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Almost sure asymptotic stability analysis of the θ-Maruyama method applied to a test system with stabilising and destabilising stochastic perturbations

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Abstract

We perform an almost sure linear stability analysis of the θ-Maruyama method, selecting as our test equation a two-dimensional system of Itô differential equations with diagonal drift coefficient and two independent stochastic perturbations which capture the stabilising and destabilising roles of feedback geometry in the almost sure asymptotic stability of the equilibrium solution. For small values of the constant step-size parameter, we derive close-to-sharp conditions for the almost sure asymptotic stability and instability of the equilibrium solution of the discretisation that match those of the original test system. Our investigation demonstrates the use of a discrete form of the Itô formula in the context of an almost sure linear stability analysis.

1. *Introduction*

The field of numerical analysis is often concerned with the effect of a numerical method on the qualitative behaviour of solutions of differential equations. To investigate this effect, one can apply the method to a test problem which is simple enough to allow for a precise analytical treatment, but which nonetheless exhibits the property of interest.

Numerical analysis for ordinary differential equations has produced several concepts associated with the stability of numerical methods, for example $A(\alpha)$-stability, A-stability and L-stability: see Hairer and Wanner [**10**, Chapter IV.3]. These concepts are based on analysing a method's properties after it has been applied to the scalar linear test equation $x'(t) = \lambda x(t)$. The power of this approach is that a large class of general nonlinear systems may be linked analytically to the scalar linear test equation via a process of linearisation and diagonalisation: again see [**10**, Chapter IV.2].

For the linear stability analysis of stochastic numerical methods, the focus in the literature has been on a scalar linear test equation with a single Wiener process of the form $dX(t) = \lambda X(t)\,dt + \mu X(t)\,dW(t)$. A review of that literature may be found in Buckwar and Kelly [**5**], along with precise definitions for various forms of asymptotic stability of equilibria of continuous- and discrete-time stochastic processes. We additionally note that convergence to zero in probability of solutions of linear random difference equations is addressed in [**7**] and [**14**] as an application of the general analysis of products of random matrices.

Stochastic models of systems arising in such applications as neuroscience or electrical circuit engineering typically consist of many coupled stochastic differential equations (SDEs) driven by many sources of noise. Therefore, it seems appropriate to investigate the effect of multi-dimensional noise in systems of SDEs on the stability properties of numerical methods. However, results reported in [**5, 6**] indicate that a scalar linear test equation with a single Wiener process is not an appropriate representative for general systems of nonlinear SDEs.

In this paper we perform an almost sure linear stability analysis of the θ-Maruyama method for small step sizes using a two-dimensional test system with diagonal drift and almost sure stabilising and destabilising diffusion coefficients. This analysis has three stages.

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(1) Transform the discretised test system into a scalar stochastic difference equation with an equilibrium solution that has identical almost sure stability properties. This transformed equation has the form

$$Z_{n+1} = R_n Z_n, \quad n = 1, 2, \ldots,$$

where $\{R_n\}$ is an independent and identically distributed (i.i.d.) stochastic sequence with a special structure.

(2) Use the strong law of large numbers to express an almost sure asymptotic stability condition for the transformed system in terms of the constant $\mathbb{E}[\ln(R_n)]$.

(3) Apply a discrete Itô formula developed in [**2**] to $\mathbb{E}[\ln(R_n)]$ in order to derive almost sure asymptotic stability and instability conditions (for small step sizes) in terms of the system parameters.

Stage (2) was used previously by Higham [**11**] in an almost sure linear stability analysis of the θ-Maruyama method, where he showed in particular that the θ-Maruyama method does not possess the property of A-stability in the almost sure sense when applied to a linear scalar test equation. By contrast, it was pointed out in [**5**] that A-stability in the almost sure sense is not ruled out for systems of SDEs with more than one dimension and a stochastic perturbation which is purely almost surely destabilising. These results support the idea that linear systems of SDEs provide a more appropriate setting within which to study questions of almost sure stability.

To date, it has not been possible to derive necessary and sufficient almost sure stability conditions which definitively confirm A-stability in the almost sure sense for a stochastic numerical method. However, Higham, Mao and Yuan [**12**] were able to show that, if the equilibrium of a scalar test equation is almost surely asymptotically stable, then that property is preserved by a θ-Maruyama discretisation for *sufficiently small step sizes*. This represents a weaker notion than A-stability, but may be viewed as a significant intermediate step, particularly as their results generalise to a broad class of nonlinear SDE systems. In this paper we refine their approach to generate close-to-sharp conditions for almost sure asymptotic stability and instability in a test equation that encodes the effect of almost sure stabilising and destabilising perturbations. Further, we introduce a discrete form of the Itô formula, developed in Appleby, Berkolaiko and Rodkina [**2**], as a useful tool in this regard. A more detailed comparison with the existing literature is given in § 2.1.

