maslov indices are invariant under σ_* . Therefore the Maslov indices vanish by Lemma 2.2.

In particular, we may choose the symmetric Lagrangian subspace $\lambda \oplus \lambda \in H_1(X \sqcup (-X), \mathbb{R}) = H_1(X, \mathbb{R}) \oplus H_1(X, \mathbb{R})$, for any Lagrangian subspace λ of $H_1(X, \mathbb{R})$. Under these circumstances we may identify $\mathcal{H}(\hat{X}, \lambda \oplus \lambda) = \mathcal{H}(X, \lambda) \otimes \mathcal{H}(-X, \lambda)$ and write

$$C_{\lambda \oplus \lambda}(X) = \sum_j b_j(X, \lambda) \otimes b_j(-X, \lambda),$$

for any basis $b_j(X, \lambda)$ of $\mathcal{H}(X, \lambda)$ and dual basis $b_j(-X, \lambda)$ with respect to the pairing (6). In this form, the modular invariance is less apparent.

EXAMPLE. Let $X = S^1 \times S^1$ be a torus with no marked arcs. Let $D^2 \subset \mathbb{C}$ be the unit disk and view X as the boundary of a solid torus $H = D^2 \times S^1$. Take $\lambda = \lambda(X)$ to be the kernel of the map induced by the inclusion $X \hookrightarrow H$. Then a basis of $\mathcal{H}(X)$ is given by $\chi_j(X) = Z((H, j), \emptyset, X), j \in I$, see 2.5.2. Let $S(z, w) = (w^{-1}, z), T(z, w) = (zw, z)$ be the standard generators of the mapping class group $SL(2, \mathbb{Z})$ of $X = S^1 \times S^1$. Then S_{\sharp}, T_{\sharp} are represented in this basis by the matrices $s = (\mathcal{D}^{-1} s_{i,j})$ and $t = (v_i^{-1} \delta_{i,j})$, respectively.

The map $\theta(z, w) = (\bar{z}, w)$ is a homeomorphism $X \rightarrow -X$, preserving orientation and Lagrangian subspace. Therefore a basis of $\mathcal{H}(-X)$ is $\chi_j(-X) = \theta_{\sharp} \chi_{j^*}(X)$. This basis is dual to the basis $(\chi_j)_{j \in I}$, as the pairing between $\chi_j(X)$ and $\chi_k(-X)$ is the invariant of $S^2 \times S^1$ with two annuli running along S^1 . This invariant is $\delta_{j,k}$, by the trace formula. Thus

$$C_{\lambda \oplus \lambda}(X) = \sum_{j \in I} \chi_j(X) \otimes \chi_j(-X).$$

As $\hat{S} \circ \theta = \theta \circ S^{-1}$ and $\hat{T} \circ \theta = \theta \circ T^{-1}, S_{\sharp}, T_{\sharp}$ are represented in the basis $(\chi_j(-X))_{j \in I}$ by the matrices $S_{\theta} = C S^{-1} C, \hat{T}_{\theta} = C \hat{T}^{-1} C$, with

$$C_{i,j} = \begin{cases} 1, & i \simeq j^*, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

(The anomaly does not contribute as in each Maslov index there are two coinciding arguments.) The statement of modular invariance thus reduces to ${}^1\hat{T} \hat{T}_{\theta} = {}^1S S_{\theta} = 1$. These identities may also be checked directly, using the relations of S, \hat{T} of 2.5.2, the fact that S is a symmetric matrix, and the relations $v_i = v_{i^*}, S^2 = C$.

We turn to the factorization properties of correlation functions. Let X be a closed oriented surface with marked oriented labeled arcs. Suppose two of the marked arcs carry label j, j^* respectively. Let X' be obtained by removing from X a small open disk around each of the two arcs, and gluing their boundaries by an orientation reversing homeomorphism. We say that we obtain X' by gluing X at the two marked arcs. Then \hat{X}' is obtained from \hat{X} by performing this operation twice, at each of the inverse images of the two arcs. Therefore we have a gluing homomorphism $g_{\hat{X}, \hat{X}'}: \mathcal{H}(\hat{X}) \mapsto \mathcal{H}(\hat{X}')$, which is the composition of the two gluing homomorphisms (in either order) with $(\text{id}_{\lambda', \lambda_{-}(\hat{X}')})_{\sharp}$. Here λ' is the symmetric Lagrangian subspace of $H_1(\hat{X}', \mathbb{R})$ obtained from the canonical Lagrangian subspace $\lambda_{-}(\hat{X})$ of \hat{X} by the gluing prescription of 2.5, and $\text{id}_{\lambda', \lambda_{-}(\hat{X}')}$ is the identity map from \hat{X}' with Lagrangian subspace λ' to \hat{X}' with canonical Lagrangian subspace.

THEOREM 3.2 (Factorization). *Let X be as in Theorem 3.1. Let X' be obtained from X by gluing X at two marked arcs with labels j, j^* . Let X_j be the surface X , with $j \in I$*

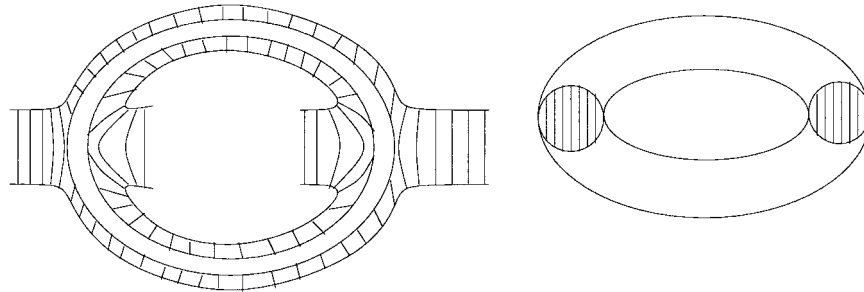


Figure 2.

arbitrary, and with all other labels fixed. Then

$$C(X') = \sum_{j \in I} \mathcal{D}^{-1} \dim(j) g_{\hat{X}_j, \hat{X}'} C(X_j).$$

Proof. Let $M_j = X_j \times [-1, 1]$. Then $g_{\hat{X}_j, \hat{X}'} C(X_j)$ is, by 2.5.3, the invariant $Z(M'_j, \emptyset, \hat{X}')$ of a cobordism. The 3-manifold M'_j is obtained from M_j by gluing two disks around the two chosen marked arcs on $X \times \{1\}$, and also on $X \times \{-1\}$. The ribbons ending at the marked arcs are then glued together to form an annulus.

It is not too hard to see that $X' \times [-1, 1]$ is homeomorphic to the manifold we obtain from M'_j by performing a surgery at this annulus. The surgery is, by definition, the following construction: we first parametrize a tubular neighborhood U of the annulus by a homeomorphism $\phi: D^2 \times S^1 \mapsto M'_j$ in such a way that the annulus is contained in $\phi([-1, 1] \times S^1)$. Then we glue $S^1 \times D^2$ to $M'_j - \text{int}(U)$ via the map ϕ restricted to $S^1 \times S^1$. Figure 2 is supposed to illustrate the homeomorphism of the resulting manifold with $X' \times [-1, 1]$: on the left, we embed the region of interest of $M'_j - \text{int}(U)$ in \mathbb{R}^3 and on the right we draw $S^1 \times D^2$. The fibers $\{x\} \times [-1, 1]$ in a plane section are drawn.

Let $M'' = M'_j - \text{int}(U)$ and $\chi_j(S^1 \times S^1) = Z((H, j), \emptyset, S^1 \times S^1)$, as in the example above. By the functoriality axiom,

$$Z(M'_j, \emptyset, \hat{X}') = Z(M', \partial U, \hat{X}') \circ \phi_{\#} \chi_j(S^1 \times S^1).$$

The Maslov indices vanish by Lemma 2.2. Indeed, the involution of the double extends to an involution of the boundary of M'' and of $S^1 \times S^1$ so that all Lagrangian subspaces involved are symmetric under the involution.

On the other hand, using the surgery presentation of $X' \times [-1, 1]$, we have

$$\begin{aligned} C(X') &= Z(X' \times [-1, 1], \emptyset, \hat{X}') \\ &= Z(M', \partial U, \hat{X}') \circ \phi_{\#} \circ S_{\#} \chi_1(S^1 \times S^1). \end{aligned}$$

Again, the Maslov indices vanish by symmetry. The claim then follows from the modular property

$$S_{\#} \chi_1(S^1 \times S^1) = \sum_{j \in I} S_{j,1} \chi_j(S^1 \times S^1). \quad \square$$

3.3. THE CASE OF THE (n, m) -POINT FUNCTION ON THE DISK

We consider the case of the correlation function for n interior points and m boundary points on the disk (the (n, m) -point function). The double of the disk is a 2-sphere S^2 , which we view as the unit sphere in \mathbb{R}^3 . The projection $p: S^2 \rightarrow D^2$ is the orthogonal projection onto the x - y plane and the involution σ is the reflection at the x - y plane.

Let X be the disk D^2 with n distinct labeled marked points in its interior and m labeled points on its boundary. Let the interior points be labeled by i_1, \dots, i_n , and the boundary points, cyclically ordered along the orientation of the circle, be labeled by j_1, \dots, j_m . The segment between the r th and the $r + 1$ st point is given a boundary condition, a simple object a_r .

Moreover, the correlation function depends also on an additional datum at each boundary point: the correlation function $C(X)$ is a linear map from a tensor product of multiplicity spaces $\bigotimes_{r=1}^m W_{a_{r-1}, a_r}(j_r)$ to $\mathcal{H}(\hat{X})$ (we set $a_0 = a_m$). The presence of these multiplicity spaces reflects the multiplicities of boundary fields. From the physical point of view, one understands these multiplicities as a consequence of the field-state correspondence of the conformal field theory, implying that they can be read off the annulus multiplicities, which for the boundary conditions of our interest coincide with fusion rule coefficients.

Accordingly, we assume that the multiplicity spaces are identified with the space of conformal blocks on the sphere with three points: $W_{a,b}(j) = \text{Hom}(b, j \otimes a)$. Using the identification $\text{Hom}(b, j \otimes a) = \text{Hom}(\mathbf{1}, b^* \otimes j \otimes a)$, we have a basis $(e_\alpha[b^*ja], \alpha = 1, \dots, N_b^{j,a})$ of each multiplicity space obeying the orthogonality properties of 2.2.

Our ansatz for the $C(X)$ evaluated on a product $e_{\alpha_1} \otimes \dots \otimes e_{\alpha_m}$ of basis elements is the element of $\mathcal{H}(\hat{X})$ associated to the graph in the 3-ball in Figure 1.

To give the precise definition we should again take the framing into account. So the marked interior points on D^2 should be taken to be disjoint oriented arcs. The boundary points are replaced by arcs on the boundary, oriented along the orientation of the boundary. The modular invariance property proven below will imply that these choices are irrelevant up to canonical isomorphisms.

The graph in Figure 1 is made into a ribbon graph as follows: the vertical lines are the cores of ribbons whose sides are the inverse images by p of the marked arcs and vertical lines connecting the endpoints of the marked arcs. The orientation of the ribbons is chosen so that they induce the orientation of the arcs on the upper hemisphere. The part of the graph connected to the equator is the core of an annulus lying in the x - y plane to which ribbons also lying in the x - y plane are glued along a

side. The opposite sides of these ribbons coincide with the marked arcs on the boundary. The orientation of the ribbon graph is such that it induces the orientation of the boundary arcs.

Let us call *equatorial graph* the component of the ribbon graph connected to the equator. The remaining components we call *vertical ribbons*.

Let M be the unit ball with this ribbon graph. Then we set

$$C(X) = Z(M, \emptyset, \hat{X}) \in \text{Hom}_{\mathbb{C}}(\otimes_{r=1}^m W_{a_{r-1}, a_r}(j_r), \mathcal{H}(\hat{X})).$$

In this formula we regard $Z(M, \emptyset, \hat{X})$ as a multilinear function of the labelings of vertices.

