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HOMOMORPHISMS OF ℓ^1-MUNN ALGEBRAS AND APPLICATIONS TO SEMIGROUP ALGEBRAS

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Abstract

In this paper, for an arbitrary ℓ^1-Munn algebra $\mathfrak{A}$ over a Banach algebra A with a sandwich matrix P, we characterise all homomorphisms from $\mathfrak{A}$ to a commutative Banach algebra B. Especially, we study the character space of this algebra. Then, as an application, its character amenability is investigated. Finally, we apply these results to certain semigroups, which are called Rees matrix semigroups.

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1. Introduction

Munn algebras were first introduced by Munn in [9] and the algebraic properties of these algebras were studied there. This concept was generalised in [5] to characterise amenability of semigroup algebras. Then some aspects of these algebras were investigated in [4]. Munn algebras present a new category of Banach algebras which can be used in the study of semigroup algebras of completely o-simple semigroups with finitely many idempotents. These applications give a strong motivation to study these algebras as abstract objects (see for example [4, 5]). In [11], the author studied the ultrapowers of these algebras. Also, in [12], by using these algebras, weighted Rees matrix semigroup algebras were investigated.

The first goal of this paper is to study all homomorphisms from a Munn algebra $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ to a commutative Banach algebra B in terms of homomorphisms from A to B. Especially, the character space of Munn algebras is investigated. Then we show that character amenability of $\mathfrak{A}$ implies character amenability of A. We also give some examples to show that the converse of this result is not true in general. We recall that for a Banach algebra A and a character ϕ on A, A is called right ϕ-amenable if there exists a right ϕ-mean on A^*; that is, a bounded linear functional m on A^* satisfying $a \cdot m = \phi(a)m$ and $m(\phi) = 1$. The 'left' version is defined in a similar way (see [7, 8]). In harmonic analysis, the interest in character amenability arises from the fact that left character amenability of both the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ are characterised by amenability of G (see [10]). Here we characterise the character space of Rees matrix semigroup algebras and present a relation between character amenability of Rees semigroup algebras and amenability of their maximal subgroups.

2. Main results

Let A be a unital Banach algebra, let I and J be nonempty sets and let $P = (p_{ji}) \in M_{J\times I}(A)$ be such that $\sup\{\|p_{ji}\| : i \in I, j \in J\} \leq 1$. Then the set $\mathfrak{A} = M_{I\times J}(A)$ of all $I \times J$ matrices $(a_{ij})_{ij}$ over A with ℓ^1-norm $\|(a_{ij})\| = \sum_{i\in I, j\in J} \|a_{ij}\| < \infty$ and product $a \circ b = aPb$, for all $a, b \in \mathfrak{A}$, is a Banach algebra, which is called the ℓ^1-*Munn algebra over* A *with sandwich matrix* P and it will be denoted by $\mathcal{M}(A, P, I, J)$. If the index sets I and J are finite with $|I| = m$ and $|J| = n$, then we use the notation $\mathcal{M}(A, P, m, n)$ rather than $\mathcal{M}(A, P, I, J)$.

Throughout, we adopt the notation as above. Also, the identity of A is denoted by e and, for $i \in I$ and $j \in J$, we follow the terminology of [4] and denote by $\mathcal{E}_{ij}$ the element of $\mathfrak{A}$ with e in the (i, j)th place and 0 elsewhere. Thus, $\|\mathcal{E}_{ij}\| = \|e\| = 1$. This notation enables us to represent an element $N = (N_{ij})_{ij} \in \mathfrak{A}$ by $N = \sum_{i\in I, j\in J} N_{ij}\mathcal{E}_{ij}$.

Let A and B be two Banach algebras. The space of all linear maps $T : A \to B$ is denoted by $\mathfrak{L}(A, B)$. If for an element $T \in \mathfrak{L}(A, B)$ and for all $a, b \in A$ we have $T(ab) = T(a)T(b)$, then T is called a homomorphism. We firstly consider the existence of a homomorphism in $\mathfrak{L}(\mathcal{M}(A, P, I, J), B)$, where B is a commutative Banach algebra.

