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RINGS WITH QUASI-CONTINUOUS RIGHT IDEALS

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Abstract. Rings in which each right ideal is quasi-continuous (right π-rings) are shown to be a direct sum of semisimple artinian square full ring and a right square free ring. Among other results it is also shown that (i) a nonlocal right continuous indecomposable right π-ring is either simple artinian or a ring of matrices of a certain type, and (ii) an indecomposable non-local right continuous ring is both a right and a left π-ring if and only if it is a right q-ring. In particular, a non local indecomposable right q-ring is a left q-ring.

0. Introduction. Rings for which every right ideal is quasi-injective (known as right q-rings) have been studied by several authors (c.f. [**4**], [**5**], [**6**], [**7**]). The purpose of our paper is to extend this line of research by studying rings in which every right ideal is quasi-continuous (right π-rings). In Section 2 we show that a π-ring is a direct sum of a semisimple artinian square full ring and a square free ring.

In Section 3 we study right continuous right π-rings. We show that a non-local, right continuous, indecomposable right π-ring R is either simple artinian or a ring of matrices of a certain type (Theorem 3.8). We show that an indecomposable, non-local, right continuous ring R is both a right and a left π-ring if and only if R is a right q-ring (Theorem 3.13). In particular, under these hypotheses, R is a right q-ring if and only if it is a left q-ring.

1. Definitions and preliminaries. Throughout the paper R will be a ring with identity and all R-modules will be unital right R-modules, unless otherwise stated. For modules M, N, the notations $N \subset' M$ and $N \subset^{\oplus} M$ respectively serve to denote that N is an essential submodule of M and that N is a direct summand of M. By $Z(M)$, $\mathrm{Soc}(M)$ we denote the singular submodule and socle of M, respectively. $J(R)$ is the Jacobson Radical of R. $\hat{M}$ (or $E(M)$) stands for the injective hull of M. $\mathrm{Hom}_R(M, N)$ stands for the set of R-homomorphisms from M to N. $\mathrm{End}_R(M)$ is the set of R-endomorphisms of M. If $N \subset M$, a *closure* of N in M is a maximal essential extension of N in M. N is said to be *closed* in M if N has no proper essential extension on M. Given right R-modules M and N, M is said to be N-injective iff for any $K \subset N$, every $\alpha \in \mathrm{Hom}_R(K, M)$ is the restriction of some $\beta \in \mathrm{Hom}_R(N, M)$.

For a module M we consider the following conditions.

(C_1) For every submodule N of M there exists a summand L of M with $N \subset' L$.
(C_2) If a submodule N of M is isomorphic to a summand of M then N itself is a summand of M.
(C_3) If A and B are summands of M with $A \cap B = 0$ the $A \oplus B$ is a summand of M.

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A module M is called (*quasi-*)*continuous* if it satisfies (C_1) and (C_2) ((C_1) and (C_3)). A module M is called a *CS* (or *extending*) module if it satisfies (C_1). Equivalently, M is a CS module if every closed submodule of M is a direct summand of M.

It is well known that a module M is quasi-continuous if and only if $eM \subset M$ for every idempotent $e \in \operatorname{End}(\hat{M})$ if and only if every decomposition $E(M) = \oplus_{i\in I} E_i$ induces $M = \oplus_{i\in I}(M \cap E_i)$ [**10**, 2.8].

Definition 1.1. A ring R is said to be a *right* $(f)\pi$*-ring* if every (finitely generated) right ideal in R is quasi-continuous.

Remark 1.2. A right uniform ring is a right $(f)\pi$-ring.

The following Lemma is a particular case of [**9**, Proposition 2.6.]. This may already be known but we have not found it anywhere in the literature.

Lemma 1.3. *A right R-module M is quasi-continuous if and only if M satisfies condition (C_1) and, whenever $M = A \oplus B$, A and B are mutually injective.*

Proof. The 'only if' part is well known (see Lemma 1.7 below). For completeness, we include here the proof of the other implication.

Let $\hat{M} = E_1 \oplus E_2$. We want to show that $M = A \oplus B$ where $A = E_1 \cap M$ and $B = E_2 \cap M$. Since A is closed in M and M satisfies (C_1), $M = A \oplus X$ for some $X \subset M$. Let π_A and π_X be the corresponding projection maps. Then $B \cong \pi_X(B) \subset X$. Define $\alpha : \pi_X(B) \to A$ by $\alpha(\pi_X(b)) = \pi_A(b)$. The map α is well-defined since $\pi_X : B \to \pi_X(B)$ is an isomorphism. But A is X-injective, so α extends to $\beta : X \to A$. Let $X^* = \{x + \beta(x) \mid x \in X\}$. Then $M = A \oplus X^*$. Also $B \subset' X^*$ and B is closed. This implies that $B = X^*$ and, consequently, $M = A \oplus B$. □

Lemma 1.3 has the following immediate consequence, an intrinsic characterization of $(f)\pi$-rings:

Theorem 1.4. *A right ring R is a right $(f)\pi$-ring if and only if every (finitely generated) right ideal of R satisfies (C_1) and, whenever A and B are right ideals of R with $A \cap B = 0$, A and B are mutually injective.*

Proof. Immediate from Lemma 1.3. □

We list below some well known results that will be used frequently.

Theorem 1.4 may be strengthened by observing that in order to check if a ring R is right $(f)\pi$, it suffices to concentrate on the essential right ideals.

Theorem 1.5. *For a ring R the following conditions are equivalent:*

(1) *R is a right $(f)\pi$-ring,*
(2) *every essential (finitely generated) right ideal of R is quasi-continuous,*
(3) *every essential (finitely generated) right ideal satisfies (C_1) and, whenever A and B are right ideals such that $A \oplus B \subset' R$, A and B are mutually injective.*

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Proof. It follows from the above characterizations of $(f)\pi$-rings since, when R satisfies (C_1), every (finitely generated) right ideal is a direct summand of a (finitely generated) essential right ideal. □

LEMMA 1.6. *Let M and N be two right R-modules. Then M is N-injective iff for any $\alpha \in \mathrm{Hom}_R(N, \hat{M})$, $\mathrm{Im}\alpha \subset M$.*

Proof. See [**10**, 1.13]. □

LEMMA 1.7. *If $M = \oplus_{i=1}^{n} M_i$ then M is quasi-continuous iff each M_i is quasi-continuous and M_i is M_j- injective for all $i \neq j$.*

Proof. See [**10**, 2.14]. □

LEMMA 1.8. *A right quasi-continuous ring R is right continuous iff $J(R) = Z(R)$ and $R/J(R)$ is a regular ring.*

Proof. See [**10**, 3.15]. □

LEMMA 1.9. *In a right quasi-continuous ring the idempotents modulo $Z(R)$ can be lifted.*

Proof. See [**10**, 3.7]. □

A right module M is called *local* if M contains a unique maximal submodule. A ring R is called *local ring* if the right R-module R_R is local. Let e be an idempotent in R. Then e is called *primitive* if the right ideal eR is indecomposable. Furthermore, e is called *local* if the ring $\mathrm{End}_R(eR)$ is local or equivalently if eJ is the unique maximal submodule of eR [**8**, 21.18].