1.1. *The θ-Maruyama discretisation method*

Consider the general linear system of stochastic ordinary differential equations given by

$$dX(t) = FX(t)\,dt + \sum_{r=1}^{m} G_r X(t)\,dW_r(t), \quad t > 0. \tag{1.1}$$

Here $F \in \mathbb{R}^{d\times d}$, $G_1, \ldots, G_m \in \mathbb{R}^{d\times d}$ and $W = (W_1, \ldots, W_m)^T$ is an m-dimensional Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geqslant 0}, \mathbb{P})$, where the natural filtration generated by W is denoted $(\mathcal{F}(t))_{t\geqslant 0}$. The initial value $X(0)$ is an $\mathcal{F}(0)$-measurable random variable with finite second moment ($X(0)$ may be constant). Equation (1.1) is a linear, autonomous stochastic differential equation with constant coefficients, and unique strong global solutions $X = \{X(t; X(0)),\ t \geqslant 0\}$ exist (see for example [**15**]). In particular, equation (1.1) has an equilibrium solution $X(t) \equiv 0$ when $X(0) = 0$.

Applying the θ-Maruyama method with step size $h > 0$ and $\theta \in [0, 1]$ to solutions of equation (1.1) yields a discrete-time approximation $X_n \approx X(n\,h)$ satisfying the stochastic difference equation

$$X_{n+1} = X_n + (1-\theta)hFX_n + \theta hFX_{n+1} + \sqrt{h}\sum_{r=1}^{m} G_r X_n \xi_{r,n+1}, \quad n \in \mathbb{N}. \tag{1.2}$$

Here each $\xi_{r,n}$, $r=1,\ldots,m$, $n\in\mathbb{N}$, represents the $\mathcal{F}(nh)$-measurable standardised Wiener increment $[W_r(nh)-W_r((n-1)h)]/\sqrt{h}$. For the analysis in this paper it is sufficient to consider each $\{\xi_{r,n}\}_{n\in\mathbb{N}}$ to be one of m independent sequences of mutually independent standard normal random variables. The natural filtration generated by the m-dimensional discrete-time stochastic process $(\xi_{1,n},\ldots,\xi_{m,n})_{n\in\mathbb{N}}$ is denoted $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Notice that the discretisation process has preserved the equilibrium solution $X_n\equiv 0$ when $X_0=0$.

1.2. *Motivating the test equation*

As discussed above, the focus in the literature for the stability analysis of stochastic numerical methods has been on the scalar linear test equation with a single Wiener process

$$dX(t)=\lambda X(t)\,dt+\mu X(t)\,dW(t),\quad t>0,\,X(0)=x_0\neq 0,\quad \lambda,\mu\in\mathbb{C}. \tag{1.3}$$

However, since the matrix coefficients of equation (1.1) are simultaneously diagonalisable only when they commute pairwise, the class of general nonlinear SDEs that can be linked analytically to equation (1.3) is small. Thus, the identification of appropriate stochastic test systems is necessary. In [**5**], two classes of test systems were proposed which seek to capture the effects of stochastic perturbation on the almost sure asymptotic stability of point equilibria, and a partial stability analysis was carried out on each. The structure of those test systems was motivated by well-established results demonstrating that multiplicative stochastic perturbations may act to stabilise or destabilise an equilibrium solution in the almost sure sense. A review of the extensive literature on the subject may be found in Appleby, Mao and Rodkina [**3**].

In this paper we work with one of the linear test systems proposed in [**5**]. Consider

$$\begin{aligned} d\begin{pmatrix}X_1(t)\\X_2(t)\end{pmatrix}&=\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\begin{pmatrix}X_1(t)\\X_2(t)\end{pmatrix}dt+\begin{pmatrix}\sigma&0\\0&\sigma\end{pmatrix}\begin{pmatrix}X_1(t)\\X_2(t)\end{pmatrix}dW_1(t)\\ &\quad+\begin{pmatrix}0&-\varepsilon\\\varepsilon&0\end{pmatrix}\begin{pmatrix}X_1(t)\\X_2(t)\end{pmatrix}dW_2(t),\quad t>0, \end{aligned} \tag{1.4}$$

with system parameters $\lambda,\sigma,\varepsilon\in\mathbb{R}$. The independent terms of the diffusion coefficient capture both the almost sure stabilising and destabilising effects of stochastic perturbation; the equilibrium solution of equation (1.4) is almost surely asymptotically stable if and only if

$$2\lambda-\sigma^2+\varepsilon^2<0. \tag{1.5}$$

Remark 1. Clearly, the equilibrium solution of equation (1.4) is almost surely unstable if $2\lambda-\sigma^2+\varepsilon^2\geqslant 0$. This regime breaks into two cases. First, when $2\lambda-\sigma^2+\varepsilon^2>0$, non-equilibrium solutions of equation (1.4) satisfy $\lim_{t\to\infty}|X(t)|=\infty$ almost surely. Second, in the threshold case where $2\lambda-\sigma^2+\varepsilon^2=0$, non-equilibrium solutions satisfy $\liminf_{t\to\infty}|X(t)|=0$ and $\limsup_{t\to\infty}|X(t)|=\infty$. The combination of the strong law of large numbers and the discrete Itô formula employed in this paper allows us to determine when the norms of the discretisation of discrete solutions of equation (1.4) tend to infinity almost surely, but does not address recurrent behaviour; therefore, we omit the threshold case from consideration. For this reason we refer to the almost sure asymptotic stability and instability conditions developed for the discretisation of equation (1.4) as being 'close-to-sharp', rather than 'sharp'.