Theorem 2.1. *Let* $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ *be the* ℓ^1*-Munn algebra over* A *with sandwich matrix* $P = (p_{ji})$ *such that* $\{p_{ji} : i \in I, j \in J\} \cap \mathrm{Inv}(A) \neq \emptyset$. *Let* B *be a commutative Banach algebra and let* $T \in \mathfrak{L}(\mathfrak{A}, B)$. *Then* T *is a nonzero homomorphism if and only if there exists a unique nonzero homomorphism* L *in* $\mathfrak{L}(A, B)$ *such that*

$$L(p_{ji}p_{lk}) = L(p_{jk}p_{li}) \quad (j, l \in J, i, k \in I) \tag{2.1}$$

and

$$T(N) = \sum_{i\in I, j\in J} L(n_{ij})L(p_{ji}) \quad (N = (n_{ij}) \in \mathfrak{A}). \tag{2.2}$$

Proof. Suppose that $T \in \mathfrak{L}(\mathfrak{A}, B)$ is a nonzero homomorphism. By our hypothesis, there exist $i_0 \in I$ and $j_0 \in J$ with $p_{j_0 i_0} \in \mathrm{Inv}(A)$. We define $L \in \mathfrak{L}(A, B)$ by

$$L(a) = T(a p_{j_0 i_0}^{-1} \mathcal{E}_{i_0 j_0}) \tag{2.3}$$

for all $a \in A$. Since T is nonzero, there exists $N = (N_{ij})_{ij}$ such that $T(N) \neq 0$. Now we may use the representation $N = \sum_{i\in I, j\in J} N_{ij}\mathcal{E}_{ij}$ to obtain $\imath \in I$, $\jmath \in J$ such that $T(N_{\imath\jmath}\mathcal{E}_{\imath\jmath}) \neq 0$. So,

$$T(N_{\imath\jmath}\mathcal{E}_{\imath\jmath}) = T(N_{\imath\jmath}\mathcal{E}_{\imath j_0} \circ p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0} \circ p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 \jmath}) = T(N_{\imath\jmath}\mathcal{E}_{\imath j_0})T(p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0})T(p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 \jmath}).$$

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This shows that $L(e) = T(p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) \neq 0$ and hence L is nonzero. Furthermore, for all $a, b \in A$,

$$\begin{aligned} L(ab) = T(abp_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) &= T(ap_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0} \circ bp_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) \\ &= T(ap_{j_0 i_0}^{-1}\mathcal{E}_{j_0 i_0})T(bp_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) \\ &= L(a)L(b). \end{aligned}$$

Therefore, L is a homomorphism.

Now we show that L satisfies the equality (2.1). For $j, l \in J$ and $i, k \in I$,

$$\begin{aligned} L(p_{ji}p_{lk}) = L(p_{ji})L(p_{lk}) = L(p_{lk}p_{ji}) &= T(p_{lk}p_{ji}p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) \\ &= T(\mathcal{E}_{i_0 l} \circ \mathcal{E}_{kj} \circ p_{j_0 i_0}^{-1}\mathcal{E}_{i j_0}) \\ &= T(\mathcal{E}_{i_0 l} \circ p_{j_0 i_0}^{-1}\mathcal{E}_{i j_0} \circ \mathcal{E}_{kj}) \quad \text{(since } B \text{ is commutative)} \\ &= T(\mathcal{E}_{i_0 l} \circ p_{j_0 i_0}^{-1}\mathcal{E}_{i j_0})T(\mathcal{E}_{kj}) \\ &= T(p_{li}p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0})T(\mathcal{E}_{kj}) \\ &= L(p_{li})T(p_{j_0 i_0}^{-1}\mathcal{E}_{k j_0} \circ \mathcal{E}_{i_0 j}) \\ &= L(p_{li})T(\mathcal{E}_{i_0 j} \circ p_{j_0 i_0}^{-1}\mathcal{E}_{k j_0}) \\ &= L(p_{li})T(p_{jk}p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) = L(p_{li}p_{jk}). \end{aligned}$$

It remains to show that L satisfies the equality (2.2). Suppose that $N = (n_{ij}) \in \mathfrak{A}$. By using the representation $N = \sum_{i \in I, j \in J} n_{ij}\mathcal{E}_{ij}$,

$$\begin{aligned} T(N) = \sum_{i \in I, j \in J} T(n_{ij}\mathcal{E}_{ij}) &= \sum_{i \in I, j \in J} T(n_{ij}p_{j_0 i_0}^{-1}\mathcal{E}_{i j_0} \circ \mathcal{E}_{i_0 j}) \\ &= \sum_{i \in I, j \in J} T(\mathcal{E}_{i_0 j} \circ n_{ij}p_{j_0 i_0}^{-1}\mathcal{E}_{i j_0}) \\ &= \sum_{i \in I, j \in J} T(p_{ji}n_{ij}p_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0}) \\ &= \sum_{i \in I, j \in J} L(p_{ji}n_{ij}) \\ &= \sum_{i \in I, j \in J} L(p_{ji})L(n_{ij}). \end{aligned}$$