LEMMA 1.10. *A primitive idempotent in a right continuous ring is local.*

Proof. Let e be a primitive idempotent in a continuous ring R. Then eRe is a continuous ring [**10**, 3.8] and contains no nontrivial idempotents [**8**, 21.8]. By Lemmas 1.7 and 1.8, eRe/eJe is a division ring. It follows that eJe is the maximal right ideal in eRe and, therefore, e is local [**8**, 21.9]. □

Local rings have nontrivial idempotents. We see next that the converse also holds when the ring is continuous.

LEMMA 1.11. *A right continuous ring R is local iff it contains no nontrivial idempotents.*

Proof. Let R be a right continuous ring with no nontrivial idempotents. Then $R/J(R)$ is a regular ring with no nontrivial idempotents. Hence $R/J(R)$ is a division ring. It follows that R is a local ring. The converse is trivial. □

Two modules M and N are said to be *orthogonal* if no submodule of M is isomorphic to a submodule of N. Let M, X be arbitrary right R-modules and N a submodule of M. Then M is said to be a *square* if $M \cong X^2$. The N is called a *square* *root* in M if N^2 is embeddable in M. M is called *square free* if M contains no square roots. M is called *square full* if every submodule of M contains a square root *in* M.

2. Rings with quasi-continuous right ideals. Right $(f)q$-rings may be characterized as those right self-injective rings for which every (finitely generated) essential right ideal is a two sided ideal [7]. For right $(f)\pi$-rings one obtains the following.

PROPOSITION 2.1. *A ring R is a right $(f)\pi$-ring if and only if R is quasi-continuous and every (finitely generated) essential right ideal is a left S-module, where S is the subring of R generated by its idempotent elements.*

Proof. From Theorem 1.5, in order to check to see if R is an $(f)\pi$-ring, it suffices to check that every (finitely generated) essential right ideal in R is quasi-continuous. We'll show here that a (finitely generated) essential right ideal I of a quasi-continuous ring R is quasi-continuous if and only if it is a left S-module. Our result will then follow. Let R be a quasi-continuous ring. The subring T generated by all the idempotents of the ring $\mathrm{End}_R(\hat{R})$ can then be viewed as the subring S generated by all the idempotents of the ring R. Let I be a (finitely generated) essential right ideal of R. Then, $I = TI = SI$ if and only if I is quasi-continuous. □

LEMMA 2.2. *Let A, B be right ideals in a right $f\pi$-ring R with $A \cap B = 0$ and $A \cong B$. Then*

(a) *every finitely generated right ideal in A or B is injective. Moreover, if R is a right π-ring then A and B are semisimple and injective, and*
(b) *the right ideals A and B are nonsingular.*

Proof. For the first part (a) we can assume that A and B themselves are finitely generated. By the quasi-continuity of R, there exists an idempotent $e^2 = e \in R$ such that $A \subset' eR \subset^{\oplus} R$. The fact that $B \cap eR = 0$ implies that B is eR-injective. And $A \cong B \Rightarrow A$ is eR-injective. But A is also $(1-e)R$-injective, therefore A is R-injective. For the second part of (a), assume that R is a π-ring and $0 \neq X \subset A$ then $X \cong \alpha(X)$ where $\alpha : A \to B$ is a given isomorphism. By a similar argument as above X is injective. Thus, A is semisimple and so is B. Let us now consider (b). Let x be an element of $Z(A)$. Since xR is injective, there exists an idempotent $e \in R$ such that $eR = xR < Z(A)$. Since $Z(R)$ contains no nonzero idempotents, it follows that $x = 0$. □

LEMMA 2.3. *Let A, B be right ideals in a ring R with $A \cap B = 0$. Let $\alpha : A \to B$ be a nonzero homomorphism.*

(a) *If R is a right π-ring then the image of α is semisimple.*
(b) *If R is a right $f\pi$-ring and B is uniform then the image of α is simple.*
(c) *Let R be a right $f\pi$-ring with a non-trivial primitive idempotent e such that $eR(1-e) \neq 0$. Then eR contains a simple right ideal.*

Proof. (a) Let $L \subset' B$. Since $L \oplus A$ is quasi-continuous, and α may be viewed as $\alpha : A \to \hat{L}$, $\mathrm{Im}\alpha \subset L$. It follows that $\mathrm{Im}\alpha \subset \mathrm{Soc}(B)$.

(b) Let a and x be a nonzero elements in A and B respectively. Then $\alpha : A \to B \subset \hat{B} = \widehat{xR}$ implies $\alpha(aR) \subset xR$ since $aR \oplus xR$ is quasi-continuous. It follows that $\mathrm{Im}\alpha \subset xR$. Then $0 \neq \mathrm{Im}\alpha \subset \mathrm{Soc}(B)$, a simple right R-module.

(c) Since eR quasi-continuous and indecomposable, eR is uniform. Thus, the result follows from (b). □

Recall that a *homogeneous component* of an R-module is a complete sum of mutually isomorphic simple submodules.