Following discretisation by the θ-Maruyama method, equation (1.4) becomes, for $n\in\mathbb{N}$,

$$\begin{pmatrix}X_{1,n+1}\\X_{2,n+1}\end{pmatrix}=\begin{pmatrix}\dfrac{1+(1-\theta)h\lambda}{1-\theta h\lambda}+\dfrac{\sqrt{h}\sigma\xi_{1,n+1}}{1-\theta h\lambda}&\dfrac{-\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda}\\\dfrac{\sqrt{h}\varepsilon\xi_{2,n+1}}{1-\theta h\lambda}&\dfrac{1+(1-\theta)h\lambda}{1-\theta h\lambda}+\dfrac{\sqrt{h}\sigma\xi_{1,n+1}}{1-\theta h\lambda}\end{pmatrix}\begin{pmatrix}X_{1,n}\\X_{2,n}\end{pmatrix}, \tag{1.6}$$

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where additionally $\theta \in [0,1]$, $h > 0$ and $\{\xi_{1,n}\}_{n\in\mathbb{N}}$, $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ are independent sequences of mutually independent standard normal random variables. Notice that the equilibrium solution $(X_{1,n}, X_{2,n}) \equiv (0,0)$ has been preserved in equation (1.6). In order that equation (1.6) be well defined for all $n \in \mathbb{N}$, we assume for the duration of the paper that $1 - \theta h\lambda \neq 0$.

The main result of this paper shows that for sufficiently small h, with the possible exception of the threshold case where $2\lambda - \sigma^2 + \varepsilon^2 = 0$, we recover precisely Condition (1.5) and therefore for small step sizes the almost sure asymptotic stability or instability of the equilibrium of equation (1.4) is preserved exactly through a θ-Maruyama discretisation.

1.3. *Outline of the paper*

In § 2.1 we review some of the relevant literature and discuss the context of our analysis within it. In § 2.2 we introduce the discrete Itô formula published in [**2**], and reformulate it in a manner that makes it useful for our purposes. In §§ 2.3 and 2.4 we transform equation (1.6) in such a way that the discrete Itô formula may be applied. In § 3 we present our main results, first writing almost sure asymptotic stability and instability conditions in terms of the expectation of the logarithm of the stochastic coefficient of the transformed system. These conditions apply for all step-size parameters $h > 0$. Second, we apply the discrete Itô formula in order to state (for sufficiently small h) close-to-sharp almost sure asymptotic stability and instability criteria explicitly in terms of the system parameters. Finally in § 4 we summarise and discuss our work and indicate possible future lines of investigation.

2. *Preliminaries*

2.1. *The context of our work in the literature*

A method of characterising the almost sure asymptotic stability of a prototypical linear stochastic difference equation in terms of an expectation may be found in Berkolaiko and Rodkina [**4**]. Consider

$$X_{n+1} = X_n(1 + \xi_{n+1}), \quad n = 1, 2, \ldots, X_0 = x_0 \in \mathbb{R}, \tag{2.1}$$

where ξ_i are i.i.d. random variables with $\mathbb{E}\xi_i = 0$ and $\mathrm{Var}[\ln|1+\xi_i|] < \infty$. Since one can write

$$|X_n| = |X_0| \prod_{i=1}^{n} |1 + \xi_{i+1}| = |X_0| \exp\left\{\sum_{i=0}^{n} \ln|1 + \xi_{n+1}|\right\},$$

almost sure asymptotic stability (respectively instability) follows if

$$\sum_{i=1}^{\infty} \ln|1 + \xi_{i+1}| = -\infty \quad (\text{respectively } +\infty).$$

One can then apply Kolmogorov's strong law of large numbers (see for example Shiryaev [**16**, Chapter IV, p. 389] and Theorem 2.1 below) to show that almost sure asymptotic stability of the equilibrium solution of equation (2.1) corresponds precisely to $\mathbb{E}[\ln|1 + \xi_i|] < 0$.

THEOREM 2.1 (Kolmogorov's strong law of large numbers). *Let $\{\xi_i\}_{i\in\mathbf{N}}$ be a sequence of independent random variables with $\mathrm{Var}[\xi_k] = \theta_k^2 < \infty$. Let $S_n = \xi_1 + \ldots + \xi_n$ and define the positive sequence $\{b_n\}$ such that $\lim_{n\to\infty} b_n = \infty$ and*

$$\sum_{i=1}^{\infty} \frac{\theta_i^2}{b_i^2} < \infty.$$

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Then

$$\lim_{n\to\infty} \frac{S_n - \mathbb{E}[S_n]}{b_n} = 0 \quad \textit{almost surely.}$$

The same technique appears earlier in the numerical analysis literature. In [**11**, Lemma 5.1], Higham characterises the almost sure asymptotic stability of the equilibrium of the θ-Maruyama discretisation of a scalar linear test equation in exactly this way.