Let us check that this ansatz is modular invariant. If $f: D^2 \mapsto D^2$ is a degree one homeomorphism of the disk preserving the marked arcs, then there exists a unique degree one homeomorphism \hat{f} of S^2 , the *lift* of f , so that $p \circ \hat{f} = p \circ f$. It preserves the inverse images of the arcs.

THEOREM 3.3 (Modular invariance). *Let X be a disk with n marked arcs in the interior and m marked arcs on the boundary. Let $f: X \mapsto X$ be a degree one homeomorphism preserving the marked arcs and their orientations. Let \hat{f} be its lift to \hat{X} . Then $\hat{f}_{\#} C(X) = C(X)$.*

Proof. Denote by x, y, z the standard coordinates of \mathbb{R}^3 . View D^2 as the unit disk in the x - y plane. Let $F(x, y, z) = (f(x, y), r(x, y)z)$, with $r(x, y) = (1 - x^2 - y^2)^{-1/2} (1 - |f(x, y)|^2)^{1/2}$. Then F is a degree one homeomorphism of the ball D^3 whose restriction to S^2 is \hat{f} . The image of the ribbon graph γ in D^3 by F is isotopic to γ . Therefore, by the naturality axiom, $\hat{f}_{\#} Z(M, \emptyset, \hat{X}) = Z(M, \emptyset, \hat{X})$. \square

We turn to the factorization properties. In the case of surfaces with boundary there are two kinds of factorizations. One may either cut the surface along a loop in the interior, as in the case of closed surfaces, or along a path joining points on the boundary. In the first case, factorization is analyzed analogously as for closed surfaces. Thus we consider here only the latter possibility.

Let X', X'' be two oriented disks with marked labeled arcs as above. Let n', n'' denote the numbers of interior arcs and m', m'' the numbers of boundary arcs. Suppose that a marked arc x' on the boundary of X' has label j and that a marked arc x'' on the boundary of X'' has dual label j^* . Then we may glue X' to X'' along an orientation reversing homeomorphism from x' to x'' . We assume that the labels of the boundary segments on the sides of these two arcs match, so that the gluing results in a disk X with $n' + n''$ interior marked arcs and $m' + m'' - 2$ boundary marked arcs. We say that X is obtained by gluing X', X'' at x', x'' . Then the double \hat{X} may be obtained by gluing the extended surface $\hat{X}' \sqcup \hat{X}''$ at the inverse image of x', x'' , and we have a gluing homomorphism $g_{\hat{X}' \sqcup \hat{X}'', \hat{X}}: \mathcal{H}(\hat{X}') \otimes \mathcal{H}(\hat{X}'') \mapsto \mathcal{H}(\hat{X})$.

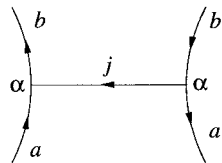
We need not care about Lagrangian subspaces since the first homology groups are trivial in this case.

THEOREM 3.4 (Factorization). *Let X be a disk obtained by gluing disks X', X'' at the marked arcs x', x'' with labels j, j^* . Let the labels of the boundary segment preceding and following x' be a, b respectively. Let X'_j be the surface X' , with j running over I with all other labels fixed. Similarly, let X''_{j^*} be the surface X'' , with j running over I with all other labels fixed. Let us order the boundary arcs in a way compatible with the cyclic ordering, so that x' is the last arc of X' and x'' is the first arc of X'' . Then for any choice of basis $e_\alpha[b^*ja]$ of $H^{b^*j,a}$ and dual basis $e_\alpha[a^*j^*b]$ of $H_{a^*j^*,b}$ as in 2.2,*

$$\begin{aligned}
 & C(X)(u_1 \otimes \cdots \otimes u_{m'+m''-2}) \\
 &= \sum_{j \in I, \alpha} \dim(j) g_{\hat{X}'_j \cup \hat{X}''_{j^*}, \hat{X}} [C(X'_j)(u_1 \otimes \cdots \otimes u_{m'-1} \otimes e_\alpha[b^*ja]) \otimes \\
 & \quad \otimes C(X''_{j^*})(e_\alpha[a^*j^*b] \otimes u_{m'} \otimes \cdots \otimes u_{m'+m''-2})].
 \end{aligned}$$

Proof. By the gluing construction (see 2.4.3), the summand labeled by j, α on the right-hand side is $\dim(j) Z(M_{j,\alpha}, \emptyset, \hat{X})$, where $M_{j,\alpha}$ is a ball obtained by gluing two balls defining $C(X'_j)$ and $C(X''_{j^*})$. $M_{j,\alpha}$ contains a ribbon graph $\gamma_{j,\alpha}$ obtained by the gluing prescription. Thus $\gamma_{j,\alpha}$ has vertical lines and a piece lying in the x - y plane obtained by gluing two equatorial graphs. We have to show that the sum over the labelings j and α of the invariant of the cobordism with this graph gives the same as if we replace it by an equatorial graph.

In the vicinity of the point at which the gluing was performed the ribbon graph looks like



Applying (4), after summing over j and α , we can replace this part of the graph by two horizontal bands. In this way $\gamma_{j,\alpha}$ is replaced by the equatorial graph appearing on the left-hand side as desired. □

3.4. THE GENERAL CASE

We now turn to the general case of a compact surface, possibly with boundary, possibly nonorientable. There is a subtlety that arises when one considers nonorientable surfaces. Namely a label of a point by a simple object is only defined if one chooses a local orientation. This is formalized by the following definitions of labeled surfaces and their doubles. The correlation functions will then be defined for *labeled surfaces* and they will take values in the space of states of their doubles, which are *extended surfaces* (see 2.5).

3.4.1. Labeled Surfaces and their Doubles

A *labeled surface* is a compact two-dimensional manifold X with (possibly empty) oriented boundary and with marked disjoint arcs (embedded closed intervals). The arcs lie either in the interior or on the boundary and carry labels. Boundary arcs are labeled by simple objects. The connected components of the complement in ∂X of the boundary arcs also carry labels, called *boundary conditions*, which are also simple objects. The label of an interior arc z is an equivalence class of triples (i, or, or') where i is a simple object, or is a local orientation of the surface at z , and or' is an orientation of the arc. Two triples are equivalent if they are equal or if one is obtained from the other by taking the dual object and reversing the orientations or, or' .

We call *boundary segments* the connected components of the complement in ∂X of the boundary arcs. If a boundary arc x lies between two boundary segments labeled by boundary conditions a, b in the order given by the orientation of the boundary, we say that x *changes the boundary conditions from a to b* .

The *double* \hat{X} of a labeled surface is an extended surface associated to a labeled surface. It is, as an oriented 2-manifold, the double \hat{X} of X , with projection $p: \hat{X} \mapsto X$ and orientation reversing involution $\sigma: \hat{X} \mapsto \hat{X}$. The double is made into an extended surface, by taking as arcs the inverse images of the arcs of X . The boundary arcs have one inverse image and are labeled by the labels in X . Their orientation is inherited from the orientation of the boundary of X . Each interior arc z of X has two inverse images. They are labeled and oriented by the two labels in the equivalence class labeling z , in such a way that the local orientation or appearing in the label agrees with the orientation of \hat{X} . The Lagrangian subspace of $H_1(\hat{X}, \mathbb{R})$ is the eigenspace of σ_* to the eigenvalue -1 . It is called canonical Lagrangian subspace and denoted by $\lambda_-(\hat{X})$.

This definition makes sense because of the following lemma.

LEMMA 3.5. *Let $\sigma: \hat{X} \mapsto \hat{X}$ be an orientation reversing homeomorphism of a surface \hat{X} such that $\sigma \circ \sigma = \text{id}$. Then the induced map $\sigma_*: H_1(\hat{X}, \mathbb{R}) \mapsto H_1(\hat{X}, \mathbb{R})$ is diagonalizable and its eigenspaces are Lagrangian.*

Proof. The induced map σ_* is a linear involution of a real vector space. Thus $p_{\pm} = \frac{1}{2}(1 \pm \sigma_*)$ are projections onto the eigenspaces λ_{\pm} corresponding to the eigenvalues ± 1 . Since $p_+ + p_- = \text{id}$, σ_* is diagonalizable. Let ω denote the intersection pairing on $H_1(\hat{X}, \mathbb{R})$. Since σ reverses the orientation, we have $\omega(\sigma_* a, \sigma_* b) = -\omega(a, b)$ for all $a, b \in H_1(\hat{X}, \mathbb{R})$. Therefore ω vanishes identically on λ_+ and λ_- . Since $\lambda_+ \oplus \lambda_- = H_1(\hat{X}, \mathbb{R})$, the subspaces λ_{\pm} are of maximal dimension with this property, i.e., Lagrangian. \square

PROPOSITION 3.6. *Let \hat{X} be a double of the labeled surface X . Let $\text{Aut}(\hat{X}, \sigma)$ be the group of degree one homeomorphisms of \hat{X} preserving the marked arcs with their orientation and commuting with the involution σ . Then $f \mapsto f_{\sharp}$ defines a representation (not just a projective representation) of $\text{Aut}(\hat{X}, \sigma)$ on $\mathcal{H}(\hat{X})$.*

Proof. Let $f \in \text{Aut}(\hat{X}, \sigma)$. Since f and σ commute, the induced maps f_* , σ_* also commute. It follows that the eigenspaces of σ_* are preserved by f_* . In particular, the Lagrangian subspace of \hat{X} is preserved by elements of $\text{Aut}(\hat{X}, \sigma)$. Under these circumstances the anomaly is trivial and we have the representation property $(f \circ g)_\# = f_\# \circ g_\#$. \square

3.4.2. Connecting 3-Manifolds

To each compact surface X we associate a connecting 3-manifold M_X . The connecting 3-manifold M_X is a three-dimensional oriented manifold with boundary \hat{X} . It is used to construct the correlation functions and reduces to the cylinder $X \times [-1, 1]$ if X is closed and orientable and to the ball if X is a disk.

We first describe M_X as an oriented manifold. If X has no boundary, M_X is $(\hat{X} \times [-1, 1])/\mathbb{Z}_2$, where \mathbb{Z}_2 acts on the first factor by the involution σ and on the second by $t \mapsto -t$. This action preserves the product orientation, so that M_X is naturally an oriented manifold. It comes with a projection $[(x, t)] \mapsto p(x)$ to X . The fiber of this projection over $y \in X$ is an interval, the *connecting interval over y* , connecting the two inverse images of y in \hat{X} . If X has a boundary, M_X is obtained from $(\hat{X} \times [-1, 1])/\mathbb{Z}_2$ by contracting the fibers over the boundary to single points.

Alternatively, let $\rho: X \mapsto [0, \infty)$ be any nonnegative function such that $\rho(x) = 0$ if and only if $x \in \partial M$. Then we may define M_X to consist of $[(x, t)] \in (\hat{X} \times \mathbb{R})/\mathbb{Z}_2$ such that $t^2 \leq \rho(p(x))$. The points with $t^2 = \rho(p(x))$ form the boundary which is obviously homeomorphic to \hat{X} . Connecting manifolds corresponding to different choices of ρ are canonically homeomorphic. The homeomorphism commutes with p and reduces to the identity on $\hat{X} \simeq \partial M_X$.

PROPOSITION 3.7. *Let \hat{X} be the double of X , $p: \hat{X} \mapsto X$ the projection, $\lambda_-(\hat{X})$ the canonical Lagrangian subspace of $H_1(\hat{X}, \mathbb{R})$.*

- (i) M_X is a compact manifold with boundary $\partial M_X = \hat{X}$.
- (ii) The restriction of $\pi: M_X \rightarrow X$, $[(x, t)] \mapsto p(x)$ to $\pi^{-1}(X - \partial X)$ is a fiber bundle whose fiber over y is an interval with boundary $p^{-1}(y)$.
- (iii) $\lambda_-(\hat{X})$ is the kernel of the homomorphism $H_1(\hat{X}, \mathbb{R}) \rightarrow H_1(M_X, \mathbb{R})$ induced by inclusion.
- (iv) The involution $\sigma: \hat{X} \rightarrow \hat{X}$ extends to the involution $[(x, t)] \mapsto [(x, -t)]$ of M_X . Its fixed point set is the image of X under the embedding $i: y \mapsto [(x, 0)]$ for any x with $p(x) = y$.