LEMMA 2.4. *For a right π-ring R, we have the following.*
(1) *Let A, B be independent right ideals of R with B an epimorphic image of A. Then B is cyclic.*
(2) *Let $\{A\} \cup \{B_i : i \in I\}$ be an independent family of right ideals in R with $\oplus_{i\in I} B_i$ an epimorphic image of A. Then the index set I is finite.*

Proof. Let α be a homomorphism from A onto B. There exists an idempotent e in R such that eR is a closure of A in R. As R is a π-ring, α extends to an epimorphism from eR onto B. Hence B is cyclic and the proof of (1) follows. The proof of (2) is immediate from (1). □

THEOREM 2.5. *A right π-ring R has only finitely many nonsimple homogeneous components and each one of them is injective.*

Proof. Let $\{H_i : i \in I\}$ be the family of homogeneous components of a right π-ring R. For each $i \in I$, let S_i and T_i be minimal right ideals such that $H_i = [S_i] = [T_i]$ and $S_i \cap T_i = 0$. Then there exists an isomorphism $\oplus_{I\in i} S_i \to \oplus_{i\in I} T_i$. By Lemma 2.4(2), $|I| < \infty$. Hence R has only finitely many non-simple homogeneous components. Now let H be a non-simple homogeneous component of R. If H is not finitely generated then there exist independent submodules A, B of H such that $A \cong H \cong B$. By Lemma 2.2, A, B are injective. But then H is finitely generated, a contradiction. So, H is finitely generated. □

THEOREM 2.6. *A right π-ring is a direct sum of a square full semisimple artinian ring and a right square free ring.*

Proof. Let R be a right π-ring and H the direct sum of all square full homogeneous components of R. By Lemma 2.3, each simple submodule of H is injective, and by Theorem 2.5, H is injective. Let $H = eR$ for some idempotent $e \in R$. Since eR and $(1-e)R$ are orthogonal, it follows that $(1-e)Re = 0 = eR(1-e)$. Hence the decomposition $R = eR \oplus (1-e)R$ is a ring decomposition. Now, suppose $(1-e)R$ contains a square. Then there exist right ideals A, B in $(1-e)R$ with $A \cap B = 0$ and $A \cong B$. By Lemma 2.3, A, B are semisimple. It follows that $(1-e)R$ contains a square full homogeneous component, a contradiction. Hence, $(1-e)R$ is square free and we get the desired decomposition. □

COROLLARY 2.7. *An indecomposable ring R containing a square is a right π-ring iff R is simple Artinian.*

PROPOSITION 2.8. *Let R be a right π-ring. Suppose $\{A_i\}_{i\in I}$ is an independent family of right ideals in R. If for each $i \in I$ there exists a right ideal B_i in R that is a homomorphic image of A_i with $A_i \cap B_i = 0$ then I is finite.*

Proof. Suppose I is infinite. In view of Theorem 2.6 we can assume that R is square free. By Lemma 2.3 we can assume the B_i are simple. Since R is a π-ring, any homomorphism from A_i onto B_i can be extended to a homomorphism from e_iR to B_i, where e_iR is a closure of A_i in R and e_i is an idempotent in R. As R is square free, using the projectivity of e_iR one can assume that B_i are independent. Pick an A_{i_1}. By Lemma 2.4, images of the A_i are contained in A_{i_1} for only finitely many i. Pick an A_{i_2} whose image is not contained in A_{i_1}. Again by Lemma 2.4, images of the A_i are contained in A_{i_2} for only finitely many i. Hence there exists an A_{i_3} whose image is not contained in $A_{i_1} \oplus A_{i_2}$. Clearly this process is inductive.

Hence there exists an infinite subset $I' \subset I$ such that $(\oplus_{i\in I'} A_i) \cap (\oplus_{i\in I'} B_i) = 0$. Now for each $i \in I'$, the homomorphisms from A_i onto B_i induce an epimomorphism from $\oplus_{i\in I'} A_i$ onto $\oplus_{i\in I'} B_i$. By Lemma 2.4, I' is finite, a contradiction. Hence the proof follows. □

3. Continuous $(f)\pi$-rings. In this section we will study the $(f)\pi$-rings that are continuous. Since, by Lemma 1.9, a continuous ring is local if and only if it contains no non-trivial idempotents, a continuous, local $(f)\pi$-ring is uniform. Conversely, all right uniform rings are π-rings (Remark 1.2.). Consequently, from now on, we shall only consider non-local rings. We completely characterize indecomposable, non-local, right π-rings that are right continuous. We prove that such rings are either simple artinian or a certain type of rings of matrices.

LEMMA 3.1. *Let R be a right continuous right $f\pi$-ring, e a primitive idempotent in R and $\alpha : eR \to R$ a nonzero homomorphism with* $\mathrm{Im}\alpha \cap eR = 0$. *Then*

(a) $\mathrm{Im}\alpha$ *is simple;*

(b) $eR(1-e) \neq 0$.

In particular, for a right continuous, right $f\pi$-ring R and primitive idempotent $e \in R$, if $(1-e)Re \neq 0$ then $eR(1-e) \neq 0$.

Proof. (a). Let $\alpha : eR \to R$. By Lemma 1.8, eR is local and therefore $\alpha(eR)$ is local. It follows that $\alpha(eR)$ is indecomposable and therefore uniform ($\alpha(e)R$, being finitely generated, is quasi-continuous). Thus, by Lemma 2.3(b), $\mathrm{Im}\alpha$ is simple.

(b). Let $\mathrm{Im}\alpha \subset' fR$. Then, $fR \cap eR = 0$ and, therefore, $R = (fR \oplus eR) \oplus L$ for some $L \subset R$, since R is quasi-continuous. Thus, $(1-e)R \cong \dfrac{R}{eR} \cong fR \oplus L$ and, therefore, A is embedded in $(1-e)R$. Hence we can assume that $\alpha : eR \to (1-e)R$.

Assume $eR(1-e) = 0$. By Lemma 1.8, $eRe = eR$ is local ring with its unique maximal ideal eJ where $J = J(R)$. If $eJ = 0$, eR is simple right R-module and $\alpha(eR) \cong eR$. The inverse of α from eR into $\alpha(eR)$ extends to some nonzero homomorphism from $(1-e)R$ to eR, proving our claim. Let x be a nonzero element in eJ. Since eRe is local there exists an eRe-epimorphism $\beta : xeR \to \dfrac{eR}{eJ}$. Furthermore, since $eR = eRe$, β is an R-homomorphism as well.

