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BLOCKS OF CONSECUTIVE INTEGERS IN SUMSETS $(A+B)_t$

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Let $A, B \subseteq \{1, \dots, n\}$. For $m \in \mathbf{Z}$, let $r_{A,B}(m)$ be the cardinality of the set of ordered pairs $(a, b) \in A \times B$ such that $a+b=m$. For $t \geqslant 1$, denote by $(A+B)_t$ the set of the elements m for which $r_{A,B}(m) \geqslant t$. In this paper we prove that for any subsets $A, B \subseteq \{1, \dots, n\}$ such that $|A|+|B| \geqslant (4n+4t-3)/3$, the sumset $(A+B)_t$ contains a block of consecutive integers with the length at least $|A|+|B|-2t+1$, and that (a) for any two subsets A and B of $\{1, \dots, n\}$ such that $|A|+|B| \geqslant (4n)/3$, there exists an arithmetic progression of length n in $A+B$; (b) for any $2 \leqslant r \leqslant (4n-1)/3$, there exist two subsets A and B of $\{1, \dots, n\}$ with $|A|+|B|=r$ such that any arithmetic progression in $A+B$ has the length at most $(2n-1)/3+1$.

1. INTRODUCTION

Let G be an Abelian group written additively. For two subsets A and B of G, the sumset and the restricted sumset of A and B are defined as

$$A+B=\{a+b : a \in A, b \in B\}, \quad A \hat{+} B=\{a+b : a \in A, b \in B, a \neq b\},$$

respectively. We abbreviate $2A = A+A$, $2^\wedge A = A \hat{+} A$, and we use the standard notation $|A|$ for the cardinality of the set A. For $m \in G$, let $r_{A,B}(m)$ be the cardinality of the set of ordered pairs $(a, b) \in A \times B$ such that $a+b=m$. For $t \geqslant 1$, denote by $(A+B)_t$ the set of the elements $m \in A+B$ for which $r_{A,B}(m) \geqslant t$. Obviously, $(A+B)_1 = A+B$.

Pollard [7] first studied the sumset $(A+B)_t$ and extended the well known Cauchy-Davenport theorem by showing that for $A, B \subseteq \mathbf{Z}/p\mathbf{Z}$ and $t \leqslant \min(|A|, |B|)$,

$$\sum_{i=1}^{t} \left|(A+B)_i\right| \geqslant \min\Big(tp,\ t\big(|A|+|B|-t\big)\Big).$$

Caldeira and Dias da Silva gave in [2] an extension of the Pollard's theorem to an arbitrary field, and in [1] an analogue for restricted sums.

For a positive integer n, let $[1, n] = \{1, 2, \dots, n\}$. Many problems in combinatorial number theory have the following character: for a given arithmetic property P, find $f(n)$

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such that if $A \subseteq [1,n]$ with $|A| > f(n)$, then A has property P. For related research, one may refer to [**3, 4, 5, 6, 7, 9**] and the references therein.

Lev [4] proved that if $A \subseteq [1,n]$ with $|A| \geqslant (2n+3)/3$ then $2^\wedge A$ contains a block of at least $2|A|-3$ consecutive integers. In this paper, we study the blocks of consecutive integers in $(A+B)_t$ and prove the following results.

THEOREM 1. *Let n, t be positive integers, and let $A, B \subseteq [1,n]$ be such that $|A| + |B| \geqslant n+2t-1$. Then $(A+B)_t$ contains either a block of consecutive integers with the length at least $|A|+|B|-2t+1$ or three blocks of consecutive integers with the lengths of two blocks at least $2\big(|A|+|B|-n-2t\big)+3$ and the length of one block at least $2\big(|A|+|B|-n-t\big)+1$.*

THEOREM 2. *Let n, t be positive integers. For any $A, B \subseteq [1,n]$ such that $|A| + |B| \geqslant (4n+4t-3)/3$, the sumset $(A+B)_t$ contains a block of consecutive integers with the length at least $|A|+|B|-2t+1$.*

Clearly, for a set A of integers, $(A+A)_2 = 2^\wedge A$, and the above result of Lev follows from Theorem 2 immediately.