More recently, Higham, Mao and Yuan [**12**] have shown that, when the scalar test equation is almost surely asymptotically stable, the θ-Maruyama discretisation preserves that stability for small step sizes. They do so by (as above) characterising almost sure asymptotic stability in terms of the expectation of the logarithm of a stochastic sequence, applying the inequality

$$\ln(1+u) \leqslant u - \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3, \quad u \geqslant -1 \tag{2.2}$$

and exploiting the moment properties of normal random variables. They also show how their results generalise to nonlinear systems of stochastic differential equations.

In this paper, instead of using equation (2.2) to estimate the logarithm function and hence showing that for sufficiently small h

Equilibrium of equation (1.4) almost surely asymptotically stable

$\Rightarrow$ Equilibrium of equation (1.6) almost surely asymptotically stable,

we employ the discrete Itô formula from [**2**] to show that for sufficiently small h this statement is in fact necessary and sufficient.

By restricting attention to the test system equation (1.4) and its discretisation equation (1.6), our analysis shows that the θ-Maruyama method captures the effects of almost sure stabilising and destabilising perturbations when the step size is small. This complements the main result in [**5**] on almost sure asymptotic stability, which may be stated as follows.

THEOREM 2.2. *The equilibrium solution of equation* (1.6) *is globally almost surely asymptotically stable if*

$$\lambda + \tfrac{1}{2}(\sigma^2 + \varepsilon^2) + \tfrac{1}{2}h(1-2\theta)\lambda^2 < 0.$$

The statement of Theorem 2.2 gives a sufficient condition which is unlikely to be sharp, due to the positive sign in front of the σ^2 term. However, it demonstrates that A-stability is not ruled out for systems of SDEs with more than one dimension and a stochastic perturbation which is purely almost surely destabilising. We refer the reader to [**5**] for further discussion.

2.2. *A discrete form of the Itô formula*

An introduction to discrete forms of the Itô formula may be found in Shiryaev [**16**, Chapter VII, p. 389]; note also the papers by Akahori [**1**], Kannan and Zhang [**13**] and the survey of discrete stochastic calculus presented in Gzyl [**9**]. Although [**1**] demonstrated the use of a discrete Itô formula to prove convergence of the Euler–Maruyama method, to the best of our knowledge such a formula has never been used to perform an almost sure linear stability analysis.

The particular discrete Itô formula that we are interested in was developed in [**2**] and is reproduced here in Theorem 2.3.

ASSUMPTION 1. $\{\zeta_n\}$ is a sequence of $\mathcal{F}_n$-measurable random variables, where

$$\mathbb{E}\zeta_n = 0, \quad \mathbb{E}\zeta_n^2 = 1, \quad \mathbb{E}|\zeta_n|^3 \text{ are uniformly bounded} \tag{2.3}$$

and each ζ_n has density function $y = p_n(x)$ satisfying

$$x^3 p_n(x) \to 0 \quad \text{as } |x| \to \infty \text{ uniformly in } n. \tag{2.4}$$

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THEOREM 2.3. *Consider $\phi:\mathbb{R}\to\mathbb{R}$ such that there exist $\delta>0$ and $\tilde{\phi}:\mathbb{R}\to\mathbb{R}$ satisfying:*

(i) $\tilde{\phi}\equiv\phi$ *on* $U_\delta=[1-\delta,1+\delta]$;

(ii) $\tilde{\phi}\in C^3(\mathbb{R})$ *and* $|\tilde{\phi}'''(x)|\leqslant M$ *for some* M *and all* $x\in\mathbb{R}$;

(iii) $\int_{\mathbb{R}}|\phi-\tilde{\phi}|\,dx<\infty$.

For each n, let f_n and g_n be $\mathcal{F}_n$-measurable uniformly bounded random variables, $|f_n|,|g_n|<K$, and let ζ_{n+1} be an $\mathcal{F}_n$-independent random variable satisfying (2.3) and (2.4) in Assumption 1. Then

$$\mathbb{E}[\phi(1+f_nh+g_n\sqrt{h}\zeta_{n+1})|\mathcal{F}_n]$$
$$=\phi(1)+\phi'(1)f_nh+\frac{\phi''(1)}{2}g_n^2h+hf_nO(h)+hg_n^2O(h), \tag{2.5}$$

where the error term $O(h)\to 0$ as $h\to 0$, uniformly in n, f_n and g_n.

REMARK 2. Theorem 2.3 provides an explicit expression for the conditional expectation of a transformed stochastic mapping typically found in θ-Maruyama discretisations of stochastic differential equations. Hence, the authors of [**2**] refer to it as a discrete Itô formula. The requirement for the auxiliary function $\tilde{\phi}$ satisfying Conditions (i)–(iii) arises from [**2**, Proof of Theorem 2.3], to which we refer interested readers.

Theorem 2.3 may be reformulated as a corollary more appropriate for the analysis in this paper, as follows.