Proof. Choose a function ρ as above. Let $\{\phi_\alpha: U_\alpha \rightarrow \mathbb{R}^2\}$ be an atlas of X with connected charts U_α . Let $\varepsilon_{\alpha,\beta}$ be the sign of the Jacobian of $\phi_\alpha \circ \phi_\beta^{-1}$. Then M_X is homeomorphic to

$$\bigsqcup_\alpha \{(y, t) \in U_\alpha \times \mathbb{R} \mid t^2 \leq \rho(y)\} / \sim$$

with equivalence relation $(x \in U_\alpha, t) \sim (x \in U_\beta, \varepsilon_{\alpha,\beta} t)$. The projection to X is $p(y, t) = y$ and the involution is $\sigma(y, t) = (y, -t)$. Therefore, we have a surjective map $\hat{X} \times [-1, 1]$ onto M given by $((y, t), s) \mapsto (y, ts)$. The fibers of this map consist of two points related by the \mathbb{Z}_2 action, except if $y \in \partial X$ where the fiber is an interval.

This presentation of M_X implies (i) and (ii).

To prove (iii), notice that if a is a loop on \hat{X} , then $a - \sigma \circ a$ is the boundary of a surface in M_X consisting of connecting intervals ending at points of a . Thus if c is a cycle on \hat{X} such that $\sigma_* c$ is homologous to $-c$, then c is homologous to zero in M_X . This shows that $\lambda_-(\hat{X})$ is contained in the kernel K of the homomorphism induced by inclusion. On the other hand, it is a general fact that the intersection form vanishes on K . Therefore the dimension of K cannot be larger than the dimension of the Lagrangian subspace $\lambda_-(\hat{X})$.

(iv) is obvious. □

3.4.3. Multiplicity Spaces

Suppose that X is a labeled surface. If x is a marked arc on the boundary labeled by a simple object j and changing the boundary condition a to the boundary condition b , the *multiplicity space* of x is $W_{a,b}(j) = \text{Hom}(b, j \otimes a)$. The *multiplicity space* $W_{\partial X}$ of a labeled surface X is the (unordered) tensor product of the multiplicity spaces of its boundary arcs. If there are no boundary arcs, we set $W_{\partial X} = \mathbb{C}$.

3.4.4. Construction of Correlation Functions

We are ready to define correlation functions for general labeled surfaces. Let X be a labeled surface, M_X be its connecting manifold and $\hat{X} = \partial M_X$ the double of X , with its structure of extended surface. Let $i: X \mapsto M_X$ be the inclusion of X as zero section (Proposition 3.7(iv)). We construct a ribbon graph in M_X . It consists of vertical ribbons and an equatorial graph for each connected component of ∂X .

The *vertical ribbons* are associated to interior arcs of \hat{X} : if z, z' are interior arcs projecting to an interior arc of X , the corresponding vertical ribbon is the union of the connecting intervals ending at z and z' . It is an embedded rectangle with two sides equal to z, z' . The orientation of the vertical ribbon is chosen so as to induce the orientations of z, z' . If we orient the core from z to z' , the label of the ribbon is equal to the label of z . The *equatorial graphs* consist of annuli and joining ribbons. The annuli lie in the zero section $i(X)$ of M_X and their cores are obtained by moving $i(\partial X)$ into M_X by a short amount, where ‘short’ means away from the vertical ribbons. The joining ribbons are short ribbons in $i(X)$ connecting boundary arcs to the annuli at trivalent vertices. They are labeled by the label of the corresponding boundary arcs and their cores are oriented inward. The labels of the parts of the annuli between trivalent vertices are the boundary conditions between the corresponding arcs. The orientation of the equatorial graphs is chosen so as to induce the orientation of the boundary arcs. This does not fix the orientation

of the annuli that are not connected to the boundary; but this does not matter since the correlation function will not depend on the choice of that orientation.

Then the correlation function of the labeled surface X is

$$C(X) = Z(M_X, \emptyset, \hat{X}) : W_{\partial X} \rightarrow \mathcal{H}(\hat{X}),$$

considered as a multilinear function of the labels of the trivalent vertices. Thus if $u = \otimes_x u_x \in W_{\partial X}$ with x running over the boundary arcs, $C(X)u = Z(M_{X,u}, \emptyset, \hat{X})$, where $M_{X,u}$ is the connecting 3-manifold with its ribbon graph, such that the trivalent vertex connected to x by a joining ribbon is labeled by u_x .

3.4.5. Modular Invariance

If $f: X \rightarrow X$ is a homeomorphism of the labeled surface X , preserving the orientation and the marked arcs of the boundary and mapping interior marked arcs to interior marked arcs with the same label,* then there exists a unique degree one homeomorphism \hat{f} of \hat{X} , the lift of f , so that $p \circ \hat{f} = f \circ p$. It preserves the inverse images of the arcs and their orientation. f commutes with σ and therefore preserves the canonical Lagrangian subspace of $H_1(\hat{X}, \mathbb{R})$.

THEOREM 3.8 (Modular invariance). *Let X be a labeled surface. Let $f: X \mapsto X$ be a homeomorphism preserving the orientation of the boundary and mapping marked arcs to marked arcs with the same label and boundary segments to boundary segments with the same boundary condition. Let \hat{f} be its lift to \hat{X} . Then $\hat{f}_\# C(X) = C(X)$.*

Proof. Let $F: M_X \mapsto M_X$ be the map $[(x, t)] \mapsto [(f(x), t)]$ of $M_X \subset (\hat{X} \times [-1, 1])/\mathbb{Z}_2$. It is clear that F is a well-defined degree one homeomorphism of M_X . It maps vertical ribbons to vertical ribbons with the same label. The equatorial graphs are mapped to slight deformations of the equatorial graphs. As the boundary arcs are fixed, we may compose F with a homeomorphism G of M_X with support in the vicinity of $i(\partial X)$ and restricting to the identity on the boundary, in such a way that the equatorial graphs are also kept fixed. Then $G \circ F$ preserves the ribbon graph and restricts to \hat{f} on the boundary. Therefore, by the naturality axiom, $\hat{f}_\# Z(M, \emptyset, \hat{X}) = Z(M, \emptyset, \hat{X})$. \square

Remark. We may relax the condition that f preserves the boundary arcs and the orientation of the boundary. We may just assume that f maps boundary arcs to boundary arcs with the same label, or the dual label, depending on whether f preserves the local orientation of the boundary. Similarly f should be compatible with the labeling of boundary segments. Then the modular invariance reads $\hat{f}_\# C(X) = C(X)\rho(f)$, for a suitable action on the multiplicity spaces. We leave the details to the reader.

* Recall that a label of an interior arc is an equivalence class of triples $[(i, or, or')]$. The condition for an interior arc z means that f maps z to an arc z' , and if z has label $[(i, or, or')]$, z' has label $[(i, f_*or, f_*or')]$.

3.4.6. Factorization

If X is a surface, we can obtain a new surface X' by cutting and pasting in two basic ways: either we can cut out disks around two interior marked arcs and glue their boundaries together or we can glue two boundary arcs.

In both cases we want to relate the correlation functions on X' to the correlation functions on X . In the first case the relation between correlation functions is called bulk factorization. In the second case it is called boundary factorization.

Let z_1, z_2 be two interior arcs of a labeled surface. We say that the labels of z_1, z_2 match if they are of the form $[(i, or_1, or'_1)]$ and $[(i^*, or_2, or'_2)]$, respectively. In this case we construct a labeled surface X' as follows. Choose representatives $(i, or_1, or'_1), (i^*, or_2, or'_2)$ so that the arcs are oriented and we have local orientations around the arcs. Let $\phi, \phi': D^2 \rightarrow X$ be orientation preserving disjoint embeddings of the unit disk $D^2 \subset \mathbb{C}$ such that their restriction to $[-1, 1]$ are parametrizations of the oriented arcs z_1, z_2 . Then X' is obtained from X by removing the interiors of the disks $\phi(D^2)$ and $\phi'(D^2)$, and gluing their boundaries by identifying $\phi(z)$ with $\phi'(-\bar{z})$, for $z \in S^1$. The arcs of X' are the remaining arcs of X .

We say that X' is obtained from X by gluing z to z' .

The double of X' is then obtained from the double of X by gluing the inverse images of z_1 to the inverse images of z_2 : the inverse image of z_1 with orientation or_1 is glued to the inverse image of z_2 with orientation or_2 and the inverse image of z_1 with the opposite orientation $-or_1$ is glued to the inverse image of z_2 with the orientation $-or_2$.

Then we have a gluing homomorphism $g_{\hat{X}, \hat{X}'}: \mathcal{H}(\hat{X}) \rightarrow \mathcal{H}(\hat{X}')$, which is the composition of the two gluing homomorphisms (in either order) with $(\text{id}_{\lambda', \lambda_-(\hat{X}')})_{\sharp}$. Here λ' is the symmetric Lagrangian subspace of $H_1(\hat{X}', \mathbb{R})$ obtained from the canonical Lagrangian subspace $\lambda_-(\hat{X})$ of \hat{X} by the gluing prescription of 2.5, and $\text{id}_{\lambda', \lambda_-(\hat{X}')}$ is the identity map from \hat{X}' with Lagrangian subspace λ' to \hat{X}' with canonical Lagrangian subspace.

EXAMPLE. Let z_1, z_2 be two marked arcs on a sphere X . Choose an orientation or of the sphere and let $(i_1, or, or'_1), (i_2, or, or'_2)$ be representatives of the labels of these two arcs chosen to agree with the global orientation. If $i_2 = i_1^*$, the labeled surface obtained by gluing z to z' is a torus. If $i_2 = i_1$, we may take the other representative $(i_2^* = i_1^*, -or, -or'_2)$ and obtain a Klein bottle as a result of gluing. In the first case, \hat{X}' is obtained from the disjoint union of the two spheres by gluing pairs of arcs on the same connected component. In the second case, \hat{X}' is obtained by gluing arcs on one connected component to arcs of the other.

THEOREM 3.9 (Bulk factorization). *Let X be a labeled surface. Let X' be obtained from X by gluing X at two interior marked arcs with labels $[(j, or_1, or'_1)], [(j^*, or_2, or'_2)]$. Let X_j be the surface X , with j running over I with all other labels fixed. Then*

$$C(X') = \sum_{j \in I} \mathcal{D}^{-1} \dim(j) g_{\hat{X}_j, \hat{X}'} C(X_j).$$

The proof of this theorem is the same as the proof of Theorem 3.2.

We now turn to the boundary factorization. Let x, x' be boundary arcs with the orientation induced by the orientation of the boundary. Suppose that the label of x is j and that the label of x' is j^* . Assume that x changes the boundary conditions from a to b and that x' changes the boundary conditions from b to a . Under these circumstances, we may glue x to x' via an orientation reversing homeomorphism and obtain a surface X' . We say that X' is obtained from X by gluing x to x' .

The double \hat{X}' of X' may then be identified with the surface obtained from \hat{X} by gluing the inverse image in \hat{X}' of x to the inverse image in \hat{X}' of x' . The Lagrangian subspace of $H_1(\hat{X}', \mathbb{R})$ obtained by the gluing prescription coincides in this case with the canonical Lagrangian subspace $\lambda_-(\hat{X}')$. We thus have a gluing homomorphism $g_{\hat{X}, \hat{X}'}: \mathcal{H}(\hat{X}) \rightarrow \mathcal{H}(\hat{X}')$.

To formulate the boundary factorization properties of correlation functions, we need to compare the multiplicity spaces of X and X' .