By our method we can construct the blocks of consecutive integers contained in $(A+B)_t$. This allows us to prove the following theorem.

THEOREM 3. *Let n, t be positive integers. For any $A, B \subseteq [1,n]$ such that $|A| + |B| = (4n+4t-4)/3$, the sumset $(A+B)_t$ contains a block of consecutive integers with the length at least $|A|+|B|-2t+1$ unless*

$$A = B = \big[1, (n+t-1)/3\big] \cup \big[(2n-t+4)/3, n\big].$$

NOTE. It is clear that if $A = B = \{1,2,\ldots,n\}$, then $\big|(A+B)_t\big| = |A|+|B|-2t+1$. So $|A|+|B|-2t+1$ in Theorems 2 and 3 is best possible.

In the case when $t=1$, we have $(A+B)_1 = A+B$. By Theorems 2 and 3, we have the following theorem.

THEOREM 4. *For any subsets A and B of $[1,n]$ such that $|A|+|B| \geqslant 4n/3$, there exists an arithmetic progression of length n in $A+B$.*

For any $2 \leqslant r \leqslant (4n-1)/3$, the following theorem shows that there exist two subsets A and B of $[1,n]$ with $|A|+|B| = r$ such that any arithmetic progression in $A+B$ has the length at most $(2n-1)/3+1$.

THEOREM 5. *For any $2 \leqslant r \leqslant (4n-1)/3$, there exist two subsets A and B of $[1,n]$ with $|A|+|B| = r$ such that any arithmetic progression in $A+B$ has the length at most $\big(|A|+|B|\big)/2+1$.*

2. LEMMAS

In this section we prove several lemmas, which will be used repeatedly in the proofs of our theorems. For a set A of integers and an integer k we write $A+k$ for the set

$\{a + k : a \in A\}$. Moreover, these lemmas are of some independent interest and can be applied in certain problems.

Lemma 1. *Let n, t be positive integers, and let $A, B \subseteq [1, n]$ such that $|A| + |B| \geqslant n + t$. Then*

$$\left[2n + 1 - |A| - |B| + t,\ |A| + |B| + 1 - t\right] \subseteq (A + B)_t.$$

Proof: For $n < k \leqslant |A| + |B| + 1 - t$, by

$$|A| + |k - B| = |A| + |B| \geqslant k - 1 + t, \quad A \subseteq [1, k-1], \quad k - B \subseteq [1, k-1],$$

we have $\left|A \cap (k - B)\right| \geqslant t$. Hence $k \in (A + B)_t$.

For $2n + 1 - |A| - |B| + t \leqslant k \leqslant n$, let

$$A_1 = \{a \in A : a < k\}, \quad B_1 = \{b \in B : b < k\}.$$

Then $|A_1| \geqslant |A| - (n - k + 1)$ and $|B_1| \geqslant |B| - (n - k + 1)$. By

$$|A_1| + |k - B_1| = |A_1| + |B_1| \geqslant |A| + |B| - 2(n - k + 1) \geqslant k - 1 + t,$$

$$A_1 \subseteq [1, k-1], \quad k - B_1 \subseteq [1, k-1],$$

we have $\left|A_1 \cap (k - B_1)\right| \geqslant t$. Hence $k \in (A_1 + B_1)_t \subseteq (A + B)_t$.

Combining the above arguments, we have

$$\left[2n + 1 - |A| - |B| + t,\ |A| + |B| + 1 - t\right] \subseteq (A + B)_t.$$

This completes the proof. ∎

Lemma 2. *Let A and B be two sets of integers, and let α, β be two integers such that $\alpha \leqslant \beta$. If $\left|[\alpha, \beta] \cap A\right| + \left|[\alpha, \beta] \cap B\right| > \beta - \alpha + t$, then $\alpha + \beta \in (A + B)_t$.*

Proof: Since $\alpha + \beta - \left([\alpha, \beta] \cap A\right) \subseteq [\alpha, \beta]$ and

$$\left|\alpha + \beta - \left([\alpha, \beta] \cap A\right)\right| + \left|[\alpha, \beta] \cap B\right| = \left|[\alpha, \beta] \cap A\right| + \left|[\alpha, \beta] \cap B\right| > \beta - \alpha + t.$$

