A CONVERSE PROBLEM IN MATRIX DIFFERENTIAL EQUATIONS

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Suppose $X$ and $Y$ are $n \times n$ matrix solutions of the $n \times n$ matrix differential equation

(1) \[ X'' + P(t)X = 0 \quad \text{on } J \]

such that

(2) \[ XX^* + YY^* = I \quad \text{on } J \]

where $J$ is some interval. Here $P(t)$ is a symmetric $n \times n$ matrix and 0 and $I$ respectively the $n \times n$ zero and identity matrices. Also $X^*$ and $\text{tr } X$, denote, respectively, the transpose and the trace of $X$.

In the scalar case it is well known (1) and (2) together imply $p(t) \equiv k$, a non-negative constant. See [1] and the references therein. The lower case letters used above indicate scalars which are $1 \times 1$ matrices.

Now in the case that $n>1$, we can no longer be sure that $P(t) \equiv K$ a constant matrix, but we can obtain a result which in the scalar case implies $p(t) \equiv k$. To see that $P(t) \equiv K$ may fail when (1) and (2) are true, we consider the following example. Let

\[
U(t) = \begin{pmatrix}
\frac{1}{\sqrt{3}} \cos 4t & \frac{1}{\sqrt{3}} \sin 4t \\
\sqrt{2} \cos 4t + \frac{8}{5\sqrt{3}} \sin 4t & \frac{8}{5\sqrt{3}} \cos 4t - \frac{\sqrt{2}}{5} \sin 4t
\end{pmatrix}
\]

and

\[
V(t) = \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{5}} \cos t + \frac{2}{\sqrt{15}} \sin t & \frac{2}{\sqrt{15}} \cos t + \frac{\sqrt{2}}{\sqrt{5}} \sin t \\
\frac{1}{\sqrt{15}} \cos t & -\frac{1}{\sqrt{15}} \sin t
\end{pmatrix}
\]

Further, let

\[
C = \begin{pmatrix} 3 & \sqrt{6} \\ \sqrt{6} & 12 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

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and, note that

\[ e^{Bt} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \]

and \((e^{Bt})^* = e^{B^*t} = e^{-Bt}\). The following may be easily, but tediously, verified.

\[ UU^* + VV^* = I. \]

Next we note that \(U\) and \(V\) are solutions of \(U'' - 2BU' + (C - B^*B)U = 0\), a constant coefficient matrix differential equation. Defining \(X = e^{-Bt}U\) and \(Y = e^{-Bt}V\) we obtain that \(X\) and \(Y\) are solutions to \(X'' + P(t)X = 0\) where \(P(t) = e^{-Bt}Ce^{Bt}\) is a nonconstant matrix. Looking at \(XX^* + YY^*\), we see that

\[ XX^* + YY^* = e^{Bt}UU^*(e^{Bt})^* + e^{Bt}VV^*(e^{Bt})^* = e^{Bt}(UU^* + VV^*)e^{-Bt} = e^{Bt}Ie^{-Bt} = I. \]

Hence (1) and (2) are satisfied but \(P(t) \equiv K\) fails in this case where \(n=2\). It is easy to see that this system can be incorporated into a higher dimensional system thus preventing the conclusion that \(P(t) \equiv K\) for \(n>2\) also.

We are now ready to state and prove a theorem true for all \(n \geq 1\).

**Theorem.** Let \(X\) and \(Y\) be solutions of the matrix differential equation (1) where \(P(t)\) is a symmetric matrix. Suppose that (2) holds. Then \(P(t)\) is positive semi-definite and \(\text{tr} P(t) \equiv k\) a constant. Further \(P(t) = e^{-Bt}Ce^{Bt}\) where \(C\) is symmetric, and \(B\) is skew-symmetric.

**Corollary.** If \(x\) and \(y\) are solutions to the scalar differential equation \(x'' + p(t)x = 0\) and if \(x^2 + y^2 = 1\), then \(p(t) \equiv k \geq 0\).

**Proof of Theorem.** Differentiating (2) we have

\[ (X'X^* + Y'Y^*) + (XX'^* + YY'^*) = 0. \]

Let \(B = X'X^* + Y'Y^*\), then (3) becomes

\[ B + B^* = 0. \]

Thus \(B\) is a skew-symmetric matrix, and, since \(B\) is differentiable, \(B'\) is a skew-symmetric matrix also. On the other hand, \(B' = X'X'^* + Y'Y'^* - P(XX'^* + YY'^*) = X'X'^* + Y'Y'^* - P\) which is symmetric. Thus \(B' = 0\), since the only simultaneously symmetric and skew-symmetric matrix is the zero matrix. So, \(B\) is constant. From \(B' = 0\) we obtain

\[ X'X'^* + Y'Y'^* - P = 0 \]

Thus \(P\) is differentiable and positive semi-definite.

The third differentiation gives us

\[ P' = -(BP + PB^*) = PB - BP \]
Taking the trace in (6) we obtain

\[(\text{tr } P)' = \text{tr}(P') = \text{tr}(PB - BP) = \text{tr}(PB) - \text{tr}(BP) = 0.\]

Thus \(\text{tr } P = k\).

Since \(P\) satisfies (6), we know that \(P(t) = e^{-Bt}Ce^{Bt}\) where \(C\) is constant. But \(B\) is skew-symmetric, and, thus, \(C\) is symmetric since \(P(t)\) is.

**Remark.** An obvious specialization of the proof of the theorem yields for the corollary an easy and straightforward proof different from any found in [1] or its reference [6].

**Reference**


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