Note that $W_{\partial X'} = W_{\partial X'} \otimes W_{a,b}(j) \otimes W_{b,a}(j^*)$, and that $W_{a,b}(j)$ is dual to $W_{b,a}(j^*)$. We thus have a natural map $\gamma_{X',X}: W_{\partial X'} \rightarrow W_{\partial X}$ obtained by taking the tensor product with the canonical tensor. In terms of the bases of 2.2,

$$\gamma_{X',X}(w) = w \otimes \sum_{\alpha} e_{\alpha}[b^*ja] \otimes e_{\alpha}[a^*j^*b].$$

THEOREM 3.10 (Boundary factorization). *Let X' be obtained by gluing two marked boundary arcs x', x'' with labels j, j^* of a labeled surface X . Let X_j be the surface X , with j running over I with all other labels fixed. Then*

$$C(X') = \sum_j \dim(j) g_{\hat{X}_j, \hat{X}'} \circ C(X) \circ \gamma_{X',X}$$

This theorem is proved in the same way as Theorem 3.4.

4. Structure Constants

It is clear that using the factorization property of correlation functions (Theorems 3.9 and 3.10) the calculation of any correlation function can be reduced to four basic cases: the sphere with three points, the disk with three boundary points, the disk with one interior point and one boundary point, and the real projective plane with one point.

We compute the correlation functions in these four cases. The special cases of two points on the sphere and on the disk and one interior point on the disk can be in principle deduced by setting one of the labels to 0. But since the results are particularly simple we compute them separately.

The calculation of the two-point functions also explains the appearance of the factors of $\dim(j)$ and \mathcal{D}^{-1} in the factorization formulae.

4.1. TWO-POINT FUNCTIONS

We calculate the two-point functions on the sphere and on the disk.

Let $X = (S^2, j, j^*)$ be the unit sphere in \mathbb{R}^3 with two points, say at the north pole with label j and at the south pole with label j^* . We give S^2 the standard orientation. As usual, we rather have to specify two arcs than two points. Let the arc at the north pole be a short arc in the x - z plane oriented in the positive x direction, and let the arc at the south pole be a short arc in the x - z plane, pointing in the negative direction. X may be both viewed as a labeled surface and as an extended surface. There are no Lagrangian subspaces here since the first homology of the sphere is trivial.

The correlation function $C(X)$ on the sphere takes its values in $\mathcal{H}(X \sqcup (-X)) = \mathcal{H}(X) \otimes \mathcal{H}(-X)$. A basis of the one-dimensional vector space $\mathcal{H}(X)$ is given by

$$b(X) = Z((D^3, j), \emptyset, X),$$

associated to the unit ball $M_X = D^3$ endowed with a ribbon $D^3 \cap ([-\varepsilon, \varepsilon] \times \{0\} \times \mathbb{R})$. The ribbon has label j ; it runs vertically along the z -axis and its core is oriented from top to bottom. The orientation of the ribbon is such that it induces the orientation of the arcs.

A basis of $\mathcal{H}(-X)$ is given by $b(-X) = \theta_{\pm} b(X)$, where θ is the reflection with respect to the x - z plane.

To compute the two-point function on the sphere, we have to compute the proportionality constant c in $C(S^2, j, j^*) = c b(X) \otimes b(-X)$. This can be done by using the functoriality of the invariant Z . If we glue two balls, each with a ribbon inside, to $S^2 \times [-1, 1]$ we get S^3 with an unknot labeled by j . These two balls may be viewed as a cobordism from $S^2 \sqcup S^2$ to the empty set. Applying its invariant to $C(S^2, j, j^*)$, we get $\mathcal{D}^{-1} \dim(j)$, the invariant of S^3 with the unknot. Applying the invariant of the same cobordism to $b(X) \otimes b(-X)$, we get the invariant $\mathcal{D}^{-2} \dim(j) \dim(j^*)$ of a closed manifold with two connected components, each of which is a 3-sphere with an unknot labeled by j and j^* , respectively. As $\dim(j) = \dim(j^*) \neq 0$, we get $c = \mathcal{D} \dim(j)^{-1}$, with the result

$$C(S^2, j, j^*) = \frac{\mathcal{D}}{\dim(j)} b(S^2, j, j^*) \otimes b(-(S^2, j, j^*)).$$

Let us turn to the case of the disk with two boundary points labeled by j, j^* , and boundary conditions a, b . The two-point correlation function $C(D^2, j, j^*; a, b)$ is a map from $W_{ab}(j) \otimes W_{ba}(j^*)$ to the space $\mathcal{H}(S^2, j, j^*)$ of conformal blocks on the sphere. Evaluating the correlation function on basis vectors $e_{\alpha} \otimes e_{\beta} = e_{\alpha}[jba^*] \otimes e_{\beta}[j^*ab^*]$ we get the invariant of D^3 with an equatorial graph with two outgoing

lines. This graph may be replaced by a single ribbon using (3). The result is

$$C(D^2, j, j^*) e_\alpha \otimes e_\beta = \frac{\delta_{\alpha,\beta}}{\dim(j)} b(S^2, j, j^*).$$

Remark. Our two-point correlation functions have a somewhat nonstandard normalization, which avoids square root ambiguities. For any choice of square roots one may define ‘normalized correlation functions’. If X is a labeled surface with interior arcs labeled by i_1, \dots, i_n and m boundary arcs labeled by j_1, \dots, j_m , let

$$C_{\text{norm}}(X) = \prod_{v=1}^n \sqrt{S_{i_v,1}} \prod_{v=1}^m \sqrt{\frac{S_{j_v,1}}{S_{1,1}}} C(X). \tag{9}$$

(Recall that $\mathcal{D}^{-1} = S_{1,1}$ and that $\dim(j) = S_{j,1}/S_{1,1}$.) For these correlation functions there are no factors of $\dim(j)$ or \mathcal{D} in the two-point functions or in the factorization theorems.

4.2. THE 3-POINT FUNCTION ON THE SPHERE

Let $X = (S^2, i, j, k)$ be the sphere with three marked arcs labeled by i, j and k . To fix the conventions let us take the sphere as the unit sphere in \mathbb{R}^3 , with standard orientation and have the three marked arcs on the equator. We orient the equator counterclockwise and orient the arcs in the same direction. The correlation function then takes values in $\mathcal{H}(X \sqcup (-X)) = \mathcal{H}(X) \otimes \mathcal{H}(-X)$. A basis of $\mathcal{H}(X)$ is given by $e_\alpha(X) = Z(M_{X,\alpha}, \emptyset, X)$, $\alpha = 1, \dots, N_{k,j,i}$, where the connecting 3-manifold $M_{X,\alpha}$ is the 3-ball with a ribbon graph in the x - y plane with one trivalent vertex at the origin. The vertex is labeled by the basis element $e_\alpha[kji]$ of $H^{k,j,i}$. A basis of $\mathcal{H}(-X)$ is $e_\alpha(-X) = \theta_\alpha e_\alpha(X)$, where θ is the reflection at the x - y plane.

As in the case of the two-point function, to compute the correlation function $Z(X \times [-1, 1], \emptyset, X \sqcup (-X))$ in terms of the basis $e_\alpha(X) \otimes e_\beta(-X)$ of $\mathcal{H}(X \sqcup (-X)) = \mathcal{H}(X) \otimes \mathcal{H}(-X)$, we use the functoriality of Z and act on $C(X)$ with $Z(M_{X,\alpha,\beta}, X \sqcup (-X), \emptyset)$. Here, $M_{X,\alpha,\beta}$ is the disjoint union of two 3-balls, each with a ribbon graph with one vertex labeled by e_α, e_β respectively. The result is the invariant of S^3 with the ‘theta graph’, which has two vertices connected by three ribbons. By the orthogonality relations of basis elements this invariant is $\mathcal{D}^{-1} \delta_{\alpha,\beta}$.

On the other hand, if we act with $Z(M_{X,\alpha,\beta}, X \sqcup (-X), \emptyset)$ on $e_\gamma(X) \otimes e_\delta(-X)$, we obtain the invariant of $S^3 \sqcup S^3$ with a theta graph in each copy of S^3 , which is $\mathcal{D}^{-2} \delta_{\alpha,\gamma} \delta_{\beta,\delta}$. The result is thus

$$C(S^2, i, j, k) = \mathcal{D} \sum_{\alpha=1}^{N_{k,j,i}} e_\alpha(S^2, i, j, k) \otimes e_\alpha(-(S^2, i, j, k)).$$

As usual, $N_{k,j,i}$ is the dimension of $\text{Hom}(\mathbf{1}, k \otimes j \otimes i)$. With the normalization (9) we

then have

$$C_{\text{norm}}(S^2, i, j, k) = \frac{\sqrt{S_{i,1}S_{j,1}S_{k,1}}}{S_{1,1}} \sum_{\alpha=1}^{N_{k,j,i}} e_{\alpha}(S^2, i, j, k) \otimes e_{\alpha}(-(S^2, i, j, k)).$$

4.3. THE (1, 0)-POINT FUNCTION ON THE DISK

Let $X = (D^2, i; a)$ be the unit disk with an arc labeled by i at the origin and boundary condition a . The correlation function $C(X)$ takes values in $\mathcal{H}(\hat{X})$ with $\hat{X} = (S^2, i, i^*)$, the sphere with two marked arcs as in 4.1. We have, by construction,

$$C(X) = Z(M, \emptyset, \hat{X}).$$

The 3-manifold M is a 3-ball with a vertical ribbon and an equatorial graph consisting of an annulus labeled by a . We want to express $C(X)$ in terms of the basis $b(S^2, i, i^*)$ (the ‘Ishibashi boundary state’) of the one-dimensional vector space $\mathcal{H}(\hat{X})$. We do this as above, by gluing a ball with a vertical ribbon inside onto M and using the functoriality of Z . The resulting closed 3-manifold is a 3-sphere with two unknots with linking number 1. It has invariant $\mathcal{D}^{-1}_{S_{i,a}}$. This has to be compared with the 3-manifold obtained by gluing the same ball to D^3 with a vertical ribbon, which is S^3 with an unknot and has invariant $\mathcal{D}^{-1}_{S_{i,1}} = S_{i,1}$. The result is

$$C(D^2, i; a) = \frac{S_{i,a}}{S_{i,1}} b(S^2, i, i^*).$$

Implementing the normalization as given in Equation (9), the corresponding normalized correlation function is

$$C_{\text{norm}}(D^2, i; a) = \frac{S_{i,a}}{\sqrt{S_{i,1}}} b(S^2, i, i^*).$$

In conformal field theory, this formula for the one-point functions on the disk was first obtained in [C1].

4.4. THE (1, 1)-POINT FUNCTION ON THE DISK

We calculate the correlation function in the case of $n = 1$ point in the interior of the disk D^2 and $m = 1$ point on its boundary. The label of the interior point is i and the label of the boundary point is j . The double \hat{X} of the disk with these points is the 2-sphere S^2 with one point on the equator labeled by j and two points, say the north and south pole, labeled by i, i^* . Then the correlation function with boundary condition k maps $W_{k,k}(j) = \text{Hom}(k, j \otimes k)$ to $\mathcal{H}(\hat{X}) = \text{Hom}(\mathbf{1}, i \otimes i^* \otimes j)$.