Since $\mathrm{Im}\alpha$ is simple (by (a)), $\dfrac{eR}{eJ} \cong \mathrm{Im}\alpha \subset (1-e)R$. Let γ be a nonzero homomorphism from $\dfrac{eR}{eJ}$ to $(1-e)R$. Then $\gamma\beta$ is a nonzero R-homomorphism from xeR to $(1-e)R$. Since R is quasi-continuous, $\gamma\beta$ extends to a nonzero R-homomorphism, δ say, from eR to $(1-e)R$. But $xeR \subset eJ = \mathrm{Ker}\delta$ (by (a)). This implies that

$0 = \delta(xeR) = \gamma\beta(xeR) = \mathrm{Im}\alpha$, a contradiction since $\alpha \neq 0$. Thus $eR(1-e) \neq 0$ and the proof follows. □

LEMMA 3.2. *Let R be a continuous square free $f\pi$-ring with a primitive idempotent e such that $eR(1-e) \neq 0$. Then eRe is a division ring and $eR(1-e)$ is the only proper submodule of eR.*

Proof. By Lemma 2.3 $S = \mathrm{Soc}(eR)$ is non-zero. Since eR is continuous, S is simple. By Lemma 2.3, we get $eR(1-e) \subseteq S$. Now let $0 \neq s \in S$ and $0 \neq \alpha : (1-e)R \to eR$. Clearly, by Lemma 2.3, $\mathrm{Im}\alpha = S$. If $se \neq 0$ then $S = seR$ is a homomorphic image of eR under some $\beta : R \to sR$ given by $\beta(ex) = sex$. By the projectivity of eR there exists a homomorphism $\gamma : eR \to (1-e)R$ so that $\alpha\gamma = \beta$. By Lemma 3.1, $\mathrm{Im}\gamma$ is simple. It follows that $\mathrm{Im}\gamma \cong S$, a contradiction since R is square free. Hence $se = 0$ and $s \in eR(1-e)$. Consequently, $S = eR(1-e)$. Now, let $J = J(R)$. Then eJe is the Jacobson radical of eRe. Since R is continuous $J(R) = Z(R)$ and therefore $JS = 0$. Hence $(eJe)S = (eJ)(eS) = eJS = 0$. Thus, $eJe[eR(1-e)] = 0$, and so $eJeR = eJeRe \subseteq eJe$. Hence eJe is an R-submodule of eR. As $S = eR(1-e)$ it follows that $(eJe) \cap S = 0$. But eR is uniform, so $eJe = 0$. Thus eRe is a division ring. Now let $I \subseteq eR$. Then $S \subseteq I$ (since eR is uniform). It follows that $S \subseteq I(1-e) \subseteq eR(1-e) = S$. Hence $I(1-e) = S$. Furthermore, since Ie is an eRe-submodule of eRe, it follows that either $Ie = 0$ or $Ie = eRe$. If $Ie = 0$ then $I = I(1-e) = S$. If $Ie = eRe$ then, since $I = Ie \oplus I(1-e)$, it follows that $I = eRe \oplus eR(1-e) = eR$. Hence S is the only proper submodule of eR. □

THEOREM 3.3. *Let R be an indecomposable, non-local, right π-ring. If R is right continuous then R has essential socle.*

Proof. If R contains a square then, by Corollary 2.7, R is simple artinian and we are done. Assume R is square free. Since R is non-local and indecomposable it contains a nontrivial idempotent f and either $fR(1-f)$ or $(1-f)Rf$ is nonzero (by Lemma 1.9). Therefore, by Lemma 2.3, R has nonzero socle. Let $e^2 = e \in R$ with $\mathrm{Soc}(R) \subset' eR$. Suppose $e \neq 1$. For any non-zero idempotent $g \in (1-e)R$, since $\mathrm{Soc}(gR) = 0$, $(1-g)Rg \neq 0$ and g is not primitive (Lemmas 2.3, 3.1). Indeed, for any $\alpha : gR \to (1-g)R$, $\mathrm{Im}\alpha$ is semisimple. So $a : gR \to eR$. Furthermore, for any two non-zero orthogonal idempotents $g_1, g_2 \in (1-e)R$ and non-zero $\alpha_i : g_iR \to eR$, $(i = 1, 2)$, one gets that $\mathrm{Im}\alpha_1 \cap \mathrm{Im}\alpha_2 = 0$, for, otherwise, there exists a minimal right ideal $S \subset eR$ such that g_1R and g_2R map onto S. But then by the projectivity of g_1R there exists a non-zero $\phi : g_1R \to g_2R$. By Lemma 2.3, $\mathrm{Im}\phi$ is semisimple, a contradiction.

Let $1 - e = f_1 + g_1$, a sum of orthogonal non-zero idempotents and $0 \neq \alpha_1 : g_1R \to eR$. Write $f_1 = f_2 + g_2$, a sum of non-zero orthogonal idempotents and $0 \neq \alpha_2 : g_2R \to eR$. Continue like this writing $f_i = f_{i+1} + g_{i+1}$ and considering $0 \neq \alpha_{i+1} : g_{i+1}R \to eR$. Then the sum of $\mathrm{Im}\alpha_i$ is direct and there exists an epimorphism $\alpha : \oplus_{i=1}^{\infty} g_iR \to \oplus_{i=1}^{\infty} \mathrm{Im}\alpha_i$. This is a contradiction in view of Lemma 2.4. Therefore $e = 1$, proving that $\mathrm{Soc}(R) \subset' R$. □

LEMMA 3.4. *Let R be a continuous ring. Suppose e is a nonzero idempotent of R. If, for $a, b \in R$, aR, bR are two non-isomorphic minimal right ideals of R that are homomorphic images of eR, then there exists a nonzero idempotent f in $eR \cap a^{\perp}$ such that $f \notin b^{\perp}$, where $x^{\perp}$ is the right of x in R.*

Proof. As aR and bR are non-isomorphic, there exists a nonzero element x in $eR \cap a^{\perp}$ but not in $eR \cap b^{\perp}$. Let $\bar{R} = R/J(R)$. Then $\overline{eRe} = eRe/eJe$ is regular [**10**, 3.11]. Hence there exists $\bar{y} \in \overline{eRe}$ such that $\overline{xy}$ is a nonzero idempotent in $\overline{eRe}$. Thus there exists a nonzero idempotent f in eRe such that $f - xy \in eJe$ (Lemmas 1.8, 1.9). Thus $f \in eR \cap a^{\perp}$. Now, if $f \in b^{\perp}$, then $xy \in b^{\perp}$. But, as $x - xyx \in b^{\perp}$, it follows that $x \in b^{\perp}$, a contradiction. Hence, $f \notin b^{\perp}$. □

THEOREM 3.5. *Let R be a continuous π-ring. For any two independent right ideals A and B in R, A has only finitely many simple images in B.*

Proof. Let I be an infinite index set such that for each $i \in I$, S_i is a simple image of A in B. For $e^2 = e \in R$, let eR be a closure of A in R. As R is a π-ring, each S_i is an image of eR. Let $S_i = a_1R$ for some $a_i \in R$. For $i \neq j$, $i, j \in I$, there exists, by Lemma 3.4, a nonzero idempotent f in $eR \cap a_i^{\perp}$ such that f is not in $a_j^{\perp}$. One of fR or $(e - f)R$ maps onto infinitely many a_kR, $k \in I$. Denote the one that maps onto infinitely many a_kR' by g_1R, and the other one by f_1R.