The uniqueness of L is easily verified by (2.3). Indeed, if there exists a homomorphism L' in $\mathfrak{L}(A, B)$ which satisfies (2.1) and (2.2), we can choose $i_0 \in I$ and $j_0 \in J$ such that $p_{j_0 i_0} \in \mathrm{Inv}(A)$. For a given $a \in A$, by (2.2), we obtain $L'(a) = T(ap_{j_0 i_0}^{-1}\mathcal{E}_{i_0 j_0})$, which is equal to $L(a)$ by (2.3). This implies that $L = L'$, as required.

Conversely, suppose that $T \in \mathfrak{L}(\mathfrak{A}, B)$ and there exists a nonzero homomorphism L in $\mathfrak{L}(A, B)$ such that (2.1) and (2.2) hold. Again using the fact that $\{P_{ji} : i \in I, j \in J\} \cap \mathrm{Inv}(A) \neq \emptyset$, we may obtain $i \in I$ and $j \in J$ with $L(p_{ji}) \neq 0$. Hence, $T(\mathcal{E}_{ij}) = L(p_{ji}) \neq 0$ and therefore T is nonzero.

Now we claim that T is a homomorphism. Indeed, for $a = (a_{ij})$ and $b = (b_{ij})$ in $\mathfrak{A}$,

$$\begin{aligned}
T(a \circ b) &= T\Big(\sum_{k\in J, l\in I} a_{ik} p_{kl} b_{lj} \mathcal{E}_{ij}\Big)\\
&= \sum_{j,k\in J, i,l\in I} L(a_{ik} p_{kl} b_{lj} p_{ji})\\
&= \sum_{j,k\in J, i,l\in I} L(a_{ik} b_{lj} p_{kl} p_{ji}) \quad \text{(since } B \text{ is commutative)}\\
&= \sum_{j,k\in J, i,l\in I} L(a_{ik} b_{lj} p_{ki} p_{jl}) \quad \text{(since (2.1) holds)}\\
&= \sum_{k\in J, i\in I} L(a_{ik} p_{ki}) \sum_{l\in J, j\in I} L(b_{lj} p_{jl}) \quad \text{(again, since } B \text{ is commutative)}\\
&= T(a)T(b),
\end{aligned}$$

which completes the proof. □

Corollary 2.2. *Let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix $P = (p_{ji})$ such that $\{p_{ji} : i \in I, j \in J\} \cap \mathrm{Inv}(A) \neq \emptyset$ and let $\Phi \in \mathfrak{A}^*$. Then Φ is a character on $\mathfrak{A}$ if and only if there exists a unique character ϕ on A such that*

$$\phi(p_{ji} p_{lk}) = \phi(p_{jk} p_{li}) \quad (j, l \in J, i, k \in I) \tag{2.4}$$

and

$$\Phi(N) = \sum_{i\in I, j\in J} \phi(n_{ij})\phi(p_{ji}) \quad (N = (n_{ij}) \in \mathfrak{A}). \tag{2.5}$$

Proof. Since a character on a Banach algebra A is a homomorphism from A onto the commutative complex field $\mathbb{C}$, the result follows from Theorem 2.1. □

Remark 2.3. For a Banach algebra A, the character space of A is denoted by $\Delta(A)$. By Corollary 2.2, we obtain a bijection between a certain subset of $\Delta(A)$ and the character space of $\mathfrak{A}$. Indeed, we may write $\Delta(\mathfrak{A}) \subseteq \Delta(A)$. By choosing an appropriate sandwich matrix P, one may obtain a Munn algebra $\mathfrak{A}$ for which, up to a bijection, we have $\Delta(\mathfrak{A}) = \Delta(A)$.

Example 2.4. Let A be a Banach algebra, I and J be nonempty sets and $\{a_j : j \in J\}$ be a set of nonzero elements of the unit ball of A. For every $i \in I$, $j \in J$, define the $J \times I$ matrix P by $p_{ji} = a_j$. If $\mathfrak{A} = \mathcal{M}(A, P, I, J)$, then, for each character $\phi \in A$, (2.4) holds. Therefore, up to a bijection, we obtain $\Delta(\mathfrak{A}) = \Delta(A)$.