Its matrix elements $R_{\alpha\beta}$ with respect to the chosen bases are given by the formula

$$\begin{array}{c} j \\ \downarrow \\ \alpha \\ \downarrow \\ i \end{array} \quad \begin{array}{c} i \\ \downarrow \\ k \\ \downarrow \\ \downarrow \end{array} = \sum_{\beta} R_{\alpha\beta} \begin{array}{c} j \\ \downarrow \\ \beta \\ \downarrow \\ i \end{array} .$$

To calculate $R_{\alpha\beta}$, we compose this morphism with the basis element $e_{\gamma}[j^*i^*]$ of $\text{Hom}(j \otimes i, i) \simeq \text{Hom}(\mathbf{1}, j^* \otimes i \otimes i^*)$ and obtain

$$\begin{array}{c} i \\ \downarrow \\ j \\ \downarrow \\ \alpha \\ \downarrow \\ i \end{array} \quad \begin{array}{c} i \\ \downarrow \\ k \\ \downarrow \\ i \end{array} = \sum_{\beta} R_{\alpha\beta} \begin{array}{c} i \\ \downarrow \\ \gamma \\ \downarrow \\ \beta \\ \downarrow \\ i \end{array} = \frac{1}{\dim(i)} R_{\alpha\gamma} \begin{array}{c} i \\ \downarrow \\ i \end{array} .$$

The left-hand side may be evaluated in terms of fusing matrices, with the result:

$$\frac{1}{\dim(i)} R_{\alpha\gamma} \begin{array}{c} i \\ \downarrow \\ i \end{array} = \sum_{l,\delta,\varepsilon} \begin{array}{c} i \\ \downarrow \\ l \\ \downarrow \\ \varepsilon \\ \downarrow \\ k \\ \downarrow \\ i \end{array} \left\{ \begin{array}{c} i \quad i^* \quad j \\ k \quad k \quad l \end{array} \right\}_{\delta\varepsilon}^{\gamma\alpha} .$$

The expression appearing on the right may be further simplified:

$$\begin{array}{c} i \\ \downarrow \\ l \\ \downarrow \\ \varepsilon \\ \downarrow \\ k \\ \downarrow \\ i \end{array} = \begin{array}{c} i \\ \downarrow \\ l \\ \downarrow \\ \delta \\ \downarrow \\ k \\ \downarrow \\ i \end{array} = \begin{array}{c} i \\ \downarrow \\ \varepsilon \\ \downarrow \\ k \\ \downarrow \\ \delta \\ \downarrow \\ i \end{array} \\
 = \frac{v_l}{v_i v_k} \begin{array}{c} i \\ \downarrow \\ \varepsilon \\ \downarrow \\ k \\ \downarrow \\ \delta \\ \downarrow \\ i \end{array} = \delta_{\delta\varepsilon} \frac{1}{\dim(i)} \frac{v_l}{v_i v_k} \begin{array}{c} i \\ \downarrow \\ i \end{array} .$$

Putting everything together, we arrive at the result

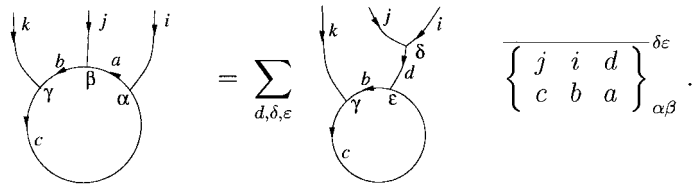
$$R_{\alpha\beta} = \sum_{l,c} \frac{v_l}{v_i v_k} \left\{ \begin{matrix} i & i^* & j \\ k & k & l \end{matrix} \right\}_{\varepsilon\varepsilon}^{\beta\alpha}.$$

4.5. THREE BOUNDARY POINTS ON THE DISK

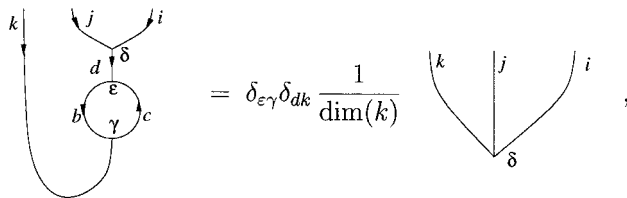
The correlation function for three points on the boundary of a disk can be computed in an analogous way. In this case \hat{X} is the 2-sphere with three points on the equator. Let us orient the boundary of the unit disk counterclockwise and denote the labels of the three points i, j, k . The boundary conditions are labeled by a, b, c . Then $C(X)$ is a map from $W_{c,a}(i) \otimes W_{a,b}(j) \otimes W_{b,c}(i)$ to $\mathcal{H}(\hat{X}) = \text{Hom}(\mathbf{1}, k \otimes j \otimes i)$. The structure constants $C_{\alpha\beta\gamma}^\delta$ are in this case defined by

$$C(X) e_\alpha[a^*ic] \otimes e_\beta[b^*ja] \otimes e_\gamma[c^*kb] = \sum_\delta C_{\alpha\beta\gamma}^\delta e_\delta[kji],$$

in terms of the bases of $W_{c,a}(i) \simeq \text{Hom}(\mathbf{1}, a^* \otimes i \otimes c)$, etc. The connecting 3-manifold is in this case a ball with an equatorial graph. The computation of the structure constants then goes as follows.



The diagram appearing on the right-hand side can be deformed to



with the result

$$C_{\alpha\beta\gamma}^\delta = \frac{1}{\dim(k)} \left\{ \begin{matrix} j & i & k \\ c & b & a \end{matrix} \right\}_{\alpha\beta}^{\delta\gamma}.$$

The normalized correlation function is then

$$C_{\text{norm}}(X) e_\alpha[a^*ic] \otimes e_\beta[b^*ja] \otimes e_\gamma[c^*kb] = \sqrt{\frac{S_{i,1}S_{j,1}}{S_{k,1}S_{1,1}}} \sum_\delta \overline{\begin{Bmatrix} j & i & k \\ c & b & a \end{Bmatrix}}_{\alpha\beta}^{\delta\gamma} e_\delta[kji],$$

cf. [R, BPPZ, FFFS1].

4.6. ONE POINT ON THE PROJECTIVE PLANE

We consider the one-point correlation function on the projective plane. Thus our labeled surface $X = (\mathbb{R}P^2, i)$ is $\mathbb{R}P^2 = S^2/\mathbb{Z}_2$ with a marked arc labeled by $i \in I$ and some local orientation. The connecting 3-manifold is here $(S^2 \times [-1, 1])/\mathbb{Z}_2$. View S^2 as the unit sphere in \mathbb{R}^3 , and let us put the marked arc z at the north ($=$ south) pole $\pm(0, 0, 1) \in S^2/\mathbb{Z}_2$ of the projective plane. The local orientation is obtained by identifying, locally around the arc, the projective plane with the upper hemisphere of S^2 . Then the correlation function is the element of $\mathcal{H}(\hat{X})$ associated to the ribbon graph given by the fiber over z , i.e., the image in $(S^2 \times [-1, 1])/\mathbb{Z}_2$ of the interval $(0, 0, 1) \times [-1, 1]$. The framing is determined by taking a neighboring point, say $(\varepsilon, 0, \sqrt{1-\varepsilon^2})$ and taking at each point of the fiber a vector pointing to the fiber over the neighboring point.

The space of states $\mathcal{H}(\hat{X})$ is one-dimensional in this case. A basis of this space is given by the ‘Ishibashi cross cap state’ ψ_i . Before giving its definition we notice that there are two natural candidates for a basis. Namely, we can take any of the two states ψ_i^\pm associated to the ribbon graphs in the 3-ball D^3 of Figure 3. The two states differ by a twist, so $\psi_i^+ = v_i \psi_i^-$. To define the cross cap state we choose square roots of v_i and normalize the cross cap state salomonically as

$$\psi_i = v_i^{1/2} \psi_i^- = v_i^{-1/2} \psi_i^+.$$

Our task is to express the one-point function $C(X)$ on the projective plane in terms of

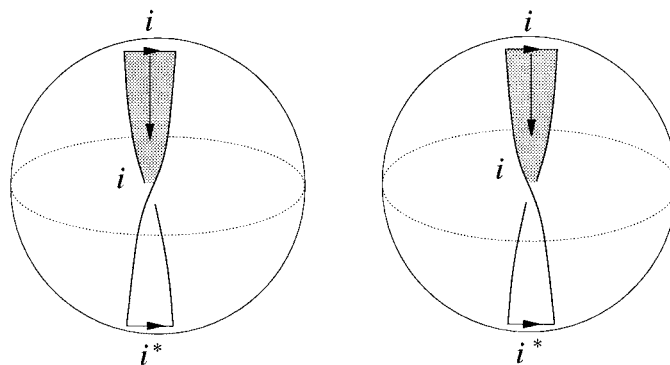


Figure 3. The vectors ψ_i^+, ψ_i^- are associated to the ribbon graphs in D^3 on the left and on the right, respectively.

this basis:

$$C(X) = c_i \psi_i = c_i v_i^{1/2} \psi_i^-.$$

In other words, we have to compute the constant of proportionality $c_i v_i^{1/2}$ between two states in $\mathcal{H}(\hat{X})$ given by ribbon graphs in 3-manifolds with the same boundary S^2 . This can be done using the functoriality axiom by attaching a 3-ball with a ribbon graph consisting of a single ribbon connecting the north pole with the south pole to S^2 , and comparing the corresponding (scalar) invariants of ribbon graphs in closed 3-manifolds.

If we attach a 3-ball to a 3-ball we get a 3-sphere. If we choose the ribbon graph in the 3-ball properly, we get in S^3 an unknotted circle with zero framing. Its invariant is $\mathcal{D}^{-1} \dim(i)$, where \mathcal{D}^{-1} is the invariant of S^3 . This proper choice of the ribbon may be described as follows. Suppose for definiteness that the upper and lower sides of the ribbon defining ψ_i^- are centered at the poles $(0, 0, \pm 1)$ of S^2 and lie in the x_2 - x_3 plane. The ribbon graph in the 3-ball we attach to S^2 should be chosen so that it can be deformed to a ribbon lying on the surface S^2 whose midline follows a ‘tennis ball pattern’ joining the north pole to the south pole: this pattern may be parametrized by $\rho \in [0, \pi]$:

$$x(\rho) = \left(\frac{1}{2} \sin 2\rho, \frac{1}{2} (1 - \cos 2\rho), \cos \rho\right).$$

If we attach a 3-ball to the boundary of $(S^2 \times [-1, 1])/\mathbb{Z}_2$ we get the real projective space $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$. Indeed the map $S^2 \times [-1, 1] \hookrightarrow S^3$:

$$(x, t) \mapsto (x \cos \pi t/4, \sin \pi t/4) \in S^3 \subset \mathbb{R}^4,$$

defines an embedding of the connecting manifold into $\mathbb{R}P^3$. Its image is the complement of the ball in $\mathbb{R}P^3$ determined by the equation $x_4^2 > 1/2$.

It is known that $\mathbb{R}P^3$ can be obtained from S^3 by surgery on the unknot with framing -2 , see Appendix B. If we follow how the ribbon graph is mapped by the surgery and view S^3 as the one point compactification of \mathbb{R}^3 , we may describe the situation as in Figure 4. The region depicted is contained in a ball in \mathbb{R}^3 which after surgery on the annulus drawn as horizontal is mapped to the connecting 3-manifold $(S^2 \times [-1, 1])/\mathbb{Z}_2$. The vertical line is mapped to the ribbon graph in the connecting 3-manifold. If we attach the 3-ball onto S^2 and compute with the formulae of Appendix B the image of the tennis ball pattern, we see that the vertical line matches the ribbon graph in the 3-ball to give an unframed unknot linked to the horizontal unknot. According to the Reshetikhin–Turaev surgery formula (12) the resulting invariant is

$$\Delta^{-1} \mathcal{D}^{-1} \sum_{j \in I} v_j^{-2} \dim(j) s_{i,j},$$

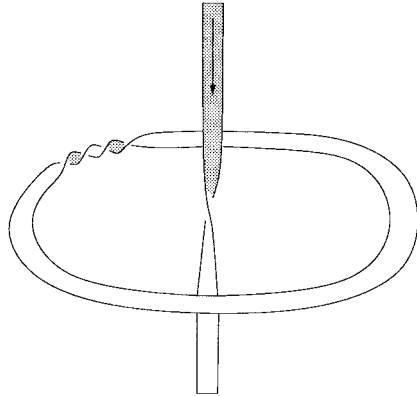


Figure 4.

with $\Delta = \sum_{i \in I} v_i^{-1} \dim(i)^2$. By using $\kappa = \Delta \mathcal{D}^{-1}$ we then get

$$c_i = v_i^{-1/2} \Delta^{-1} \sum_{j \in I} v_j^{-2} \frac{\dim(j)}{\dim(i)} s_{i,j} = v_i^{-1/2} \kappa^{-1} \sum_{j \in I} S_{i,j} v_j^{-2} \frac{S_{j,1}}{S_{i,1}}.$$

This result can be expressed in terms of the matrix P [BS], which is defined in terms of the representation of $SL(2, \mathbb{Z})$:

$$P = \kappa^{-1} \hat{T}^{1/2} S \hat{T}^2 S \hat{T}^{1/2} = T^{1/2} S T^2 S T^{1/2}.$$

The square root is defined using the choice of square roots of the v_i . The matrix P is symmetric. Its square is the conjugation matrix $C = (\delta_{i,j^*})$.