Now $eR = f_1R \oplus g_1R$. Since $f \notin a_j^{\perp}$ and $e - f$ is not in $a_i^{\perp}$, it follows that f_1R has a nonzero simple image in B. As R is quasi-continuous and $f_1R \cap B = 0$, $(1 - f_1)Rf_1 \neq 0$. Now, since g_1R has infinitely many simple images in B, repeating the same process we get idempotents f_2 and h_2 in g_1R such that $(1 - f_2)Rf_2 \neq 0$ and h_2R has infinitely many simple images in B. Now $eR = (f_1R \oplus f_2R) \oplus h_2R$. Continuing this process, we get an infinite family $\{f_n : n \in N\}$ of orthogonal idempotents in R such that $(1 - f_n)Rf_n \neq 0$, a contradiction to Proposition 2.8. Hence the index set I is finite. □

THEOREM 3.6. *A continuous, indecomposable, non-local π-ring R has finitely generated essential socle.*

Proof. In view of Theorem 2.6, we can assume that R is square free. By Theorem 3.3, $\mathrm{Soc}(R) \subset' R$. Suppose $\mathrm{Soc}(R)$ is not finitely generated. Let $\{S_i : i \in I\}$ be the infinite family of minimal right ideals in R. As R is square free, this family is independent. For each $i \in I$, let e_iR be a closure of S_i, where $e_i^2 = e_i \in R$. By Proposition 2.8, there are only finitely many simple $S_{i_1}, \ldots, S_{i_n}$ in R such that $(1 - e_{i_k})Re_{i_k} \neq 0$, $k = 1, \ldots, n$. We can pick an idempotent $e \in R$ such that both eR and $(1 - e)R$ contain infinitely many minimal right ideals of R.

By Theorem 3.5, both eR and $(1 - e)R$ have only finitely many simple images in each other. Of these simple images, consider only those that are not S_{i_k} for $k = 1, \ldots, n$. Now, take the closures fR and gR of these simple images of eR and $(1 - e)R$ in $(1 - e)R$ and eR respectively, where $f^2 = f \in (1 - e)R$ and $g^2 = g \in eR$. Since R is quasi-continuous, there exist primitive orthogonal idempotents $f_1, \ldots, f_1$, $g_1 \ldots, g_m$ such that $fR = \oplus_{i=1}^{l} f_iR$ and $gR = \oplus_{i=1}^{m} g_iR$. Then fR and gR do not map outside themselves. Now there exist idempotents $f' \in (1 - e)R$ and $g' \in eR$ such that $R = (gR \oplus g'R) \oplus (fR \oplus f'R) = (gR \oplus f'R) \oplus (fR \oplus g'R)$. Since R is a π-ring and any nonzero image of $(1 - e)R$ in eR lie inside gR, any nonzero image of $f'R$ in eR must lie inside gR. If there is a nonzero homomorphism from $f'R$ into fR, composing with a projection map we'll get a nonzero homomorphism from $f'R$ onto a simple right module S in fR. But every simple in fR is an image of eR. As eR is projective, S would be isomorphic to a simple in $f'R$. As R is square free, this is a contradiction. Hence there is no nonzero homomorphism from $gR \oplus f'R$ into $fR \oplus g'R$. Symmetrically, there is no nonzero homomorphism from $fR \oplus g'R$ into $gR \oplus f'R$. But, as R is

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indecomposable and $f'R$ and $g'R$ have infinitely many minimal right ideals, this is a contradiction. Thus, R must have finitely generated essential socle. □

PROPOSITION 3.7. *Let R be an indecomposable, right continuous, right square free ring. Then R is a right $f\pi$-ring with finite uniform right dimension if and only if R is a right artinian right π-ring.*

Proof. Let R be an $f\pi$-ring with finite uniform dimension. Then it contains an independent family of uniform right ideals $U_1, \ldots, U_n$ with $\oplus_{i=1}^n U_i \subset' R$. As R is quasi-continuous, there exist nonzero idempotents $e_1, \ldots, e_n$ in R such that $U_i \subset' e_iR$ and $R = \oplus_{i=1}^n e_iR$. Since each U_i is uniform it follows that each e_iR is primitive. As R is indecomposable and continuous it follows, by Lemma 3.1 that $e_iR(1 - e_i) \neq 0$. By Lemma 3.2, each e_iR is artinian. Hence R is artinian. Now, since R is an $f\pi$-ring, it follows that R is a π-ring.

Conversely, assume that R is a right artinian π-ring. If R is local, then it is uniform (Lemma 1.11) and, therefore, has finite uniform dimension. If R is non-local, then it has finite uniform dimension by Theorem 3.6. Hence the proof follows. □

The right handed version of Theorem 3 in [**4**] states that a ring R is a non-local indecomposable right q-ring containing no minimal injective right ideals if and only if R is isomorphic to a ring of $n \times n$ matrices of the form

$$M_n(D, V) = \begin{pmatrix} D & V & & & & \\ & D & V & & & \\ & & D & V & & \\ & & & \cdot & \cdot & \\ & & & & \cdot\ \ \cdot & V \\ V & & & & & D \end{pmatrix}$$

with D a division ring, V a null algebra over D with $\dim_D V = \dim V_D = 1$ and $n \geq 2$. We will consider next a larger family of rings. Let $n \geq 2$ be a natural number. For $i \in \{1, \ldots, n\}$, let D_i be a division ring. For $i \in \{1, \ldots, n-1\}$, let ${}_{D_i}V_{i,i+1}{}_{D_{i+1}}$ be a bimodule and let ${}_{D_n}V_{n1}{}_{D_1}$ be a bimodule. For convenience, we will consider here, when dealing with the subscripts, addition modulo n on the set $\{1, \ldots, n\}$ rather than on $\{0, \ldots n-1\}$ as is customary. We do this since the rows and columns of matrices are usually labeled by the first set and not the second. So, in particular, $n + 1 = 1$ and therefore it suffices to say that ${}_{D_i}V_{i,i+1}{}_{D_{i+1}}$ is a bimodule for $i = 1, \ldots, n$. By $M = M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$ we denote the set of $n \times n$ matrices with (i, i) entry from D_1, $(i, i+1)$ entry from $V_{i,i+1} (i = 1 \ldots n)$, and all other entries zero. It is straight forward to see that M is a ring under the usual matrix addition and multiplication if one assumes that $V_{i,i+1}V_{i+1,i+2} = 0$ for $i = 1, \ldots, n$.