Then

$$\left|\left(\alpha + \beta - \left([\alpha, \beta] \cap A\right)\right) \cap \left([\alpha, \beta] \cap B\right)\right| > t - 1.$$

Hence

$$\alpha + \beta \in \left([\alpha, \beta] \cap A + [\alpha, \beta] \cap B\right)_t \subseteq (A + B)_t.$$

This completes the proof of Lemma 2. ∎

Lemma 3. *Let n, t be positive integers, and let $A, B \subseteq [1, n]$ such that $|A| + |B| \geqslant n + 2t - 1$. Let $m \in [1, n]$ be the largest integer such that $m \notin (A + B)_t$ and $l \in [n + 2, 2n + 1]$ be the least integer such that $l \notin (A + B)_t$. Then*

$$\left[l - 1 + 2t - |A| - |B|,\ l + 1 - 2t + |A| + |B| - 2n\right] \subseteq (A + B)_t,$$
$$\left[m - 1 + 2t + 2n - |A| - |B|,\ m + 1 - 2t + |A| + |B|\right] \subseteq (A + B)_t.$$

PROOF: Let $A_2 = \{a \in A : a < l - n\}$ and $B_2 = \{b \in B : b < l - n\}$. Since $l \notin (A+B)_t$, by Lemma 2 we have

$$\big|A \cap [l-n, n]\big| + \big|B \cap [l-n, n]\big| \leqslant 2n - l + t,$$

and so

$$\begin{aligned} |A_2| + |B_2| &= |A| - \big|A \cap [l-n,n]\big| + |B| - \big|B \cap [l-n,n]\big| \\ &= |A| + |B| - \Big(\big|A \cap [l-n,n]\big| + \big|B \cap [l-n,n]\big|\Big) \\ &\geqslant |A| + |B| - (2n - l + t) = |A| + |B| - 2n + l - t. \end{aligned}$$

For $l - n \leqslant k \leqslant l + 1 - 2t + |A| + |B| - 2n$, we have

$$|A_2| + |k - B_2| = |A_2| + |B_2| \geqslant |A| + |B| - 2n + l - t \geqslant k + t - 1,$$
$$A_2 \subseteq [1, k-1], \quad k - B_2 \subseteq [1, k-1].$$

Hence $\big|A_2 \cap (k - B_2)\big| \geqslant t$. Thus $k \in (A_2 + B_2)_t \subseteq (A+B)_t$.

For $l - 1 + 2t - |A| - |B| \leqslant k \leqslant l - n$, let

$$A_3 = \{a \in A : a < k\}, \quad B_3 = \{b \in B : b < k\}.$$

Then

$$A_3 \subseteq [1, k-1], \quad k - B_3 \subseteq [1, k-1],$$

$$\begin{aligned} |A_3| + |B_3| &= |A_2| - \big|A_2 \cap [k, l-n-1]\big| + |B_2| - \big|B_2 \cap [k, l-n-1]\big| \\ &\geqslant |A| + |B| - 2n + l - t - 2(l - n - k) \\ &\geqslant |A| + |B| - l + 2k - t. \end{aligned}$$

It follows that

$$|A_3| + |k - B_3| = |A_3| + |B_3| \geqslant |A| + |B| - l + 2k - t \geqslant k + t - 1,$$

and so $\big|A_3 \cap (k - B_3)\big| \geqslant t$. Hence $k \in (A_3 + B_3)_t \subseteq (A+B)_t$.