We then have $c_i = P_{i,1}/S_{i,1}$.

Summarizing, our result is $C(\mathbb{R}P^2, i) = (P_{i,1}/S_{i,1})\psi_i$. The corresponding normalized correlation function agrees with the result $C_{\text{norm}}(\mathbb{R}P^2, i) = (P_{i,1}/\sqrt{S_{i,1}})\psi_i$ obtained in conformal field theory (see, e.g., [PSS1]).

5. Annulus, Klein Bottle, Möbius Strip

We consider here the three cases of surfaces whose double is a torus: the annulus, the Klein bottle and the Möbius strip. The correlation function with no marked points (partition function) can then be expressed in terms of the basis $\chi_j(S^1 \times S^1)$ of invariants of the solid torus. It is then expected on physical grounds that the coefficients of the partition functions obey certain integrality conditions. We compute the partition function in these three cases and show that these conditions are obeyed. Different ways of computing correlation functions implies remarkable properties of the $SL(2, \mathbb{Z})$ representations arising from modular categories. The most

well-known one is the Verlinde formula

$$N_{j,k,l} = \sum_{r \in I} \frac{S_{r,j} S_{r,k} S_{r,l}}{S_{r,1}},$$

which may be understood as the result of two different computation of the annulus partition function.

5.1. ANNULUS PARTITION FUNCTION

Let $X = (A, a, b)$ be an annulus whose boundary has connected components labeled by a and b . The double \hat{X} is then a torus and the connecting 3-manifold is a solid torus $D^2 \times S^1$ with two equatorial graphs without outgoing edges. To be concrete, let us think of A as the region between two circles centered at the origin of the x - y plane. The two circles forming the boundary are oriented counterclockwise. Then \hat{X} may be thought of as the surface obtained by revolution around the z -axis of a circle \mathcal{C} in the x - z plane with center on the x -axis. Then the projection $p: \hat{X} \mapsto X$ is the orthogonal projection onto the x - y plane and the involution σ is the reflection at the x - y plane. The connecting 3-manifold M is the solid obtained by revolution of the disk in the x - z plane with boundary \mathcal{C} . It contains two annuli in the x - y plane oriented counterclockwise and labeled by a, b . A well-known calculation using (4) and (3) shows that $Z(M, \emptyset, S^1 \times S^1) = \sum_k N_{a,b}^k \chi_k(S^1 \times S^1)$. Thus

$$C(A, a, b) = \sum_{k \in I} N_{a,b}^k \chi_k(S^1 \times S^1).$$

The alternative way of doing this calculation is to glue another solid torus with an annulus graph to obtain the invariant of S^3 with three unknots. The identity between the two results is the Verlinde formula, see [W]. Our result can therefore be understood as a three-dimensional version of the derivation of the Verlinde formula in [C1], and shows that at this level the arguments of [C1] are completely equivalent to those given in [W].

A similar reasoning will be used below for the Möbius strip.

5.2. THE PARTITION FUNCTION OF THE KLEIN BOTTLE

Let \mathbb{Z}_2 act on $S^1 \times S^1$ via the involution $\sigma: (z, w) \mapsto (z^{-1}, -w)$. The quotient space is the Klein bottle $K = (S^1 \times S^1)/\mathbb{Z}_2$, with double $\hat{K} = S^1 \times S^1$. We compute the correlation function $C(K) \in \mathcal{H}(\hat{K})$ of the Klein bottle with no marked points. A basis of $\mathcal{H}(\hat{K})$ is given by

$$\chi_j(S^1 \times S^1) = Z((H, j), \emptyset, S^1 \times S^1), \quad j \in I,$$

with $H = D^2 \times S^1$ containing an annulus labeled by j , as above.

So

$$C(K) = \sum_{i \in I} c_i(K) \chi_i(S^1 \times S^1),$$

for some complex coefficients $c_i(K)$. The convention we have chosen here is that the first factor of S^1 in $\hat{K} = S^1 \times S^1$, which becomes a contractible cycle in the solid torus H , generates the kernel of $p_*: H_1(\hat{K}, \mathbb{R}) \rightarrow H_1(K, \mathbb{R})$.

We compute $c_j(K)$ by composing both sides of the above equation with the invariant $Z((-H, j^*), S^1 \times S^1, \emptyset)$. If we compose this invariant with $\chi_i(S^1 \times S^1)$ we get $\delta_{i,j}$, the invariant of $S^2 \times S^1$ with two annuli $z_1 \times S^1$, $z_2 \times S^1$ labeled by i, j^* . Therefore the right-hand side becomes $c_j(K)$.

The left-hand side is then the invariant of the 3-manifold M' obtained by gluing $(-H, j^*)$ to the connecting manifold M_K of the Klein bottle. We claim that this 3-manifold is homeomorphic to $S^2 \times S^1$ with a certain annulus labeled by j .

To see this, let us identify S^2 with $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. The connecting manifold M_K consists of classes $[(z, w, t)]$ of triples $(z, w, t) \in S^1 \times S^1 \times [-1, 1]$ modulo

$$(z, w, t) \sim (z^{-1}, -w, -t).$$

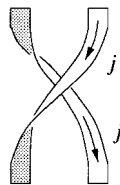
Then we have the embedding $M_K \hookrightarrow \mathbb{C}P^1 \times S^1$ given by

$$\iota: [(z, w, t)] \mapsto \left(2w \frac{e^t z - 1}{e^t z + 1}, w^2 \right).$$

The complement of $\iota(K)$ in $S^2 \times S^1$ is the interior of a solid torus $D^2 \times S^1$, embedded via $(z, w) \mapsto (2w(e^{-1}z - 1)/(e^{-1}z + 1), w^2)$. The image of the ribbon graph $[-\varepsilon, \varepsilon] \times S^1 \subset D^2 \times S^1$ in $M' = S^2 \times S^1$ is an annulus. The intersection of this annulus with the fiber over $u \in S^1$ consists of two segments centered at $\pm 2\sqrt{u}$ and is contained in the straight line connecting these two points. As u runs over the unit circle, the two points rotate around the origin by 180 degrees. By the trace formula (7), the invariant of M' is the trace

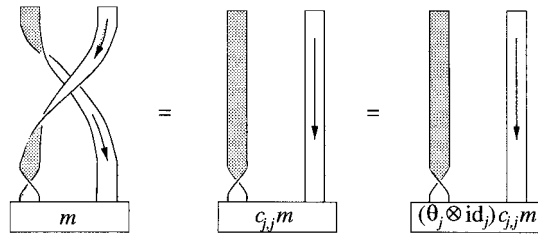
$$Z(M') = \text{Tr}_{\mathcal{H}(S^2; j, j)}(\Phi_j)$$

over the space of states for the sphere with two marked arcs of the morphism Φ_j represented by the graph



in $S^2 \times [0, 1]$, where we think of S^2 as the $x - y$ plane in \mathbb{R}^3 by stereographic projection. If $j \neq j^*$, $\mathcal{H}(S^2; j, j) = \{0\}$ and $Z(M')$ vanishes. If $j \simeq j^*$, $\mathcal{H}(S^2; j, j)$ is one-dimensional with a basis given by the invariant of a ball with a ribbon graph

consisting of ribbons connecting the two arcs to a two-valent vertex labeled by any nonzero morphism $m \in \text{Hom}(\mathbf{1}, j \otimes j)$. Any such morphism may be written as $m = (\text{id}_j \otimes \phi) \circ b_j$ for any isomorphism $\phi: j^* \rightarrow j$. Then we have $(\theta_j \otimes \text{id}_j) \circ c_{j,j} \circ m = v(j)m$, see 2.3, where $v(j) = \pm 1$ is the Frobenius–Schur indicator of j . The morphism Φ_j is then the Frobenius–Schur indicator times the identity, as can be seen by acting on the basis element:



Therefore $Z(M') = v(j)$. We conclude that

$$C(K) = \sum_{j \simeq j^*} v(j) \chi_j(S^1 \times S^1),$$

in agreement with [HSS]. The summation is over all self-dual objects $j \in I$.

5.3. THE MÖBIUS STRIP

We now consider the correlation function of the Möbius strip with no marked point and boundary condition $a \in I$. The double of the Möbius strip is $S^1 \times S^1$ with involution $\sigma: (z, w) \mapsto (w/z, w)$. Thus if $M\ddot{o}$ is the Möbius strip, its connecting manifold is $M_{M\ddot{o}} = (S^1 \times S^1 \times [-1, 1])/\mathbb{Z}_2$ and is degree one homeomorphic via

$$(z, w, t) \mapsto \left(\frac{1+t}{2} z + \frac{1-t}{2} wz^{-1}, w \right),$$

to $D^2 \times S^1$. The equatorial graph consists of an annulus lying in the zero section $t = 0$ and running close to the boundary, see Figure 5. The correlation function has then the form

$$C(M\ddot{o}; a) = \sum_{j \in I} m_{a,j} \chi_j(S^1 \times S^1).$$

The coefficients $m_{a,j}$ may be computed by composing both sides of the equation with the invariant of a solid torus with a ribbon graph consisting of an annulus labeled by $l \in I$, in such a way that the manifold obtained by gluing is the 3-sphere. The right-hand side becomes $\sum_j m_{a,j} \mathcal{D}^{-1} s_{j,l}$, and the left-hand side is \mathcal{D}^{-1} times the invariant of the link represented in Figure 6. The invariant may be further simplified,

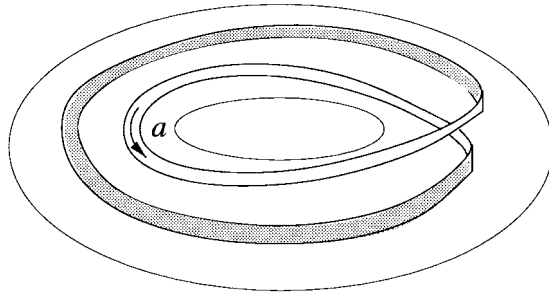


Figure 5. A solid torus with a ribbon graph, whose invariant is the correlation functions of the Möbius strip with boundary condition a .

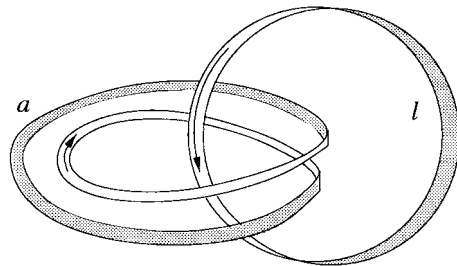


Figure 6. A link in S^3 used to compute the partition function of the Möbius strip.

by first flattening the ribbon onto a plane:

$$\begin{aligned}
 \text{Diagram 1} &= \sum_{k, \alpha} \dim(k) \text{Diagram 2} \\
 &= \sum_{k, \alpha} \dim(k) \text{Diagram 3}
 \end{aligned}$$

The diagrams are as follows:
 Diagram 1: A flattened version of the link in Figure 6, showing two overlapping loops 'a' and 'l' with arrows.
 Diagram 2: A diagram where the loops are separated by a vertical line. The left side is labeled 'a' and the right side 'l'. A vertical line with a double-headed arrow is labeled 'k' and 'α'.
 Diagram 3: A diagram where the loops are separated by a vertical line. The left side is labeled 'a' and the right side 'l'. A vertical line with a double-headed arrow is labeled 'k' and 'α'. There are small circles on the right side of the line.