In the following theorem we show that, under certain conditions, right π-rings are precisely those rings of the form

$$M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1}) = \begin{pmatrix} D_1 & V_{12} & & & & \\ & D_2 & V_{23} & & & \\ & & D_3 & V_{34} & & \\ & & & \cdot & \cdot & \\ & & & & \cdot\ \ \cdot & V_{n-1,n} \\ V_{n1} & & & & & D_n \end{pmatrix}$$

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for which $\dim(V_{i,i+1_{D_{i+1}}}) = 1$.

THEOREM 3.8. *Let R be an indecomposable, non-local ring. Then the following conditions are equivalent:*

(1) *R is right continuous and a right π-ring.*
(2) *Every right ideal in R is right continuous.*
(3) *Either R is simple artinian or R is right continuous, square free and there exist orthogonal primitive idempotents $e_1, \ldots e_n$ in R such that $e_iRe_j \neq 0$ if and only if either $i = j$ or $j - 1 (\mathrm{mod}\ n)$, each e_iR has length two and $R = \oplus_{i=1}^n e_iR$.*
(4) *R is either simple artinian or isomorphic to a ring of the form*

$$M = M_n(D_1, \ldots, D_n; V_{12}, \ldots V_{n-1,n}, V_{n1})$$

for some natural number n and with entries such that $\dim(V_{i,i+1_{D_{i+1}}}) = 1$.
(5) *R is right continuous and every right ideal in R containing the* $\mathrm{Soc}(R)$ *is two-sided.*
(6) *R is right continuous and every essential right ideal in R is two-sided.*

Proof. The implications (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (6) are trivial, and (6) $\Rightarrow$ (1) following from Proposition 2.1. It remains only to prove the following implications: (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2). Suppose (1) holds. If R contains a square then R is simple artinian, by Corollary 2.7. Suppose R is square free. By Theorem 3.6, R has finitely generated nonzero essential socle. Let $S_1, \ldots S_n$ be the minimal right ideals in R. Let $I = \{1, \ldots, n\}$. For each $i \in I$, let $S_i \subset' e_iR$, where e_i is a primitive idempotent in R. Clearly, $R = \oplus_{i=1}^n e_iR$. As R is indecomposable, $e_iR(1 - e_i) \neq 0$ for each $i \in I$ (Lemma 3.1). Hence there exists some $j \neq i, j \in I$ such that $e_iRe_j \neq 0$. As R is square free and e_jR is projective, $e_iRe_k = 0$ for all $k \neq i, j, k \in I$. Since R is indecomposable, there must exist some $k \in I$, $k \neq i$ (unless $n = 2$) such that $e_jRe_k \neq 0$. Hence, there exists a permutation ϕ on $I = \{1, \ldots, n\}$ such that $e_iRe_{\phi(i)} \neq 0$ and $\phi(i) \neq i$ for all $i \in I$. Write $\phi = \phi_1\phi_2 \ldots \phi_k$, a composition of disjoint cycles. Since R is indecomposable, $k = 1$ and ϕ is a cycle. Renumbering if necessary, we can write $\phi^i(1) = i + 1$, for $i = 1, \ldots, n - 1$, and $\phi^n(1) = 1$. Therefore, for each $i \in I$, we may consider the sequence of homomorphisms $\alpha_{i+1} : e_{i+1}R \to e_iR$ as follows: $e_nR \to e_{n-1}R \to \ldots \to e_2R \to e_1R \to e_nR$. Now, for each $i \in \{1, \ldots, n - 1\}$, $\mathrm{Im}\alpha_{i+1} = S_i$ is simple. Moreover, for each $i < j, i, j \in \{1, \ldots n\}$, $e_iRe_j \neq 0$ iff $i = j$ or $i = j - 1 (\mathrm{mod}\ n)$. By Lemma 3.2, each e_iR has length two. Hence we have proved (1)$\Rightarrow$(3). Now, assume (3) holds. If R is not simple artinian then, by (3), R is right continuous and there exist orthogonal primitive idempotents $e_1, \ldots e_n$ in R such that $e_iRe_j \neq 0$ if and only if either $i = j$ or $i = j (\mathrm{mod}\ n)$, each e_iR has length two and $R = \oplus_{i=1}^n e_iR$. By Lemma 3.2, e_iRe_i is a division ring for all $i \in \{1, \ldots n\}$. By Lemma 3.2, $S_i = e_iRe_{i+1}$. Thus S_i can be viewed as a left vector space over e_iRe_i and a right vector space over $e_{i+1}Re_{i+1}$. Denote each division ring e_iRe_i by D_i and each vector space e_iRe_{i+1} by $V_{i,i+1}$. We will show that $\dim(V_{i,i+1_{D_{i+1}}}) = 1$. Let e_{ii} be the unit matrix in M whose only nonzero entry is the (i, i) entry and equals 1. Then $e_{ii}M = (0 \ldots 0\, D_iV_{i,i+1}\, 0 \ldots 0) \cong e_iR$. It is easy to check that the proper M-submodules of $e_{ii}M$ are precisely $(0 \ldots 0\, W_{i,i+1}\, 0 \ldots 0)$ where $W_{i,i+1}$ is a right D_{i+1}-subspace of $V_{i,i+1}$. Therefore, it is clear that the $e_{ii}M$ has no non-trivial summands. Since $e_{ii}M$, being isomorphic to e_iR, is quasi-continuous it follows that it is uniform as a right M-module. Thus, $V_{i,i+1}$ is uniform as a right D_{i+1}-module. Hence,

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$\dim(V_{i,i+1_{D_{i+1}}}) = 1$. Finally by defining $V_{i,i+1}V_{i+1,i+2} = 0$, $i \in I$, we get the following matrix representation of R:

$$R = \oplus_{i=1}^{n} e_i R \cong \begin{pmatrix} D_1 & V_{12} & & & & \\ & D_2 & V_{23} & & & \\ & & D_3 & V_{34} & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & V_{n-1,n} \\ V_{n1} & & & & & D_n \end{pmatrix}$$

$$= M_n(D_1, \dots D_n; V_{12}, \dots, V_{n-1,n}, V_{n1}).$$

This proves (4).