Combining the above arguments, we have

$$\big[l - 1 + 2t - |A| - |B|,\, l + 1 - 2t + |A| + |B| - 2n\big] \subseteq (A+B)_t.$$

Now, we define $A' = n + 1 - A$, $B' = n + 1 - B$ and apply the already proved part of Lemma 3 to $A' + B'$ to conclude that

$$\big[2n + 2 - m - 1 + 2t - |A| - |B|,\, 2n + 2 - m + 1 - 2t + |A| + |B| - 2n\big] \subseteq (A' + B')_t.$$

Therefore

$$\big[m - 1 + 2t + 2n - |A| - |B|,\, m + 1 - 2t + |A| + |B|\big] \subseteq (A+B)_t.$$

This completes the proof. ∎

3. Proofs of Theorems

Proof of Theorem 1: By Lemma 1, we have

$$\left[2n+1-|A|-|B|+t,\ \ |A|+|B|+1-t\right] \subseteq (A+B)_t.$$

It is clear that $n+1 \in (A+B)_t$. Let $m \in [1,n]$ be the largest integer such that $m \notin (A+B)_t$ and $l \in [n+2, 2n+1]$ be the least integer such that $l \notin (A+B)_t$. It follows from Lemma 3 that

$$\left[l-1+2t-|A|-|B|,\, l+1-2t+|A|+|B|-2n\right] \subseteq (A+B)_t,$$
$$\left[m-1+2t+2n-|A|-|B|,\, m+1-2t+|A|+|B|\right] \subseteq (A+B)_t$$

and

$$\left[2n+1-|A|-|B|+t,\ \ |A|+|B|+1-t\right] \subseteq [m+1,\ l-1] \subseteq (A+B)_t.$$

If $m > l+1-2t+|A|+|B|-2n$, then $m-1+2t+2n-|A|-|B| > l$. Hence $\left[l-1+2t-|A|-|B|,\, l+1-2t+|A|+|B|-2n\right]$, $[m+1,\, l-1]$ and $\left[m-1+2t+2n-|A|-|B|,\, m+1-2t+|A|+|B|\right]$ are disjoint from each other.

If $m \leqslant l+1-2t+|A|+|B|-2n$, by the definition of m, we have $m < l-1+2t-|A|-|B|$. Hence

$$\left[l-1+2t-|A|-|B|,\, l-1\right] \subseteq (A+B)_t.$$

This completes the proof of Theorem 1. ∎

Proof of Theorem 2: Since $(4n+4t-3)/3 \leqslant |A|+|B| \leqslant 2n$, it follows that $n \geqslant 2t-1$, and so

$$|A|+|B| \geqslant \frac{4n+4t-3}{3} = n + \frac{n+4t-3}{3} \geqslant n+2t-\frac{4}{3}.$$

Hence $|A|+|B| \geqslant n+2t-1$.

Let $l \in [n+2, 2n+1]$ be the least integer such that $l \notin (A+B)_t$. By Lemma 1 and Lemma 3, we have $l \geqslant |A|+|B|+2-t$, and

$$(1)\quad \left[l-1+2t-|A|-|B|,\ l+1-2t+|A|+|B|-2n\right] \cup \left[2n+1-|A|-|B|+t,\ l-1\right] \subseteq (A+B)_t.$$

Since $|A|+|B| \geqslant (4n+4t-3)/3$, it follows that

$$l+1-2t+|A|+|B|-2n+1 \geqslant 2n+1-|A|-|B|+t.$$

Hence

$$\left[l-1+2t-|A|-|B|,\, l-1\right] \subseteq (A+B)_t.$$

This completes the proof of Theorem 2. ∎

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PROOF OF THEOREM 3: Since $(4n+4t-4)/3 = |A|+|B| \leqslant 2n$, it follows that $n \geqslant 2t-2$ and $3 \mid (n+t-1)$. If $n = 2t-2$, then $|A|+|B| = 2n$. Therefore $A = B = [1, n]$,

$$(A+B)_t = [n/2+2,\ 2n-n/2].$$

Clearly, Theorem 3 holds in this case.

In the case when $n \geqslant 2t-1$, by $3 \mid (n+t-1)$ we have $n \geqslant 2t+1$. It follows that $|A|+|B| \geqslant n+2t-1$. Let $l \in [n+2, 2n+1]$ be the least integer such that $l \notin (A+B)_t$. By Lemma 1 and Lemma 3, we have $l \geqslant |A|+|B|+2-t$, and

(1) $\big[l-1+2t-|A|-|B|,\ l+1-2t+|A|+|B|-2n\big] \cup \big[2n+1-|A|-|B|+t,\ l-1\big] \subseteq (A+B)_t.$

Since $|A|+|B| = (4n+4t-4)/3$, we may denote $n+t-1 = 3u$. Then $|A|+|B| = 4u$.