The expression in the sum is then $v_k^2 v_l^{-2} \dim(k) \delta_{a, \alpha}$. The sum over the $N_{a,l}^k$ possible values of α may be performed, with the result

$$\sum_j m_{a,j} s_{j,l} = \sum_k v_k^2 v_l^{-2} \dim(k) N_{a,l}^k.$$

By using the relation $\sum_l s_{i,l} s_{l,j^*} = \mathcal{D}^2 \delta_{i,j}$, and expressing our result in terms of the matrix $S = \mathcal{D}^{-1} s$, we obtain:

$$C(M\ddot{o}; a) = \sum_{j \in I} m_{a,j} \chi_j(S^1 \times S^1),$$

with

$$m_{a,j} = \sum_{k,l \in I} v_k^2 v_l^{-2} S_{1,k} N_{a,l}^k S_{l,j^*}. \tag{11}$$

An alternative way to do the computation is to cut out a ball in the solid torus of Figure 5 intersecting the ribbon graph in two segments, and use (4) with $i = j = a$. Then the ribbon graph may be replaced by a ribbon labeled by the summation index k which starts and ends at a two-valent vertex after going once around the solid torus. The two-valent vertex is labeled by $\dim(k)$ times the morphism represented by the graph



The sum over α is $\sum_{\alpha} e_{\alpha}[a^*ka^*] \Phi_{a,k} e_{\alpha}[ak^*a]$, with $e_{\alpha}[ak^*a]$, $\alpha = 1N_{a,k^*,a}$, regarded as a basis of $\text{Hom}(k, a \otimes a)$ and $e_{\alpha}[a^*ka^*]$ as a basis of $\text{Hom}(a \otimes a, k)$. $\Phi_{a,k}$ is the linear endomorphism of $\text{Hom}(k, a \otimes a)$ given by $\Phi_{a,k}(x) = (\theta_a \otimes \text{id}_a) \circ c_{a,a} \circ x$. Since $e_{\alpha}[ak^*a] \circ e_{\beta}[a^*ka^*] = \dim(k)^{-1} \delta_{\alpha,\beta}$ (Equation (3)), the quantum dimensions cancel, and we are left with $m_{a,k} = \text{Tr}_{\text{Hom}(k, a \otimes a)} \Phi_{a,k}$. Moreover, $\Phi_{a,k}^2 = v_k \text{id}$, as may be seen by deforming the graph representing this morphism put on top of itself or by using axioms (iv), (v) of Appendix A. It follows that $\Phi_{a,k}$ is diagonalizable with eigenvalues $\pm \sqrt{v_k}$. Therefore we have

$$m_{a,k} = M_{a,k} v_k^{1/2}, \quad M_{a,k} \in \mathbb{Z}, \quad M_{a,k} \equiv N_{a,a}^k 2, \quad |M_{a,k}| \leq N_{a,a}^k$$

The two different ways of calculating the Möbius strip partition function implies the following result, essentially due to Bantay [B], on representations of $SL(2, \mathbb{Z})$ arising from modular categories.

THEOREM 5.1. *Let $S = (\mathcal{D}^{-1} s_{j,k})$, $\hat{T} = (v_j^{-1} \delta_{j,k})$ the matrices defining the projective representation of $SL(2, \mathbb{Z})$ associated to a modular category with rank \mathcal{D} , see 2.5.2, and let $N_{j,k}^i = N_{i^*,j,k} \in \mathbb{Z}_{\geq 0}$ be the corresponding Verlinde numbers (10). Let*

$Q = S\hat{T}^2S^{-1}$. Then the numbers

$$M_{a,k} = v_k^{-1/2} \sum_{r \in I} (Q^{-1})_{1,r} \frac{S_{a,r}}{S_{1,r}} Q_{k,r},$$

are integers and obey $M_{a,k} \equiv N_{a,a}^k 2, |M_{a,k}| \leq N_{a,a}^k$.

The above expression for $M_{a,k}$ was obtained from (11) by using the Verlinde formula (10).

Remarks. (1) The above formula amounts to not completely trivial identities even for the simple \mathbb{Z}_{2N} example of 2.1. There we have

$$\frac{S_{a,r}}{S_{1,r}} = e^{-\pi i ar/N}, \quad Q_{k,r} = \begin{cases} \frac{1}{\sqrt{N}} e^{\frac{i\pi}{4} - \frac{i\pi}{4N}(k-r)^2}, & \text{if } k-r + N \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

and, with $v_k^{-1/2} = \exp(\pi i k^2/4N)$,

$$M_{a,k} = \begin{cases} 1, & \text{if } k \text{ is even and } a \equiv k/2 \pmod{2N}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that being even is well defined in \mathbb{Z}_{2N} , and that if k is even the above choice of square root of v_k is unambiguous.

(2) More generally one expects ([PSS1]) the numbers

$$Y_{a,j}^k = v_k^{-1/2} v_j^{1/2} \sum_r (Q^{-1})_{j,r} \frac{S_{a,r}}{S_{1,r}} Q_{k,r}$$

to be integer for any j . This has been recently shown to be true under some additional assumptions (see [Gan]).

Appendix A. Modular Categories

We give here the precise definition of modular categories, following [T]. A monoidal (= tensor) category with product \otimes and unit $\mathbf{1}$ for the product is called *strict* if for any objects U, V, W , we have $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, and $V \otimes \mathbf{1} = \mathbf{1} \otimes V = V$. A monoidal Ab-category is a monoidal category such that morphisms between any two objects U, V form an additive Abelian group $\text{Hom}(U, V)$, and such that compositions and tensor products of morphisms are bilinear. In particular, the endomorphisms of the unit $\mathbf{1}$ of the tensor product form a ring with unit. This ring is called the ground ring. The groups of morphisms are naturally modules over the ground ring. An object V of a monoidal Ab-category is called simple if $\text{Hom}(V, V) \simeq K$ as a K -module.

A *ribbon category* is a strict monoidal category with additional data: a braiding, a twist and a duality.

A *braiding* associates to any pair of objects V, W an isomorphism $c_{V,W} \in \text{Hom}(V \otimes W, W \otimes V)$. A *twist* associates to any object V an isomorphism $\theta_V \in \text{Hom}(V, V)$. A *duality* associates to any object V a dual object V^* and morphisms $b_V \in \text{Hom}(\mathbf{1}, V \otimes V^*)$, $d_V \in \text{Hom}(V^* \otimes V, \mathbf{1})$.

These data obey the following set of axioms:

- (i) $c_{U,V \otimes W} = (\text{id}_V \otimes c_{U,W})(c_{U,V} \otimes \text{id}_W)$.
- (ii) $c_{U \otimes V, W} = (c_{U,W} \otimes \text{id}_V)(\text{id}_U \otimes c_{V,W})$.
- (iii) $g \otimes f c_{V,W} = c_{V',W'} f \otimes g$.
- (iv) $\theta_{V \otimes W} = c_{W,V} c_{V,W} (\theta_V \otimes \theta_W)$.
- (v) $\theta_V f = f \theta_V$.
- (vi) $(\text{id}_V \otimes d_V)(b_V \otimes \text{id}_V) = \text{id}_V$.
- (vii) $(d_V \otimes \text{id}_{V^*})(\text{id}_{V^*} \otimes b_V) = \text{id}_{V^*}$.
- (viii) $(\theta_V \otimes \text{id}_{V^*})b_V = (\text{id}_V \otimes \theta_{V^*})b_V$.

Here U, V, \dots are arbitrary objects and $f \in \text{Hom}(V, V')$, $g \in \text{Hom}(W, W')$ are arbitrary morphisms.

Finally a *modular category* is a ribbon Ab-category with a finite family I of simple objects such that:

- (ix) $\mathbf{1} \in I$.
- (x) For every $i \in I$, i^* is isomorphic to an object in I .
- (xi) Every morphism $f: V \rightarrow V'$ may be decomposed into a finite sum $\sum_r g_r h_r$, where $h_r \in \text{Hom}(V, i)$ and $g_r \in \text{Hom}(i, V')$ for some $i = i(r)$.
- (xii) The matrix $(s_{i,j}) = (\text{tr}(c_{j,i} c_{i,j}))$ indexed by $i, j \in I$ is invertible.

Appendix B. Surgery on the Unknot

Here we describe explicitly the homeomorphism between $\mathbb{R}P^3$ and the 3-manifold obtained by surgery on the unknot in S^3 with framing ± 2 .

Let us first fix some conventions about orientations. We give an orientation to the boundary of an oriented manifold M by the ‘outward normal first’ rule. This means that a basis (b_1, \dots, b_n) of the tangent space $T_x \partial M$ at a boundary point x is positively oriented if and only if (b_0, b_1, \dots, b_n) is a positively oriented basis of $T_x M$ for any b_0 pointing outwards. The orientations of the disks $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ are inherited from \mathbb{R}^n . They define orientations of the spheres $S^{n-1} = \partial D^n$. We orient \mathbb{C}^n via the isomorphism $(x_1 + iy_1, \dots, x_n + iy_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$ with \mathbb{R}^{2n} , and view odd dimensional spheres S^{2n-1} as subsets of \mathbb{C}^n .

Recall that if L is the image of a smooth embedding of S^1 in S^3 , and n is an integer, then the surgery on L with framing n is the following construction. Let U be a closed tubular neighborhood of L . Fix an embedding $j: S^1 \times D^2 \hookrightarrow S^3$ with image U so that $S^1 \times \{0\}$ is sent to L , and $S^1 \times \{1\}$ is sent to a knot L' whose linking number with L is n . The linking number is calculated using the orientations of L, L' coming from the orientation of S^1 via j . We may think that L with framing n as an embedded annulus

with boundary $L \cup L'$. Let $S^3_{(L,n)}$ be the manifold obtained by gluing the complement $S^3 - \text{int } U$ of the interior of U to $D^2 \times S^1$ via the restriction to $S^1 \times S^1$ of the map $j: S^3_{(L,n)} = (S^3 - \text{int } U) \sqcup (D^2 \times S^1)/(x \sim j(x), x \in S^1 \times S^1)$. The orientation of $S^3_{(L,n)}$ is defined to be the orientation that extends the standard orientation of $S^3 - U \subset S^3$.

If L is the unknot defined, say, by the embedding $z \mapsto (z, 0)$ of $S^1 \subset \mathbb{C}$ into $S^3 \subset \mathbb{C}^2$, and the framing is n , then we may take, for small $\varepsilon > 0$,

$$j(z, w) = \frac{1}{\sqrt{1 + |\varepsilon w|^2}} (z, \varepsilon w z^n), \quad (z, w) \in S^1 \times D^2.$$

The image is $U = \{(u, v) \in S^3 \subset \mathbb{C}^2 \mid \varepsilon|u| \geq |v|\}$. The 3-manifold $S^3_{(L,n)}$ obtained by surgery on this link is (for $n \neq 0$) homeomorphic to the lens space $S^3/\mathbb{Z}_{|n|}$, with $\mathbb{Z}_{|n|}$ action generated by $(u, v) \mapsto (\zeta u, \zeta v)$, $\zeta = \exp(2\pi i/|n|)$.

Indeed the map

$$i_n : (u, v) \mapsto \left(\frac{v}{|v|}\right)^{-1/n} (u, |v|),$$

is a homeomorphism $S^3 - L \rightarrow (S^3 - L)/\mathbb{Z}_{|n|}$ (L is invariant under the $\mathbb{Z}_{|n|}$ action), with inverse map $(u, v) \mapsto (uv^{-1}|v|, v^{-n}|v|^{n+1})$. This homeomorphism extends to a homeomorphism from $S^3_{(L,n)}$ onto $S^3/\mathbb{Z}_{|n|}$. This follows from the fact that

$$i_n \circ j \Big|_{S^1 \times S^1} : (z, w) \mapsto \frac{w^{-1/n}}{\sqrt{1 + \varepsilon^2}} (1, \varepsilon z^{-1})$$

extends to the homeomorphism

$$(z, w) \mapsto \frac{w^{-1/n}}{\sqrt{1 + \varepsilon^2|z|^2}} (1, \varepsilon \bar{z})$$

from $D^2 \times S^1$ onto a tubular neighborhood of $L/\mathbb{Z}_{|n|}$ in $S^3/\mathbb{Z}_{|n|}$.