Now, assume (4) holds. If R is simple artinian then (5) holds trivially. Assume R is square free and isomorphic to the matrix ring M. It is easy to check that the only proper M-submodules of $e_{ii}M$ are $(0 \dots 0\, W_{i,i+1}\, 0 \dots 0)$, where $W_{i,i+1}$ is a right D_{i+1}-subspace of $V_{i,i+1}$. As $\dim(V_{i,i+1_{D_{i+1}}}) = 1$, it is clear that $e_{ii}M$ is uniform. It is also clear that $e_{ii}M$ is not isomorphic to any of its proper submodules. It follows that each $e_{ii}M$ is continuous. It is easy to check that $\mathrm{Soc}(e_{ii}M) = (0 \dots 0\, V_{i,i+1}\, 0 \dots 0)$. Therefore, as M is square free, there is no nonzero homomorphism from a proper right ideal of $e_{ii}M$ to a right ideal of $e_{jj}M$ for $i \neq j$. Hence, for $i \neq j$, $e_{ii}M$ is $e_{jj}M$-injective. Hence M is right continuous.

Now, let a, m be two nonzero elements of M. For $i = 1, \dots, n$, there exist d_i, $\Delta_i \in D_i$ and $v_{i,i+1}$, $w_{i,i+1} \in V_{i,i+1}$ such that

$$a = \begin{pmatrix} d_1 & v_{12} & & & & \\ & d_2 & v_{23} & & & \\ & & d_3 & v_{34} & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & v_{n-1,n} \\ v_{n1} & & & & & d_n \end{pmatrix} \text{and } m = \begin{pmatrix} \Delta_1 & w_{12} & & & & \\ & \Delta_2 & w_{23} & & & \\ & & \Delta_3 & w_{34} & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & w_{n-1,n} \\ w_{n1} & & & & & \Delta_n \end{pmatrix}.$$

For $i = 1 \dots n-1$, define $\delta_i = \Delta_i^{-1} d_i \Delta_i$, if $\Delta_i \neq 0$, and $\delta_i = 0$ otherwise. Define $u_{i,i+1} = \Delta_i^{-1} d_i w_{i,i+1}$ if $\Delta_i \neq 0$ and $u_{i,i+1} = 0$ otherwise. Define $s_i = v_{i,i+1}\Delta_{i+1} - w_{i,i+1}\delta_{i+1}$ if $\Delta_i \neq 0$ and $s_i = d_i w_{i,i+1} + v_{i,i+1}\Delta_{i+1} - w_{i,i+1}\delta_{i+1}$ if $\Delta_i = 0$. For $i = n$, define $u_{n1} = \Delta_n^{-1} d_n w_{n1}$ if $\Delta_n \neq 0$ and $s_n = v_{n1}\Delta_1 - w_{n1}\delta_1$ if $\Delta_n \neq 0$, and $s_n = v_{n1}\Delta_1 + d_n w_{n1} - w_{n1}\delta_1$ if $\Delta_n = 0$.

It is straightforward to verify that $ma = s + am'$, where

$$s = \begin{pmatrix} 0 & s_1 & & & & \\ & 0 & s_2 & & & \\ & & 0 & s_3 & & \\ & & \cdot & \cdot & & \\ & & \cdot & \cdot & s_{n-1} & \\ s_n & & & & & 0 \end{pmatrix} \text{and } m' = \begin{pmatrix} \delta_1 & u_{12} & & & & \\ & \delta_2 & u_{23} & & & \\ & & \delta_3 & u_{34} & & \\ & & \cdot & \cdot & & \\ & & \cdot & \cdot & u_{n-1,n} & \\ u_{n1} & & & & & \delta_n \end{pmatrix}.$$

Now, as $\mathrm{Soc}(e_{ii}M) = (0 \dots 0\, V_{i,i+1}\, 0 \dots 0)$, $\mathrm{Soc}(M)$ consists of all matrices in M with zero diagonal. Thus, $ma \in \mathrm{Soc}(M) + aM$. Hence every right ideal of M containing the $\mathrm{Soc}(M)$ is two-sided, proving (5).

Now, suppose (1) holds. Let I be a right ideal of R isomorphic to a summand eR of R where $e^2 = e \in R$. If e is primitive then, by Lemmas 3.1 and 3.2, eR has only one proper right ideal $S = \mathrm{Soc}(eR)$. Since eR is indecomposable, I is indecomposable. Let fR be a closure of I in R, where $f^2 = f \in R$. Then fR is indecomposable and again, by Lemmas 3.1, 3.2, fR has only one proper right ideal $T = \mathrm{Soc}(fR)$. As eR is not simple, I is not simple. Thus, $I = fR$. Now suppose e is not primitive. As R is quasi-continuous and has finite uniform dimension (Theorem 3.6), we can write $eR = f_1R \oplus \ldots \oplus f_kR$, where the f_i are primitive idempotents. There exist submodules I_i of I, $i = 1, \ldots k$, such that $I = I_1 \oplus \ldots \oplus I_k$ and $I_i \cong f_iR$. By the above arguments, each I, is a summand of R. Since R is quasi-continuous, it follows that $I \subset^{\oplus} R$. Hence R is continuous. Thus, (2) holds. □

Example 3.9. In the above Theorem, the left dimension of $V_{i,i+1}$ over D_i is not necessarily 1, as the following example shows.

Let F be any field, $F(x)$ the field of rational functions over F on the variable x. Let $V = F(x)$ be the $F(x)$-bimodule with left action of $F(x)$ on V given by $f(x)\cdot g(x) = f(x^2)g(x)$ and the right action given by multiplication in $F(x)$. Consider the ring

$$R = \begin{pmatrix} F(x) & V & 0 \\ 0 & F(x) & V \\ V & 0 & F(x) \end{pmatrix}$$

where $V^2 = 0$. This ring satisfies all the conditions in Theorem 3.6 but $\dim({}_{F(x)}V) = 2$. □

Let

$$M' = M'_n(D_1, \ldots, D_n; V_{21}, \ldots V_{n,n-1}, V_{1n}) = \begin{pmatrix} D_1 & & & & & V_{1n} \\ V_{21} & D_2 & & & & \\ & V_{32} & D_3 & & & \\ & & V_{43} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & V_{n,n-1} & D_n \end{pmatrix},$$

where for $i = 1 \ldots n$, D_i is a division ring and $V_{i+1,i}$ is a D_{i+1} - D_i bimodule. Notice that, as we did before, when dealing with subscripts we are considering addition modulo n on the set $\{1, \ldots, n\}$ rather than on $\{0, \ldots, n-1\}$.