COROLLARY 2.4. *Let the assumptions of Theorem 2.3 hold and additionally let the terms of $\{\zeta_n\}_{n\in\mathbf{N}}$ be identically distributed random variables. Let $f_n=F_h$ and $g_n=G_h$ depend on h and have the following representation:*

$$F_h=f(h)+O(h),\quad G_h^2=g^2(h)+O(h),\quad \text{where } O(h)\to 0 \text{ as } h\to 0.$$

Then equation (2.5) takes the form

$$\mathbb{E}[\phi(1+hF_h+\sqrt{h}G_h\zeta_{n+1})|\mathcal{F}_n]=\phi(1)+h\phi'(1)f(h)+h\frac{\phi''(1)}{2}g^2(h)+hO(h). \tag{2.6}$$

2.3. *Tail behaviour of probability distributions*

In order to apply Corollary 2.4, we will transform the system equation (1.6) into a linear scalar stochastic difference equation in § 2.4. However, the stochastic perturbation of the transformed system will no longer be normal, and we will have to prove that the density functions of the transformed perturbations satisfy the conditions of Assumption 1. In this section we present Lemmas 2.5 and 2.6, which will be necessary to achieve this.

Lemma 2.5 is a collection of standard results which may be proved by applying the change of variable technique. See Grimmett and Stirzaker [**8**] for details.

LEMMA 2.5. *Let U and V be independent random variables with probability density functions $y=f_U(x)$ and $y=f_V(x)$, respectively.*

(i) *Let $f_{U^2}(x)$ be the probability density function of U^2. Then*

$$f_{U^2}(x)=\frac{1}{\sqrt{x}}f_U(\sqrt{x}),\quad x>0.$$

(ii) *Let $a,b\in\mathbb{R}$, and $a\neq 0$. Let $f_{aU+b}(x)$ be the probability density function of $aU+b$. Then*

$$f_{aU+b}(x)=\frac{1}{|a|}f_U\left(\frac{x-b}{a}\right),\quad x\in\mathbb{R}.$$

(iii) *Let* $f_{U+V}(x)$ *be the probability density function of* $U+V$. *Then*

$$f_{U+V}(x) = \int_{-\infty}^{\infty} f_U(u) f_V(x-u)\, du, \quad x \in \mathbb{R}.$$

Lemma 2.6 below requires the following assumption.

ASSUMPTION 2. Z is a random variable with probability density function $y = f_Z(x)$ satisfying

$$|x|^{k_Z} f_Z(x) \to 0 \quad \text{as } |x| \to \infty,$$

uniformly in x, for some $k_Z > 1$.

LEMMA 2.6. *Let* U *and* V *be independent random variables with probability density functions* $y = f_U(x)$ *and* $y = f_V(x)$ *satisfying Assumption* 2 *with* $k_U > 1$ *and* $k_V > 1$, *respectively.*

(1) *Let* $c \in \mathbb{R}$. *Then*

$$|x|^{k_U} f_{U+c}(x) \to 0 \quad \text{as } |x| \to \infty.$$

(2) *Let* $k_1 < \min\{k_U, k_V\}$. *Then*

$$|x|^{k_1} f_{U+V}(x) \to 0 \quad \text{as } |x| \to \infty.$$

(3) *Let* $k_2 < (k_U+1)/2$. *Then*

$$x^{k_2} f_{U^2}(x) \to 0 \quad \text{as } x \to \infty.$$

Proof. Part (i).
By part (ii) of Lemma 2.5, we have

$$\begin{aligned} |x|^{k_U} f_{U+c}(x) &= |x-c+c|^{k_U} f_U(x-c) \\ &\leqslant |x-c|^{k_u} f_U(x-c) + |c|^{k_U} f_U(x-c) \\ &= |y|^{k_U} f_U(y) + |c|^{k_U} f_U(y), \end{aligned}$$

where $y = x - c$. Now, by Assumption 2,

$$0 \leqslant \lim_{|x|\to\infty} |x|^{k_U} f_{U+c}(x) \leqslant \lim_{|y|\to\infty} (|y|^{k_U} f_U(y) + |c|^{k_U} f_U(y)) = 0,$$

giving the result.

Part (ii).
By Assumption 2, there exist sufficiently large constants $H > 0$ and $a_0 > 0$ such that

$$f_U(u) \le H u^{-k_U}, \quad f_V(v) \le H u^{-k_V}, \quad \text{when } |u| > a_0. \tag{2.7}$$

Let $z > 2a_0$, and decompose the density function of $U+V$ as

$$\begin{aligned} f_{U+V}(z) &= \int_{-\infty}^{\infty} f_U(u) f_V(z-u)\, du \\ &= \underbrace{\int_{-\infty}^{z/2} f_U(u) f_V(z-u)\, du}_{J_1} + \underbrace{\int_{z/2}^{\infty} f_U(u) f_V(z-u)\, du}_{J_2}. \end{aligned} \tag{2.8}$$