The sign of n is connected with the orientation. Since the $\mathbb{Z}_{|n|}$ action on S^3 preserves the orientation, the lens space $S^3/\mathbb{Z}_{|n|}$ inherits an orientation from S^3 . The map i_n is then orientation preserving if $n < 0$ and orientation reversing if $n > 0$, as can be seen by linearizing i_n at $(0, 1)$.

Let us summarize the results.

PROPOSITION B.1. *Let L be the image of the embedding $z \mapsto (z, 0)$ of $S^1 \mapsto S^3$. Then, for each $n \in \mathbb{Z} - \{0\}$, the map $i_n: S^3 - L \mapsto (S^3 - L)/\mathbb{Z}_{|n|}$*

$$(u, v) \mapsto \left(\frac{v}{|v|}\right)^{-1/n} (u, |v|),$$

extends uniquely to a homeomorphism from the manifold $S^3_{(L,n)}$ obtained by surgery on L with framing n onto the lens space $S^3/\mathbb{Z}_{|n|}$. The degree of this homeomorphism is $-\text{sign}(n)$. In particular, i_{-2} is a degree one homeomorphism from $S^3_{(L,-2)}$ onto $\mathbb{R}P^3$.

Moreover we have the Reshetikhin–Turaev formula for the invariant of a ribbon graph in a manifold obtained from S^3 by surgery, see [T]. It can be deduced from the functoriality axiom applied to a solid torus. In the case of an unknot with framing $n < 0$, assuming that the ribbon graph does not intersect the solid torus at which we perform the surgery, it reads

$$Z(S^3_{(L,n)}; \Gamma) = \Delta \sum_{j \in I} \dim(j) Z(S^3, \Gamma_j), \quad (12)$$

with $\Delta = \sum_{i \in I} v_i^{-1} \dim(i)^2$. The ribbon graph Γ_j is obtained from Γ by adding L , viewed as an embedded annulus in S^3 , with label j . This involves a choice of orientation of L , but the result does not depend on this choice.

To visualize the results it is useful to stereographically project $S^3 - \{pt\}$ to \mathbb{R}^3 . For our application, a useful projection is

$$(x_1 + ix_2, x_3 + ix_4) \mapsto \frac{1}{1 + x_3} (x_1, x_2, x_4).$$

It preserves the orientation. Then the link L at which we do surgery is mapped to the unit circle in the x_1 - x_2 plane.

Appendix C. Physical Motivation

Two-dimensional conformal field theory plays a fundamental role in several distinct areas of physics. In particular, it can be used to describe universality classes of two-dimensional critical systems of classical statistical mechanics ([FQS]) and of quasi one-dimensional condensed matter physics ([Af]). For instance, the scaling behavior of such a system is encoded in so-called ‘critical exponents’. They are related to conformal weights, which, in turn, determine the twist in the tensor categories we consider in this paper. Similarly, the finite size scaling behavior is governed by the value of the conformal central charge, which, in turn, gives the charge κ of the tensor category. In string and superstring theory ([GSW]), all aspects involving the world sheet that is swept out by the string moving in space-time can be understood in terms of a conformal field theory model on the world sheet. The correlation functions of the conformal field theory model are the building blocks for the string scattering amplitudes.

Each physical state of a chiral conformal field theory is a ray in some irreducible module of a ‘chiral algebra’. One mathematical formalization of chiral algebras is the notion of a conformal vertex-operator algebra. It is believed, though not proven so far, that the category of representations of every rational vertex-operator algebra gives rise to a modular tensor category, as studied in the main text. As a consequence, the study of conformal field theories has two main aspects, a representation theoretic one, related to the theory of vertex-operator algebras, and another one that can be formulated entirely in terms of the tensor category

of representations. The present paper is devoted to the latter aspect of conformal field theory.

There are many reasons to consider conformal field theory also on surfaces with boundaries. Boundaries can describe, e.g., the presence of a point-like defect in systems of condensed matter physics ([OA]). In such applications space is (effectively) one-dimensional, a half-line, with the defect sitting at the end-point of the line; the second dimension is identified with (imaginary) time. Another application of boundary conformal field theory is the study of (critical) percolation probabilities ([C2]), which have been argued to coincide with certain correlation functions of a nonunitary conformal field theory on a disk.

In string theory, boundaries appear as soon as open strings are present, namely as the world lines of the end points of the open string. Moreover, (perturbative) superstring theory can be regarded as providing a perturbative quantization of certain classical field theories, supergravity theories. These theories possess nontrivial classical solutions, so-called string solitons. One important recent insight in string theory is that there exists a string perturbation theory for one class of these solitons, the so-called D-branes. It is formulated ([P]) in terms of open string world sheets and thus provides another reason to study conformal field theories on surfaces with boundaries. Finally, there is also a class of superstring theories, so-called type I theories, which forces one to analyze conformal field theory on nonorientable surfaces.

String theory amplitudes are defined as sums over world sheets of arbitrary genus. Such sums can be expected to behave reasonably only if the terms in the sum are related to each other. The factorization constraints that we prove in this paper show that nontrivial relations between correlation functions on surfaces of different topology indeed exist.

The following features have become clear in the study of conformal field theory on surfaces with boundaries: To each component of the boundary one must associate a 'boundary condition', which encodes what happens when fields are located close to the boundary. In string theory, in the regime of small curvature, many boundary conditions have a geometric interpretation in terms of submanifolds of spacetime, so-called D-branes. Associating a boundary condition to a component of the boundary of a world sheet has the geometric interpretation of constraining the image of the component, which is the world line of a string end point, to lie on some D-brane. As a consequence, the correlation functions of conformal field theory on surfaces with boundaries depend on these boundary conditions. Also, in addition to the bulk fields which correspond to insertions in the interior of the world sheet, there are also so-called boundary fields; they are inserted on the boundary of the worldsheet and may separate regions of the boundary with different boundary conditions. All boundary conditions considered in the present paper are compatible with the full chiral symmetry that is present in the bulk of the worldsheet. Such boundary conditions are labelled by irreducible representations of the chiral algebra ([C1]). More general boundary conditions are, however, possible. They preserve only a

subalgebra of the chiral algebra. If that subalgebra contains the Virasoro element, then they are conformally invariant. A general theory of conformally invariant boundary conditions remains to be developed.

Acknowledgements

G.F. is thankful to V. Turaev for useful correspondence. We thank Ph. Mathey for carefully reading the manuscript and for corrections.

References

- [Af] Affleck, I.: A current algebra approach to the Kondo effect, *Nuclear Phys. B* **336** (1990), 517.
- [Al] Alessandrini, V.: A general approach to dual multiloop diagrams, *Nuovo Cimento A* **2** (1971), 321–352.
- [B] Bantay, P.: The Frobenius-Schur indicator in conformal field theory, *Phys. Lett. B* **394** (1997), 87–88.
- [BPPZ] Behrend, R. E., Pearce, P. A., Petkova, V. B. and Zuber, J.-B., Boundary conditions in rational conformal field theories, *Nuclear Phys. B* **570** (2000), 525–589.
- [BS] Bianchi, M. and Sagnotti, A., The partition function of the SO(8192) bosonic string, *Phys. Lett. B* **211** (1988), 407–416.
- [C1] Cardy, J. L., Boundary conditions, fusion rules and the Verlinde formula, *Nuclear Phys. B* **324** (1989), 581–596.
- [C2] Cardy, J. L.: Critical percolation in finite geometries, *J. Phys. A* **251** (1992), L201.
- [EMSS] Elitzur, S., Moore, G., Schwimmer, A. and Seiberg, N.: Remarks on the canonical quantization of the Chern–Simons–Witten theory, *Nuclear Phys. B* **326** (1989), 108–134.
- [FFFS1] Felder, G., Fröhlich, J., Fuchs, J. and Schweigert, C.: The geometry of WZW branes, *J. Geom. Phys.* **34** (2000), 162–190.
- [FFFS2] Felder, G., Fröhlich, J., Fuchs, J. and Schweigert, C.: Conformal boundary conditions and three-dimensional topological field theory, *Phys. Rev. Lett.* **84** (2000), 1659–1662.
- [F] Finkelberg, M.: An equivalence of fusion categories, *Geom. Funct. Anal.* **6** (1996), 249–267.
- [FQS] Friedan, D. H., Qiu, Z. and Shenker, S. H.: Conformal invariance, unitarity and two-dimensional critical exponents, In: J. Lepowsky, S. Mandelstam, and I. M. Singer (eds), *Vertex Operators in Mathematics and Physics*, Springer-Verlag, New York, 1985, 491.
- [FK] Fröhlich, J. and King, C.: Two-dimensional conformal field theory and three-dimensional topology, *Internat. J. Modern Phys.* **4** (1989), 5321–5399.
- [FS] Fuchs, J. and Schweigert, C.: Branes: from free fields to general backgrounds, *Nuclear Phys. B* **530** (1998), 99–136.
- [Gan] Gannon, T.: Integers in the open string, *Phys. Lett. B* **473** (2000), 80–85.
- [Gaw] Gawędzki, K.: Conformal field theory: a case study, preprint hep-th/9904145.
- [GSW] Green, M. B., Schwarz, J. H. and Witten, E.: *Superstring Theory*, Cambridge Univ. Press, Cambridge, 1987.

- [H] Hořava, P.: Chern–Simons gauge theory on orbifolds: open strings from three dimensions, *J. Geom. Phys.* **21** (1996), 1–33.
- [HSS] Huiszoon, L. R., Schellekens, A. N. and Sousa, N.: Klein bottles and simple currents, preprint hep-th/9909114
- [K] Kirillov, A.: On inner product in modular tensor categories. II. Inner product on conformal blocks and affine inner product identities, *Adv. Theor. Math. Phys.* **2** (1998), 155–180.
- [KL] Kazhdan, K. and Lusztig, G.: Tensor structures arising from affine Lie algebras. I, *J. Amer. Math. Soc.* **6** (1993), 905–947; II, *J. Amer. Math. Soc.* **6** (1993), 949–1011; III; *J. Amer. Math. Soc.* **7** (1994), 335–381; IV, *J. Amer. Math. Soc.* **7** (1994), 383–453.
- [L] Lewellen, D.: Sewing constraints for conformal field theory on surfaces with boundaries, *Nuclear Phys. B* **372** (1992), 654–682.
- [MS] Moore, G. and Seiberg, N.: Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.
- [OA] Oshikawa, M. and Affleck, I.: Defect lines in the Ising model and boundary states on orbifolds, *Phys. Rev. Lett.* **77** (1996), 2604.
- [P] Polchinski, J.: Dirichlet-branes and Ramond–Ramond charges, *Phys. Rev. Lett.* **75** (1995), 4724.
- [PSS1] Pradisi, G., Sagnotti, A. and Stanev, Ya. S.: Planar duality in SU(2) WZW models, *Phys. Lett. B* **354** (1995), 279–286.
- [PSS2] Pradisi, G., Sagnotti, A. and Stanev, Ya. S.: Completeness conditions for boundary operators in 2D conformal field theory, *Phys. Lett. B* **381** (1996), 97–104.
- [RT1] Reshetikhin, N. Yu. and Turaev, V. G.: Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Phys.* **127** (1990), 1–26.
- [RT2] Reshetikhin, N. Yu. and Turaev, V. G.: Invariants of 3-manifolds via link polynomials and quantum groups, *Invent Math.* **103** (1991), 547–598.
- [R] Runkel, I.: Boundary structure constants for the A series Virasoro minimal models, *Nuclear Phys. B* **549** (1999), 563–578.
- [T] Turaev, V. G.: *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter, Berlin, 1994.
- [W] Witten, E.: Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–399.