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MATHEMATICAL NOTES

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A SIMPLER METHOD FOR GETTING SOME EXACT DISTRIBUTIONS

BY

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In [1] Anderson uses a simple random sample of p-component vectors from $N(\mu, \Sigma)$ to test the hypothesis H, $\Sigma=\sigma^2 I$, where I is the identity matrix and σ^2 is unknown. Let $W=\lambda^{2/N}$, where λ is the likelihood ratio criterion for testing H. We have

$$(1) \qquad E[W^h] = \frac{p^{ph}\Gamma\left(\frac{np}{2}\right)}{\Gamma\left(\frac{np}{2}+ph\right)} \prod_{i=1}^{p} \frac{\Gamma\left[\frac{n+1-i}{2}+h\right]}{\Gamma\left[\frac{n+1-i}{2}\right]}, \quad n = N-1$$

where N is the sample size. For $p=2$, the distribution of W is given in [1]. Consul [3] obtained the distribution of W for $p=2, 3, 4$ and 6 using the Inverse Mellin Transform and Operational Calculus. By using simple algebraic transformations, the exact distributions of W for $p=2, 3, 4, 5, 6$, and 8 are obtained in this article.

Since $0<W<1$, a moment sequence will uniquely determine the density of W. We will use this fact and apply simple algebraic methods to obtain the density of W in some particular cases.

We will illustrate the method by considering the case $p=3$. Using Gauss's Multiplication Formula

$$(\mathrm{A}) \qquad \Gamma(mZ) = (2\pi)^{(1-m)/2} m^{mZ-1/2} \prod_{k=0}^{m-1} \Gamma(Z+k/m),$$

we obtain,

$$(2) \qquad EW^h = \frac{\Gamma(n/2-1+h)\Gamma(n/2-\frac{1}{2}+h)\Gamma(n/2+\frac{1}{3})\Gamma(n/2+\frac{2}{3})}{\Gamma(n/2-\frac{1}{2})\Gamma(n/2-1)\Gamma(n/2+\frac{1}{3}+h)\Gamma(n/2+\frac{2}{3}+h)}$$

Let $W=UV$, where U and V are independent Beta variates with parameters (α_1, β_1) and (α_2, β_2) respectively. Let $f(u, v)$ denote the joint density U and V, that is

(3) $$f(u, v) = Ku^{\alpha_1-1}(1-u)^{\beta_1-1}v^{\alpha_2-1}(1-v)^{\beta_2-1}$$

where

$$K = \frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}.$$

Consider the transformation $w=uv$ and $w_2=v$. The density of W is obtained as,

(4) $$h(w) = Kw^{\alpha_1-1}\int_w^1 w_2^{-\alpha_2+\alpha_2-\beta_1}(w_2-w)^{\beta_1-1}(1-w_2)^{\beta_2-1}\,dw_2.$$

Let $w_2=(w-1)t+1$. By using Euler's formula,

(B) $$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,$$

we obtain,

(5) $$h(w) = \frac{\Gamma(\alpha_1+\beta_1)(\alpha_2+\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1+\beta_2)}w^{\alpha_1-1}(1-w)^{\beta_1+\beta_2-1}F(a, b; c; z),$$

where $a=\alpha_1-\alpha_2+\beta_1$, $b=\beta_2$, $c=\beta_1+\beta_2$, $z=1-w$ and $F(a, b; c; z)$ is Gauss's hypergeometric series. Substituting the actual values of α_1 α_2, β_1, β_2 and using the relation

(C) $$F(a, b; c; z) = (1-z)^{c-a-b}F(c-a, c-b; c; z),$$

we obtain

(6) $$h(w) = \frac{\Gamma(n/2+\frac{1}{3})\Gamma(n/2+\frac{2}{3})w^{(n-3)/2}}{\Gamma(n/2-\frac{1}{2})\Gamma(n/2-1)\Gamma(\frac{5}{2})}(1-w)^{3/2}F(\tfrac{4}{3}, \tfrac{5}{3}; \tfrac{5}{2}; 1-w),$$

which agrees with Consul's result.

The case $p=2$ is trivial. The case $p=4$ can be worked out exactly like the case $p=3$, since we find that $W^{1/2}=RT$ where R and T are independent, $R\sim B(n-1, 1)$ and $T\sim B(n-3, 7/2)$, where $B(.,.)$ means a Beta variate with parameters $(\cdot, \cdot)$.

For $p=6$, we find that $W^{1/2}=X_1X_2X_3$ where X_1, X_2, and X_3 are independently distributed as $B(n-5, \frac{17}{3})$, $B(n-3, \frac{10}{3})$, $B(n-1, 1)$ respectively. Consider the transformation $u_2=u_1x_3$ and $s=u_1$ where $u_1=x_1x_2$. The joint density of U_1 and X_3 is given as, using the case $p=3$,

(7) $$g(u_1, x_3) = k'u_1^{\alpha_1-1}(1-u_1)^{\beta_1+\beta_2-1}x_3^{\alpha_3-1}(1-x_3)^{\beta_3-1} \times F(\alpha_1-\alpha_2+\beta_1, \beta_2; \beta_1+\beta_2; 1-u_1)$$

where

$$k' = \frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)\Gamma(\alpha_3+\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_3)\Gamma(\beta_1+\beta_2)}.$$