CASE 1. $l \geqslant |A|+|B|+3-t$. Then

$$l+1-2t+|A|+|B|-2n+1 \geqslant 2n+1-|A|-|B|+t.$$

Hence

$$\big[l-1+2t-|A|-|B|,\ l-1\big] \subseteq (A+B)_t.$$

CASE 2. $l = |A|+|B|+2-t$. Then $l = 4u+2-t$ and (1) becomes

$$[t+1,\ 2u-t+1] \cup [2u-t+3,\ 4u+1-t] \subseteq (A+B)_t.$$

We shall show that $2u-t+2 \in (A+B)_t$.

Since $4u+2-t = l \notin (A+B)_t$, by Lemma 2 we have

$$\big|A \cap [u+1, 3u+1-t]\big| + \big|B \cap [u+1, 3u+1-t]\big| \leqslant 2u-t+t = 2u.$$

By $|A|+|B| = 4u$ we have

$$\big|A \cap [1,u]\big| + \big|B \cap [1,u]\big| \geqslant 2u,$$

and so $[1,\ u] \subseteq A$ and $[1,\ u] \subseteq B$. Since

$$\big|A \cap [u-t+2, u]\big| + \big|B \cap [u-t+2, u]\big| = 2(t-1) > (t-2)+t-1,$$

it follows from Lemma 2 that $2u-t+2 \in (A+B)_{t-1}$.

If $2u-t+2 \notin (A+B)_t$, then $u+1 \notin A \cup B$, $u+2 \notin A \cup B$, $\dots$, $2u-t+1 \notin A \cup B$. By $|A|+|B| = 4u$, we have

$$A = B = [1,u] \cup [2u-t+2, 3u+1-t],$$

namely

$$A = B = [1, (n+t-1)/3] \cup [(2n-t+4)/3, n].$$

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This contradicts the condition. Hence $2u - t + 2 \in (A+B)_t$.

Combining the above arguments, we have

$$[t+1, 4u+1-t] \subseteq (A+B)_t.$$

This completes the proof of Theorem 3. ∎

PROOF OF THEOREM 5: We use m to denote any positive integer. There are four cases.

CASE 1. $r = 4m$. Let $A = B = [1, m] \cup [n-m+1, n]$. Then

$$A + B = [2, 2m] \cup [n-m+2, n+m] \cup [2n-2m+2, 2n]. \tag{2}$$

Suppose that $A + B$ contains an arithmetic progression with the common difference d and the length $t \geqslant 2m+2$. By (2) we have

$$2m + 2 \leqslant t \leqslant \frac{2m-1}{d} + 1 + \frac{2m-1}{d} + 1 + \frac{2m-1}{d} + 1.$$

Hence $d \leqslant 3$. By $r \leqslant (4n-1)/3$ we have $n \geqslant 3m+1$. Since each interval in (2) contains $2m-1$ integers, we have $d \geqslant n - 3m + 2 \geqslant 3$. Thus $d = 3$ and $n - 3m + 2 = 3$. So $t \leqslant [2n/3] + 1 = 2m+1$, a contradiction. Therefore, any arithmetic progression in $A+B$ has the length not exceeding $2m+1$.

CASE 2. $r = 4m+1$. Let $A = [1, m] \cup [n-m+1, n]$ and $B = A \cup \{n-m\}$. Then

$$A + B = [2, 2m] \cup [n-m+1, n+m] \cup [2n-2m+1, 2n]. \tag{3}$$

By $r \leqslant (4n-1)/3$ we have $n \geqslant 3m+1$. Suppose that $A+B$ contains an arithmetic progression with the common difference d and the length $t \geqslant 2m+2$. By (3), $n \geqslant 3m+1$ and $t \geqslant 2m+2$, we have $m \geqslant 2$ and

$$2m + 2 \leqslant t \leqslant \frac{2m-1}{d} + 1 + \frac{2m}{d} + 1 + \frac{2m}{d} + 1.$$