Now we are ready to present an application of our result.

Theorem 2.5. *Let $\mathfrak{A} = \mathcal{M}(A, P, m, n)$ be the ℓ^1-Munn algebra over A with sandwich matrix $P = (p_{ji})$ such that $\{p_{ji} : 1 \le i \le m, 1 \le j \le n\} \cap \mathrm{Inv}(A) \neq \emptyset$. Let Φ be a character on $\mathfrak{A}$ with $\phi(p_{ji} p_{lk}) = \phi(p_{jk} p_{li})$ for all $1 \le j, l \le n$, $1 \le i, k \le m$ and*

$$\Phi(N) = \sum_{i=1}^{m} \sum_{j=1}^{n} \phi(n_{ij})\phi(p_{ji})$$

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for all $N = (n_{ij}) \in \mathfrak{A}$. *If* $\mathfrak{A}$ *is right (respectively left)* Φ*-amenable, then* A *is right (respectively left)* ϕ*-amenable.*

Proof. Suppose that $\mathfrak{A}$ is right Φ-amenable. Then $\mathfrak{A}$ has a right Φ-mean in $\mathfrak{A}^{**}$, say M, such that $M(\Phi) = 1$ and $N \cdot M = \Phi(N)M$ for all $N \in \mathfrak{A}$. By [6, Lemma 3.2], we have $\mathfrak{A}^{**} \simeq \mathcal{M}(A^{**}, P, m, n)$. (Note that both A^{**} and $\mathfrak{A}^{**}$ are equipped with the first or the second Arens product; see [1, 2] for more details.) Hence, we may suppose that $M = (m_{ij})_{ij}$ with $m_{ij} \in A^{**}$ for all i, j. Also, we may extend the equality (2.5) to $\mathfrak{A}^{**}$ by Goldstine's theorem to obtain $M(\Phi) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}(\phi)\phi(p_{ji})$. Since $M(\Phi) = 1$, there exist $1 \le i_0 \le m$ and $1 \le j_0 \le n$ such that $\phi(p_{j_0 i_0}) \neq 0$. For $N = \mathcal{E}_{i_0 j_0}$, by applying formula (1) of [6],

$$0 = (\mathcal{E}_{i_0 j_0} \cdot M - \Phi(\mathcal{E}_{i_0 j_0})M)_{i_0 j_0} = \sum_{k=1}^{m} p_{j_0 k} \cdot m_{k j_0} - \phi(p_{j_0 i_0}) m_{i_0 j_0}. \tag{2.6}$$

On the other hand, for each $a \in A$,

$$0 = (a\mathcal{E}_{i_0 j_0} \cdot M - \Phi(a\mathcal{E}_{i_0 j_0})M)_{i_0 j_0} = \sum_{k=1}^{m} a p_{j_0 k} \cdot m_{k j_0} - \phi(p_{j_0 i_0})\phi(a) m_{i_0 j_0}. \tag{2.7}$$

By (2.6) and (2.7),

$$\phi(p_{j_0 i_0})(a \cdot m_{i_0 j_0} - \phi(a) m_{i_0 j_0}) = \Big(\phi(p_{j_0 i_0}) a \cdot m_{i_0 j_0} - \sum_{k=1}^{m} a p_{j_0 k} \cdot m_{k j_0}\Big) + \Big(\sum_{k=1}^{m} a p_{j_0 k} \cdot m_{k j_0} - \phi(a)\phi(p_{j_0 i_0}) m_{i_0 j_0}\Big) = 0.$$

Set $m = m_{i_0 j_0}/m_{i_0 j_0}(\phi)$. Then $m(\phi) = 1$ and $a \cdot m - \phi(a)m = 0$ for all $a \in A$. This means that A is right ϕ-amenable. The 'left' part of the result holds analogously. □

By [5], $\mathfrak{A} = \mathcal{M}(A, P, m, n)$ is amenable if and only if A is amenable, $m = n$ and P is invertible in $M_n(A)$. Unexpectedly, this result does not hold in terms of character amenability. Indeed, if P is invertible in $M_n(A)$, then the Munn algebra is isomorphic to the matrix algebra $M_n(A)$ via $a \mapsto P^{-1}a$ and $M_n(A)$ does not have any characters when $n > 1$. Also, there are some ℓ^1-Munn algebras $\mathfrak{A} = \mathcal{M}(A, P, m, n)$ such that the character amenability of A implies the character amenability of $\mathfrak{A}$, but $m \neq n$, as the following example shows.