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Since $z/2 > a_0$, for J_1 we have $u < z/2$ and $z - u > z - z/2 = z/2 > a_0$, while for J_2 we have $u > z/2 > a_0$. Thus, equations (2.7) and (2.8) yield

$$\begin{aligned}\int_{-\infty}^{\infty} f_U(u) f_V(z-u)\,du &\le \int_{-\infty}^{z/2} f_U(u) H 2^{k_V} z^{-k_V}\,du + \int_{z/2}^{\infty} H 2^{k_U} z^{-k_U} f_V(z-u)\,du \\ &= H\left(2^{k_V} z^{-k_V}\int_{-\infty}^{z/2} f_U(u)\,du + 2^{k_U} z^{-k_U}\int_{z/2}^{\infty} f_V(z-u)\,du\right) \\ &\leqslant H\left(2^{k_V} z^{-k_V} + 2^{k_U} z^{-k_U}\right),\end{aligned}$$

since f_U and f_V are density functions. Thus, for $k_1 < \min\{k_U, k_V\}$ and $z \to \infty$,

$$z^{k_1}\int_{-\infty}^{\infty} f_U(u) f_V(z-u)\,du \to 0.$$

Similarly we can prove that, for $k_1 < \min\{k_U, k_V\}$ and $z \to -\infty$,

$$|z|^{k_1}\int_{-\infty}^{\infty} f_U(u) f_V(z-u)\,du \to 0.$$

Part (iii). Since $k_2 < (k_U + 1)/2$, part (i) of Lemma 2.5 gives

$$x^{k_2} f_{U^2}(x) = x^{k_2}\frac{1}{\sqrt{x}} f_U(\sqrt{x}) = (\sqrt{x})^{2k_2-1} f_U(\sqrt{x}) \to 0 \quad \text{as } x \to \infty.$$

□

2.4. *Transformation of system equation* (1.6)

In order to apply Corollary 2.4, we must transform it into a scalar stochastic difference equation with a trivial solution possessed of equivalent almost sure asymptotic stability properties to those of equation (1.6).

LEMMA 2.7. *Let $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ be independent sequences of mutually independent standard normal random variables. Define the sequence of random variables $\{\eta_n\}_{n\in\mathbb{N}}$ by*

$$\eta_n := \frac{2(1+(1-\theta)\lambda h)\sigma\xi_{1,n} + \sqrt{h}\sigma^2(\xi_{1,n}^2 - 1) + \sqrt{h}\varepsilon^2(\xi_{2,n}^2 - 1)}{(1-\theta\lambda h)^2}. \tag{2.9}$$

Then $\mathbb{E}(\eta_n^2) < \infty$ for each $n \in \mathbb{N}$.

Proof. We can write

$$\begin{aligned}\mathbb{E}\left(\eta_n^2\right) &= \frac{1}{(1-\theta\lambda h)^4}(4\sigma^2(1+(1-\theta)\lambda h)^2 \\ &\quad + h\sigma^4(\mathbb{E}\xi_{1,n}^4 - 1) + h\varepsilon^4(\mathbb{E}\xi_{2,n}^4 - 1) + 4\sqrt{h}(1+(1-\theta)\lambda h)\sigma^3\mathbb{E}\xi_{1,n}^3) \\ &= \frac{4\sigma^2}{(1-\theta\lambda h)^2}\left(\frac{(1+(1-\theta)\lambda h)^2}{(1-\theta\lambda h)^2}\right) \\ &\quad + \frac{h\sigma^4(\mathbb{E}\xi_{1,n}^4 - 1) + h\varepsilon^4(\mathbb{E}\xi_{2,n}^4 - 1) + 4\sqrt{h}(1+(1-\theta)\lambda h)\sigma^3\mathbb{E}\xi_{1,n}^3}{(1-\theta\lambda h)^4} \\ &= \frac{4\sigma^2}{(1-\theta\lambda h)^2}\left(\frac{(1+(1-\theta)\lambda h)^2}{(1-\theta\lambda h)^2}\right) + \frac{2h(\sigma^2+\varepsilon^2)}{(1-\theta\lambda h)^4}, \\ &< \infty.\end{aligned} \tag{2.10}$$

□

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LEMMA 2.8. *Let $\{\eta_n\}_{n\in\mathbb{N}}$ be the sequence defined in (2.9). Define the sequence of random variables $\{\zeta_n\}_{n\in\mathbb{N}}$ by*

$$\zeta_n := \frac{\eta_n}{\sqrt{\mathbb{E}\left(\eta_n^2\right)}}. \tag{2.11}$$

Then ζ_n satisfies Conditions (2.3) and (2.4) of Assumption 1, for each $n \in \mathbb{N}$.

Proof. Since the terms of the sequences $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ are independent random variables with common density function

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R},$$

we see that $x^k\varphi(x) \to 0$ as $|x| \to \infty$ for any $k \in \mathbb{N}$. In particular, take $k = 3$. Now denote

$$\alpha_h = \frac{1}{\sqrt{\mathbb{E}(\eta_n^2)(1-\theta\lambda h)^2}},$$

and note that $\alpha_h < \infty$ by equation (2.10). Then from (2.9) and (2.11) we have

$$\zeta_n = \alpha_h[\sqrt{h}\sigma^2\xi_{1,n}^2 + 2(1+(1-\theta)\lambda h)\sigma\xi_{1,n} + \sqrt{h}\varepsilon^2\xi_{2,n}^2 - 2\sqrt{h}(\sigma^2+\varepsilon^2)]. \tag{2.12}$$

Clearly, each ζ_n satisfies Condition (2.3) of Assumption 1. It remains to show that Condition (2.4) is also satisfied.