Hence $d \leqslant 3$. Since each interval in (3) contains at most $2m$ integers, we have $d \geqslant n - 3m + 1 \geqslant 2$. Thus $d = 2, 3$ and $n - 3m + 1 = 2, 3$. If $d = 3$, then $t \leqslant \left[(2n-2)/3\right] + 1 = 2m+1$, a contradiction. If $d = 2$, then, by $d \geqslant n - 3m + 1 \geqslant 2$, we have $n - 3m + 1 = 2$. By (3) we have

$$A + B = [2, 2m] \cup [2m+2, 4m+1] \cup [4m+3, 6m+2].$$

This implies that $t \leqslant 2m+1$, a contradiction. Therefore, any arithmetic progression in $A + B$ has the length not exceeding $2m+1$.

CASE 3. $r = 4m+2$. Let $A = B = [1, m] \cup [n-m, n]$. Then

$$A + B = [2, 2m] \cup [n-m+1, n+m] \cup [2n-2m, 2n]. \tag{4}$$

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Suppose that $A+B$ contains an arithmetic progression with the common difference d and the length $t \geqslant 2m+3$. By (4) we have

$$2m+3 \leqslant t \leqslant \frac{2m-1}{d}+1+\frac{2m}{d}+1+\frac{2m+1}{d}+1.$$

Hence $d \leqslant 3$. By $r \leqslant (4n-1)/3$ we have $n \geqslant 3m+2$. Since each interval in (4) contains at most $2m+1$ integers, we have $d \geqslant n-3m \geqslant 2$. Thus $d=2,3$ and $n-3m=2,3$. If $d=3$, then $t \leqslant \lceil (2n-1)/3 \rceil + 1 \leqslant 2m+2$, a contradiction. If $d=2$, then, by $d \geqslant n-3m \geqslant 2$, we have $n-3m=2$. By (4) we have

$$A+B=[2,2m]\cup[2m+3,4m+2]\cup[4m+4,6m+4].$$

This implies that $t \leqslant 2m+2$, a contradiction. Therefore, any arithmetic progression in $A+B$ has the length not exceeding $2m+2$.

CASE 4. $r=4m+3$. Let $A=[1,m]\cup[n-m,n]$ and $B=A\cup\{n-m-1\}$. Then

$$A+B=[2,2m]\cup[n-m,n+m]\cup[2n-2m-1,2n]. \tag{5}$$

Suppose that $A+B$ contains an arithmetic progression with the common difference d and the length $t \geqslant 2m+3$. By $r \leqslant (4n-1)/3$ we have $n \geqslant 3m+3$. By (5), $t \geqslant 2m+3$ and $n \geqslant 3m+3$, we have $m \geqslant 2$ and

$$2m+3 \leqslant t \leqslant \frac{2m-1}{d}+1+\frac{2m+1}{d}+1+\frac{2m+2}{d}+1.$$

Hence $d \leqslant 3$. Since each interval in (5) contains at most $2m+2$ integers, we have $d \geqslant n-3m-1 \geqslant 2$. Thus $d=2,3$ and $n-3m-1=2,3$. If $d=3$ and $n-3m-1=2$, then $t \leqslant \lceil (2n-1)/3 \rceil + 1 \leqslant 2m+2$, a contradiction. If $d=3$ and $n-3m-1=3$, then

$$A+B=[2,2m]\cup[2m+4,4m+4]\cup[4m+7,6m+8]$$

and $t \leqslant (4m+4)/3+1 \leqslant 2m+2$, a contradiction. If $d=2$, then, by $d \geqslant n-3m-1 \geqslant 2$, we have $n-3m-1=2$. By (5) we have

$$A+B=[2,2m]\cup[2m+3,4m+3]\cup[4m+5,6m+6].$$

This implies that $t \leqslant 2m+2$, a contradiction. Therefore, any arithmetic progression in $A+B$ has the length not exceeding $2m+2$. This completes the proof of Theorem 5. ∎

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