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Hence the density is obtained as:

(8)
$$f(u_2) = k'u_2^{\alpha_3-1}\int_{u_2}^{1} s^{\alpha_1-\alpha_3-\beta_3}(1-s)^{\beta_1+\beta_2-1}(s-u_2)^{\beta_3-1} \times F(\alpha_1-\alpha_2+\beta_1, \beta_2; \beta_1+\beta_2; 1-s)\, ds.$$

Let $s=(u_2-1)t+1$, then by applying the Monotone convergence theorem we obtain,

(9)
$$f(u_2) = k'u_2^{\alpha_3-1}(1-u_2)^{\beta_1+\beta_2+\beta_3-1}\sum_{k=0}^{\infty}(\gamma)_k(1-u_2)^k \times \int_0^1 t^{\beta_1+\beta_2+k-1}(1-t)^{\beta_3-1}[1-(1-u_2)t]^{\alpha_1-\alpha_3-\beta_3}\, dt,$$

where

$$(\gamma)_k = \frac{(\alpha_1-\alpha_2+\beta_1)_k(\beta_2)_k}{(\beta_1+\beta_2)_k k!}$$

and

$$(a)_k = (a)(a+1)\cdots(a+k-1).$$

Now, apply B and we get:

(10)
$$f(u_2) = Cu_2^{\alpha_3-1}(1-u_2)^{\beta_1+\beta_2+\beta_3-1}\sum_{k=0}^{\infty}(\alpha)_k(1-u_2)^k \times F(-\alpha_1+\alpha_3+\beta_3, \beta_1+\beta_2+k; \beta_1+\beta_2+\beta_3+k; 1-u_2)$$

where

$$(\alpha)_k = \frac{(\alpha_1-\alpha_2+\beta_1)_k(\beta_2)_k}{(\beta_1+\beta_2+\beta_3)_k k!}$$

and

$$C = \frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)\Gamma(\alpha_3+\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1+\beta_2+\beta_3)}.$$

Now substitute the values for the parameters and put $W^{1/2}=U_2$.

For $p=5$, W is distributed as a product of four Beta variates, X_1, X_2, X_3, X_4, with parameters $(n/2-\frac{1}{2}, \frac{7}{10})$, $(n/2-1, \frac{7}{5})$, $(n/2-\frac{3}{2}, \frac{21}{10})$, $(n/2-2, \frac{14}{5})$ respectively. To find the density of $W=X_1X_2X_3X_4$ consider the transformation $w=u_2x_4$ and $r=u_2$ where $u_2=x_1x_2x_3$. Apply the technique of case $p=6$ and we obtain,

(11)
$$h(w) = C'w^{n/2-3}(1-w)^{33/2}\sum_{k=0}^{\infty}(\alpha)'_k(1-w)^k \times \sum_{j=0}^{\infty}\frac{\gamma'(k,j)\Gamma(\frac{42}{10}+k+j)}{\Gamma(\frac{35}{2}+k+j)}(1-w)^j F(\tfrac{23}{10}, \tfrac{42}{10}+k+j; \tfrac{35}{2}+k+j; 1-w)$$

where

$$C' = \frac{\Gamma(n/2+\frac{1}{5})\Gamma(n/2+\frac{2}{5})\Gamma(n/2+\frac{3}{5})\Gamma(n/2+\frac{4}{5})}{\Gamma(n/2-\frac{1}{2})\Gamma(n/2-1)\Gamma(n/2-\frac{3}{2})\Gamma(n/2-2)\Gamma(\frac{42}{10})},$$

$$\gamma'(k,j) = \frac{(\frac{8}{5})_j(\frac{21}{10}+k)_j}{j!(\frac{42}{10}+k)_j} \quad \text{and} \quad (\alpha)'_k = \frac{(\frac{6}{5})_k(\frac{7}{5})_k}{k!\,(\frac{42}{10})_k}.$$

For $p=8$, $W^{1/2}=R$ is distributed as a product of four independent Beta variates: $X_1 \sim B(n-1, 1)$, $X_2 \sim B(n-3, \frac{13}{4})$, $X_3 \sim B(n-5, \frac{11}{2})$, $X_4 \sim B(n-7, \frac{31}{4})$. Using the above technique, we find

$$(12) \quad h(r) = C'' r^{n-8}(1-r)^{33/2} \sum_{k=0}^{\infty} (\alpha)''_k (1-r)^k \times \sum_{j=0}^{\infty} \frac{\gamma''(k,j)\Gamma(\frac{39}{4}+k+j)}{\Gamma(\frac{35}{2}+k+j)} (1-r)^j F(\tfrac{23}{4}, \tfrac{39}{4}+k+j; \tfrac{35}{2}+k+j; 1-r)$$

where

$$C'' = \frac{\Gamma(n)\Gamma(n+\frac{1}{4})\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{4})}{\Gamma(n-1)\Gamma(n-3)\Gamma(n-5)\Gamma(n-7)\Gamma(\frac{39}{4})},$$

$$\gamma''(k,j) = \frac{(\frac{7}{2})_j(\frac{17}{4}+k)_j}{j!(\frac{39}{4}+k)_j}$$

and

$$(\alpha)''_k = \frac{(3)_k(\frac{13}{4})_k}{k!(\frac{39}{4})_k}.$$

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