Define the independent random variables

$$A_n := \left[\xi_{1,n} + \frac{(1+(1-\theta)\lambda h)}{\sqrt{h}\sigma}\right]^2; \quad B_n := \xi_{2,n}^2. \tag{2.13}$$

Applying parts (i) and (iii) of Lemma 2.6, we see that the density function of A_n and B_n must satisfy Condition (2.4); moreover, we can rewrite equation (2.12) as a linear combination of A_n and B_n:

$$\zeta_n = \alpha_h\sqrt{h}\sigma^2 A_n + \alpha_h\sqrt{h}\varepsilon^2 B_n - \alpha_h\left[\frac{(1+(1-\theta)\lambda h)^2}{h\sigma^2} - 2\sqrt{h}(\sigma^2+\varepsilon^2)\right]. \tag{2.14}$$

A final application of parts (i) and (ii) of Lemma 2.6 gives the result. □

LEMMA 2.9. *Let $\{(X_{1,n}, X_{2,n})\}_{n\in\mathbb{N}}$ be a solution of equation (1.6), and set*

$$Z_n := X_{1,n}^2 + X_{2,n}^2, \quad n \in \mathbb{N}. \tag{2.15}$$

Then the sequence $\{Z_n\}_{n\in\mathbb{N}}$ satisfies almost surely

$$Z_{n+1} = Z_n(1 + hF_h + \sqrt{h}G_h\zeta_{n+1}), \quad n \in \mathbb{N}, \tag{2.16}$$

where each ζ_n is defined by (2.11). Moreover:

(i) *F_h can be represented in the form*

$$F_h = f(h) + O(h) := \frac{2\lambda + \sigma^2 + \varepsilon^2}{(1-\theta\lambda h)^2} + O(h); \tag{2.17}$$

(ii) *G_h can be represented in the form*

$$G_h^2 = g^2(h) + O(h) := \frac{4\sigma^2}{(1-\theta\lambda h)^2} + O(h); \tag{2.18}$$

where $O(h) \to 0$ as $h \to 0$ in each case.

Proof. Part (i).

Squaring out the individual components of equation (1.6) and summing yields equation (2.16), where

$$F_h := \frac{2\lambda}{(1-\theta\lambda h)} + \frac{\sigma^2+\varepsilon^2}{(1-\theta\lambda h)^2} + \frac{\lambda^2 h}{(1-\theta\lambda h)^2} \tag{2.19}$$

and

$$G_h := \sqrt{\mathbb{E}(\eta_n^2)}, \tag{2.20}$$

each η_n being defined by (2.9). Clearly, (2.17) follows from (2.19).

Part (ii).

Note that G_h is independent of n since terms of the sequence $\{\eta_n\}$ are mutually independent and identically distributed. It remains to show that equation (2.18) holds. Recall from the proof of Lemma 2.7 that, since $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ are mutually independent sequences of mutually independent standard normal random variables,

$$G_h^2 = \mathbb{E}(\eta_n^2) = \frac{4\sigma^2}{(1-\theta\lambda h)^2}\left(\frac{(1+(1-\theta)\lambda h)^2}{(1-\theta\lambda h)^2}\right) + \frac{2h(\sigma^2+\varepsilon^2)}{(1-\theta\lambda h)^4}.$$

The statement of the lemma now follows from the fact that

$$\frac{(1+(1-\theta)\lambda h)^2}{(1-\theta\lambda h)^2} = 1 + 2\frac{\lambda h}{(1-\theta\lambda h)} + \frac{\lambda^2 h^2}{(1-\theta\lambda h)^2}.$$

□

3. *Main results*

Our stability analysis takes place in two stages. First, Theorem 3.1 expresses the stability of the transformed system in equation (2.16) (and therefore of the test system equation (1.6)) in terms of the sign of the following quantity.

DEFINITION 1. Let $\{\zeta_n\}_{n\in\mathbb{N}}$, F_h and G_h be as defined in (2.11), (2.19) and (2.20), respectively. Then we can define

$$b := \mathbb{E}\ln(1 + hF_h + \sqrt{h}G_h\zeta_{n+1}). \tag{3.1}$$

REMARK 3. By (2.9) and (2.11), the terms of $\{\eta_n\}$, and therefore $\{\zeta_n\}$, are identically distributed, and consequently b does not depend on n for all $n\in\mathbb{N}$.

Second, and following Berkolaiko and Rodkina [**4**], we apply the discrete Itô formula in Theorem 3.2 to express this condition in terms of the test system parameters for sufficiently small values of the step size h.

THEOREM 3.1. *Let* $\{Z_n\}_{n\in\mathbb{N}}$ *be a solution of equation* (2.16). *Then:*

(1) $\lim_{n\to\infty} Z_n = 0$ *almost surely if and only if* $b<0$;

(2) $\lim_{n\to\infty} Z_n = \infty$ *almost surely if and only if* $b>0$.