Example 2.6. Let $n \in \mathbb{N}$ and choose a right (respectively left) ϕ-amenable Banach algebra A with $\phi \in \Delta(A)$. Then $\mathfrak{A} = \mathcal{M}(A, P, 1, n)$ is right (respectively left) Φ-amenable, even if $n \neq 1$. Indeed, if $m \in A^{**}$ is a right (respectively left) ϕ-mean for A, then, by [6, Lemma 3.2], we may consider $M = (m_{1j})$ with $m_{1j} = m$ $(1 \le j \le n)$ as an element of $\mathfrak{A}^{**}$. It is easily seen that M is a right (respectively left) Φ-mean for $\mathfrak{A}$.

Next we give an example which shows that the converse of Theorem 2.5 is not necessarily true.

Example 2.7. Let $A = \mathbb{C}$, $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathfrak{A} = \mathcal{M}(A, P, 2, 2)$. It is easy to see that $\phi = id_{\mathbb{C}}$ is the only character on A and A is right ϕ-amenable, but $\mathfrak{A}$ is not right Φ-amenable. Indeed, if it was, then there would exist $m = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in \mathfrak{A}^{**}$ such that $\Lambda \cdot m = \Phi(\Lambda)m$ and $m(\Phi) = 1$ for all $\Lambda \in \mathfrak{A}$ and we may suppose that for each $i \in \{1, 2, 3, 4\}$, $m_i \in A^{**}$. For each $\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \mathfrak{A}$,

$$\Lambda \cdot m = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} \lambda_2 m_3 & \lambda_2 m_4 \\ \lambda_4 m_3 & \lambda_4 m_4 \end{pmatrix}.$$

On the other hand, by Corollary 2.2, $\Phi(\Lambda) = \lambda_4$ and so we obtain

$$\Phi(\Lambda)m = \begin{pmatrix} \lambda_4 m_1 & \lambda_4 m_2 \\ \lambda_4 m_3 & \lambda_4 m_4 \end{pmatrix}.$$

Since $\Phi(\Lambda)m = \Lambda \cdot m$,

$$\begin{pmatrix} \lambda_4 m_1 & \lambda_4 m_2 \\ \lambda_4 m_3 & \lambda_4 m_4 \end{pmatrix} = \begin{pmatrix} \lambda_2 m_3 & \lambda_2 m_4 \\ \lambda_4 m_3 & \lambda_4 m_4 \end{pmatrix}.$$

But Λ is arbitrary, so we see that $m_1 = m_2 = m_3 = m_4 = 0$, in contradiction to $m(\Phi) = m_4 = 1$.

3. Applications to semigroup algebras

In this section, we apply our results to some special semigroups, which are called Rees matrix semigroups, mainly to characterise their character space.

For a group G and $m, n \in \mathbb{N}$, we consider the set

$$S = \{(g)_{ij} : g \in G, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{o\},$$

where $(g)_{ij}$ denotes the element of $M_{m\times n}(G^o)$ with g in the (i, j)th place and o elsewhere, o is the zero matrix and $G^o = G \cup \{o\}$. Let $P = (p_{ji})$ be an $n \times m$ matrix over G^o. Then the set S with the composition

$$(a)_{ij} \circ o = o \circ (a)_{ij} = o \quad \text{and} \quad (a)_{ij} \circ (b)_{lk} = (ap_{jl}b)_{ik} \quad ((a)_{ij}, (b)_{lk} \in S)$$

is a semigroup, which is called a *Rees matrix semigroup with a zero over G*. We denote it by $S = \mathcal{M}^o(G, P, m, n)$. By [3, Lemma 2.46 and Theorem 3.5], G is a maximal subgroup of G. Further, by [5, Proposition 5.6],

$$\ell^1(S)/\mathbb{C}\delta_o = \mathcal{M}(\ell^1(G), P, m, n),$$

where the zero of G^o is identified with the zero of the ℓ^1-Munn algebra $\mathcal{M}(\ell^1(G), P, m, n)$ and P is considered as a matrix over $\ell^1(G)$ (for more details, see [5]).