It is straightforward to see that M' is a ring under the usual matrix addition and multiplication if we assume that $V_{i+1,i}V_{i,i-1} = 0$ for $i = 1 \ldots n$.

Certainly, there is a symmetric, left-handed version of Theorem 3.8 to characterize left continuous indecomposable left π rings in terms of rings of the form $M' = M'_n(D_1, \ldots, D_n; V_{21}, \ldots V_{n,n-1}, V_{1n})$, as follows.

Theorem 3.10. *Let R be an indecomposable, non-local ring. Then the following conditions are equivalent:*

(1) *R is left continuous and a left π-ring;*

(2) *R is either simple artinian or isomorphic to a ring of the form*

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$$M' = M'_n(D_1, \ldots, D_n; V_{21}, \ldots, V_{n,n-1}, V_{1n})$$

for some natural number n and with entries such that $\dim({}_{D_{i+1}}V_{i+1,i}) = 1$.

Proof. Similar to that of Theorem 3.8. □

However, it is important to point out that one can also characterize right continuous, indecomposable, non-local rings in terms of rings of the form M', as follows.

THEOREM 3.11. *Let R be a right continuous, indecomposable, non-local ring. Then the following conditions are equivalent:*

(1) *R is a right π-ring*
(2) *either R is simple artinian or R is square free and there exist orthogonal primitive idempotents $f_1, \ldots, f_n$ in R such that $f_iRf_j \neq 0$ if and only if either $i = j$ or $i = j + 1 (\mathrm{mod}\ n)$, each f_iR has length two and $R = \oplus_{i=1}^n f_iR$;*
(3) *either R is simple artinian or R is square free and isomorphic to a ring of the form*

$$M' = M_n(D_1, \ldots, D_n, V_{21} \ldots; V_{n,n-1}, V_{1n}) = \begin{pmatrix} D_1 & & & & & V_{1n} \\ V_{21} & D_2 & & & & \\ & V_{32} & D_3 & & & \\ & & V_{43} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & V_{n,n-1} & D_n \end{pmatrix}$$

for some natural number n and $\dim(V_{i+1,i_{D_i}}) = 1$.

Proof. Suppose (1) holds. Assume R is not simple artinian. By Theorem 3.8, R is right continuous, square free and there exist orthogonal primitive idempotents $e_1, \ldots e_n$ in R such that $e_iRe_j \neq 0$ if and only if either $i = j$ or $i = j - 1 (\mathrm{mod}\ n)$, each e_iR has length two and $R = \oplus_{i=1}^n e_iR$. Define $f_i = e_{n-i+1}$. Then $f_iRf_j = e_{n-i+1}Re_{n-j+1} \neq 0$ if and only if either $n - i + 1 = n - j + 1$ or $n - i + 1 = n - j + 1 - 1 (\mathrm{mod}\ n)$ if and only if either $i = j$ or $i = j + 1 (\mathrm{mod}\ n)$. Hence the sequence $e_nR \to e_{n-1}R \to \ldots \to e_2R \to e_1R \to e_nR$ is the same as the sequence $f_1R \to f_2R \to \ldots \to f_nR \to f_1R$. This gives rise to the matrix representation M' of the ring R. A proof similar to the proof of Theorem 3.8 will prove the equivalence of all the statements in Theorem 3.11.

COROLLARY 3.12. *If R is indecomposable are non-local, then the condition of being a right q-ring is equivalent to being a left q-ring.*

Proof. A right handed version of [**4**, Theorem 3] states that an indecomposable, non-local right q-ring is either simple artinian or a ring of the form

$$M_n(D, V) = \begin{pmatrix} D & V & & & \\ & D & V & & \\ & & \cdot & \cdot & \\ & & & \cdot\ \cdot & V \\ V & & & & D \end{pmatrix},$$

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where D is a division ring and V is a null algebra over D with $\dim({}_D V) = 1 = \dim(V_D)$. Following the proof of Theorem 3.11, it is easy to check that an indecomposable, non-local, right q-ring R is either simple artinian or a ring of the form

$$M'_n(D, V) = \begin{pmatrix} D & & & & & V \\ V & D & & & & \\ & V & D & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & V & D \end{pmatrix}$$

with D and V as above. By [**4**, Theorem 3], $M'_n(D, V)$ is a left q-ring. □

If the ring R in Theorem 3.8 is both a left and right π-ring, then both the left and right dimensions of $V_{i,i+1}$, over D_i and D_{i+1} respectively, are equal to one and R is a (two-sided) q-ring, as the following Theorem shows:

THEOREM 3.13. *Let R be a right continuous, indecomposable, non-local ring. Then the following conditions are equivalent:*
(1) *R is a right and left π-ring;*
(2) *R is a left quasi-continuous right π-ring;*
(3) *R is either simple artinian or isomorphic to a ring of the form*

$$M = M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1}),$$

for some natural number n and with entries such that $\dim(V_{i,i+1_{D_{i+1}}}) = 1 = \dim({}_{D_i} V_{i,i+1})$;
(4) *R is a right q-ring;*
(5) *R is a right and a left q-ring.*

Proof. Clearly (1) implies (2). Now suppose (2) holds. Then R is a right π-ring. By Theorem 3.8(4), R is either simple artinian or of the form $M_n(D_1, \ldots D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$. If R is simple artinian then (3) holds trivially. Otherwise, every left summand $Re_i \cong \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ V_{i+1,i} \\ D_i \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$, where e_i is the unit matrix e_{ii}, has as its only

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submodules Re_i and 'columns' of the form $\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ W_{i-1,i} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$ where $W_{i-1,i}$ is a D_{i-1}-subspace of $V_{i-1,i}$. Therefore, Re_i is indecomposable and, since it is also left quasi-continuous, the dimension of $V_{i-1,i}$ as left D_{i-1}-space is 1, proving (3). Now, suppose (3) holds. Let $0 \neq x \in V_{i,i+1}$. Then $V_{i,i+1} = xD_{i+1} = D_i x$. The assignment $d \rightarrow d'$ if and only if $dx = xd'$ defines a ring isomorphism between D_i and D_{i+1}. Hence the D_i' are all isomorphic and we can view them as a division ring D. It is now easy to check that the V_i' are all isomorphic as well. Hence R is a right q-ring [**4**, Theorem 3], proving (4). Now (4) $\Rightarrow$ (5) is clear by Corollary 3.12, (5) $\Rightarrow$ (1) is trivial. □

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