Proof. By (2.15) and equation (2.16), both Z_n and $(1 + hF_h + \sqrt{h}G_h\zeta_{n+1})$ are almost surely non-negative for all $n\in\mathbb{N}$. Hence, we can write

$$Z_{n+1} = Z_0 e^{S_n}, \tag{3.2}$$

where

$$S_n := \sum_{i=0}^{n} \ln(1 + hF_h + \sqrt{h}G_h\zeta_{n+1}).$$

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By (3.1), we see that $\mathbb{E}S_n = nb$ and, applying Theorem 2.1, we have

$$\lim_{n\to\infty} \frac{S_n}{n} = b, \quad \text{almost surely.} \tag{3.3}$$

Part (i), *sufficiency.*

Let $b < 0$ and set $\varepsilon = -b/2$. By equation (3.3), we can find, for almost all $\omega \in \Omega$, $N_1 = N_1(\omega, \varepsilon)$, where $n > N_1$ gives

$$\frac{S_n(\omega)}{n} - b \leqslant -\frac{b}{2}$$

and, therefore, for $n > N_1$,

$$S_n(\omega) \leqslant \frac{b}{2} n$$

and

$$\lim_{n\to\infty} S_n = -\infty, \quad \text{almost surely.} \tag{3.4}$$

By equation (3.2), this is equivalent to $\lim_{n\to\infty} Z_n = 0$, almost surely.

Part (i), *necessity.*

We proceed by contradiction, assuming that $b \geqslant 0$. Suppose that equation (3.4) holds, and again choose $\varepsilon = -b/2$. Then, for almost all $\omega \in \Omega$, there is $N_2 = N_2(\omega, \varepsilon)$ such that

$$\frac{S_n(\omega) - nb}{n} \geqslant -\frac{b}{2}, \quad n > N_2,$$

and therefore

$$\frac{b}{2} n \leqslant S_n(\omega), \quad n > N_2.$$

So, if $b > 0$,

$$\lim_{n\to\infty} S_n = \infty, \quad \text{almost surely}$$

and, if $b = 0$,

$$\lim_{n\to\infty} S_n \geqslant 0, \quad \text{almost surely,}$$

both of which statements contradict equation (3.4). We therefore conclude that $b < 0$. Part (ii) is proved similarly. □

Finally we apply Corollary 2.4 to express the stability conditions of Theorem 3.1 in terms of the coefficients of equation (1.6), for sufficiently small step size h.

THEOREM 3.2. *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a solution of equation (2.16). Then there exists $h_0 = h_0(\lambda, \varepsilon, \theta, \sigma)$ such that for all $h \leq h_0$:*

(i) $\lim_{n\to\infty} Z_n = 0$ *almost surely if and only if*

$$2\lambda - \sigma^2 + \varepsilon^2 < 0;$$

(ii) $\lim_{n\to\infty} Z_n = \infty$ *almost surely if and only if*

$$2\lambda - \sigma^2 + \varepsilon^2 > 0.$$

Proof. Set $\phi(u) = \ln u$, for $u \in (0, \infty)$, and define $\tilde{\phi}$ as follows:

$$\tilde{\phi}(x) = \begin{cases} \ln|x|, & |x| \geqslant 1/e, \\ -\frac{1}{4}e^4x^4 + e^2x^2 - \frac{7}{4}, & |x| \leqslant 1/e. \end{cases}$$

Clearly, $\tilde{\phi}$ is continuous, bounded and has a bounded third derivative on $[0,\infty)$. Moreover, since

$$\int_0^{1/e} \ln u \ du < \infty,$$

we have

$$\int_0^{\infty} |\ln u - \tilde{\phi}(u)|\, du < \infty.$$

Since the coefficients F_h, G_h are independent of n and, by Lemma 2.8, each ζ_i satisfies the conditions of Assumption 1, all conditions of Theorem 2.3 hold. Therefore, we may apply equation (2.6) in Corollary 2.4 to obtain

$$\mathbb{E}[\ln(1 + hF_h + \sqrt{h}G_h\zeta_n)] = \frac{h}{(1-\theta\lambda h)^2}\left(2\lambda + \sigma^2 + \varepsilon^2 - \frac{1}{2}4\sigma^2 + (1-\theta\lambda h)^2 O(h)\right). \quad (3.5)$$

The right-hand side of equation (3.5) is negative (respectively positive) if and only if

$$2\lambda - \sigma^2 + \varepsilon^2 + (1-\theta\lambda h)^2 O(h) < 0 \quad (\text{respectively} > 0),$$

and the statement of the theorem follows. □

4. *Conclusions and future work*

In this paper we apply the discrete form of the Itô formula developed in [**2**] to a linear almost sure asymptotic stability analysis of the θ-Maruyama numerical method, using a particular test system of SDEs proposed in [**5**] that captures the almost sure stabilising and destabilising effects of stochastic perturbation on a diagonal drift coefficient.

The main result shows that, for sufficiently small step sizes, the almost sure asymptotic stability properties of the equilibrium of the test system are captured by the stochastic difference equation that results from the discretisation.

Future work will seek to refine our technique to relax the limitations on the step size, to investigate the threshold case where $2\lambda - \sigma^2 + \varepsilon^2 = 0$ and to allow for more complex drift structures to be treated.

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