Theorem 3.1. *Let G be a group and $S = \mathcal{M}^o(G, P, m, n)$ be a Rees matrix semigroup with a zero over G. Then $\Phi \in \ell^1(S)^*$ is a character on $\ell^1(S)$ if and only if there is a character ϕ on G^o such that, for each $f \in \ell^1(S)$,*

$$\Phi(f) = \sum_{g\in G, 1\leq i\leq m, 1\leq j\leq n} f((g)_{ij})\phi(g)\phi(p_{ji}) \tag{3.1}$$

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and

$$\phi(p_{ji}p_{lk}) = \phi(p_{li}p_{jk}) \quad (1 \le j, l \le n, 1 \le i, k \le m). \tag{3.2}$$

Proof. Suppose that Φ is a character on $\ell^1(S)$. Since δ_o is an idempotent of $\ell^1(S)$, two cases may happen.

Case 1. Suppose that $\Phi(\delta_o) = 1$. Then Φ is the augmentation character, that is, $\Phi(f) = \sum_{s\in S} f(s)$ for all $f \in \ell^1(S)$. Indeed, we have $\Phi(\delta_s)\Phi(\delta_o) = \Phi(\delta_o)$ and so $\Phi(\delta_s) = 1$ for all $s \in S$, as required. Now it is sufficient to define a character ϕ on G^o by $\phi(s) = 1$ for all $s \in G^o$. Then (3.1) and (3.2) hold.

Case 2. Suppose that $\Phi(\delta_o) = 0$. The equality

$$\ell^1(S)/\mathbb{C}\delta_o = \mathcal{M}(\ell^1(G), P, m, n)$$

shows that Φ induces a character of the ℓ^1-Munn algebra $\mathcal{M}(\ell^1(G), P, m, n)$, which we denote by $\widetilde{\Phi}$. Since, for at least one (i, j) $(1 \le i \le m, 1 \le j \le n)$, p_{ji} is a point mass (that is, δ_g for some $g \in G$), we have $\{p_{ji} : 1 \le i \le m, 1 \le j \le n\} \cap \mathrm{Inv}(A) \ne \emptyset$. By Corollary 2.2, there is a character ϕ on $\ell^1(G)$ such that (2.4) and (2.5) hold. Further, there exists a character ϕ_G on G such that $\phi(\sum \alpha_g\delta_g) = \sum \alpha_g\phi_G(g)$ (see [4, Chs. 3 and 4]). So, ϕ_G can be extended to a character on G^o if we define $\phi_G(o) = 0$ and this extension of ϕ_G satisfies (3.1) and (3.2) when Φ is replaced by $\widetilde{\Phi}$. The identity $\Phi(\delta_0) = 0$ ensures that Φ satisfies (3.1).

The 'only if' part holds immediately. □

Corollary 3.2. *Let G be a group and $S = \mathcal{M}^o(G, P, m, n)$ be a Rees matrix semigroup with a zero over G. If $\Phi \in \Delta(\ell^1(S))$ is such that $\Phi(\delta_o) = 0$ and $\ell^1(S)$ is right (respectively left) Φ-amenable, then there exists a character $\phi \in \Delta(\ell^1(G))$ such that G is right (respectively left) ϕ-amenable.*

Proof. Suppose that $\Phi \in \Delta(\ell^1(S))$. Define $\widetilde{\Phi} : \ell^1(S)/\mathbb{C}\delta_o \to \mathbb{C}$ with $\widetilde{\Phi}(f + \delta_o) = \Phi(f)$. Since $\Phi(\delta_o) = 0$, $\widetilde{\Phi}$ is well defined. Further, $\widetilde{\Phi} \in \Delta(\ell^1(S)/\mathbb{C}\delta_o)$ and $\widetilde{\Phi} \circ \pi = \Phi$, where $\pi : \ell^1(S) \to \ell^1(S)/\mathbb{C}\delta_o$ is a natural embedding. By [8, Proposition 3.5], $\ell^1(S)/\mathbb{C}\delta_o$ is right (respectively left) $\widetilde{\Phi}$-amenable. On the other hand, $\ell^1(S)/\mathbb{C}\delta_o = \mathcal{M}(\ell^1(G), P, m, n)$. Now Theorem 2.5 implies that $\ell^1(G)$ is right (respectively left) ϕ-amenable and so is G. □

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