ON RING PROPERTIES OF INJECTIVE HULLS

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1. Introduction. Several authors have investigated "rings of quotients" of a given ring $R$. Johnson showed that if $R$ has zero right singular ideal, then the injective hull of $R_R$ may be made into a right self injective, regular (in the sense of von Neumann) ring (see [7] and [12]). In articles by Utumi [10], Findlay and Lambek [6], and Bourbaki [2], various structures which correspond to sub-modules of the injective hull of $R$ are made into rings in a natural manner. In [8], Lambek points out that in each of these cases the rings constructed are subrings of Utumi's maximal ring of right quotients, which is the maximal rational extension of $R$ in its injective hull. Lambek also shows that Utumi's ring is canonically isomorphic to the bicommutator of the injective hull of $R_R$ if $R$ has 1. It thus appears that a "natural" definition of the injective hull of $R_R$ as a ring extending module multiplication by $R$ has been carried out only in the case that the injective hull is a rational extension of $R$. (See [12], [10], or [6] for various definitions of this concept.)

The purpose of this note is to study what may happen if one tries to make the entire injective hull of a ring $R$ into a ring extending module multiplication, rather than stopping at

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Utumi's ring of quotients. The author first exhibits an example which shows that it may be impossible to do so. Then a ring is constructed whose injective hull may be made into a ring, although this ring properly contains Utumi's ring of quotients. Finally some information about such a ring extension is derived.

In what follows, \( R \) will denote an associative ring with identity. \( M_R \) will signify that \( M \) is a unital right \( R \) module, and \( \hat{M} \) will denote its injective hull. \( \hat{M} \) is a maximal essential extension and minimal injective extension of \( M \) (see [5]). Much use will be made of the fact that \( M_R \) is injective if and only if for every right ideal \( I \) of \( R \) and every \( f \in \text{Hom}_R(I, M) \), there is an \( m \in M \) such that \( f(x) = mx \) for all \( x \in I \) (see [1], or [3] p. 8). Such an element \( m \) will be said to induce \( f \).

If \( \{a, b, \ldots\} \subseteq M_R \), \( \langle a, b, \ldots \rangle \) will denote the submodule of \( M_R \), and \( \langle a, b, \ldots \rangle \) the subgroup of \( (M, +) \) generated by \( \{a, b, \ldots\} \). \( \mathbb{Z} \) will denote the ring of rational integers, and \( \mathbb{Z}_n \) will denote \( \mathbb{Z}/n\mathbb{Z} \) for \( n \in \mathbb{Z} \).

2. An example where \( \hat{R} \) is not a ring. \(^2\) Let \( R \) be the ring

\[
\begin{bmatrix}
\mathbb{Z}_4 & \mathbb{Z}_4 \\
0 & \mathbb{Z}_4
\end{bmatrix}
\]

under usual matrix addition and multiplication.

Let

\[
I = \begin{bmatrix}
0 & 0 \\
0 & 2
\end{bmatrix}, \quad J = \begin{bmatrix}
0 & 2 \\
0 & 0
\end{bmatrix}.
\]

\(^2\) For further examples where \( \hat{R} \) may fail to be a ring, see the author's dissertation. In these other examples, the associative law rather than the distributive law fails.
One readily verifies that the map
\[ f \left( \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \]
gives an \( R \) isomorphism between \( I \) and \( J \). Then \( f \) extends to an isomorphism \( \hat{\phi}_I : \hat{I} \rightarrow \hat{J} \). One also readily verifies that \( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \) is an essential extension of \( I \), so it is contained in some injective hull of \( I \), say \( \hat{I} \).

Since \( \hat{I}_R \) is injective, the map \( f^{-1} \in \text{Hom}_R(J, \hat{I}) \) is induced by an element \( m \in \hat{I} \). Let \( m' = m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Then
\[ m' \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = m', \quad m' \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad m' \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0. \]

Assume \( 2m' \neq 0 \). Since \( \hat{I} \) is an essential extension of \( I \), \( (2m')_R \cap I \neq 0 \); but
\[ (2m')_R = <m'(2R)> = <m' \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}> = <2m'>, \]
so that \( 2m' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \), and
\[ 0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 2m' \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 2m'. \]
This contradicts our assumption that \( 2m' \neq 0 \).

Now assume \( \hat{R} \) is a ring. Then, from the above,
\[ 0 = (2m') \hat{\phi} \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = (m') \hat{\phi} \left( \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) = m' \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \]
a contradiction.
3. \( \hat{R} \) a ring properly containing Utumi's ring of quotients.

Let \( R \) be an algebra over \( \mathbb{Z}_2 \) with basis \( \{1, x, y, xy\} \) and multiplication defined by:

1. is a two sided identity,

\[
0 = x^2 = y^2 = (xy)^2 = yx = x(xy) = y(xy) = (xy)x = (xy)y .
\]

\( R \) is associative, since any triple product not involving \( 1 \) is 0.

We observe that the socle of \( R_R = \langle y \rangle \oplus \langle xy \rangle \). Hence

\[
\hat{R} = \langle \hat{y} \rangle \oplus \langle \hat{xy} \rangle \quad (\text{see [9]}).
\]

Moreover, since \( \hat{R} \) is unital, \( 2\hat{R} = 0 \).

By direct computation we obtain \( \hat{y} = \langle y, m, n, u \rangle \) where

\[
mx = y, \quad my = 0, \\
mx = y, \quad my = 0, \\
ux = n, \quad uy = 0.
\]

This may be easily verified by showing that every map from a right ideal of \( R \) into \( \langle y, m, n, u \rangle \) is induced by some element thereof, and that we indeed have an essential extension of \( \langle y \rangle \).

Since \( \langle xy \rangle \) is isomorphic to \( \langle y \rangle \), \( \langle \hat{xy} \rangle \) is isomorphic to \( \langle \hat{y} \rangle \). We then get an injective hull of \( \langle \hat{xy} \rangle \) by taking \( \langle \hat{xy}, \hat{m}, x, 1-n \rangle \) where

\[
\hat{mx} = xy, \quad \hat{my} = 0.
\]

Then a basis for \( \hat{R} \) is \( \{1, x, y, xy, m, n, u, \hat{m}\} \). We construct the following multiplication table for \( \hat{R} \) as an algebra over \( \mathbb{Z}_2 \).
That this multiplication is associative may be verified by actually computing triple products. The author was unable to find a non-computational method for proving that \( \hat{R} \) is a ring.

To prove that \( \hat{R} \) is not Utumi’s ring of quotients, we use the fact that Utumi’s ring consists precisely of those elements of \( \hat{R} \) which are annihilated by all \( \lambda \in \text{Hom}_R(\hat{R}, \hat{R}) \) such that \( \lambda(1) = 0 \) (see Lambek [8]). It is easily verified that the following induce \( R \) homomorphisms of \( \hat{R} \):

\[
\begin{align*}
  f(m) &= y; \quad f(1) = f(\hat{m}) = f(u) = 0; \\
  g(\hat{m}) &= y; \quad g(1) = g(m) = g(u) = 0; \\
  h(m) &= m; \quad h(1) = h(m) = h(\hat{m}) = 0.
\end{align*}
\]

Since each homomorphism is 0 on the identity and each element of \( <m, n, u, \hat{m}> \) is not sent into 0 by some one of \( \{f, g, h\} \), we conclude that Utumi’s ring of quotients is precisely \( R \).

4. \( \hat{R} \) is a ring. In this section we generalize a result of Lambek [8] to the case where \( \hat{R} \) may be made into a ring, although that ring may properly contain Utumi’s ring of quotients.

3) A table of these triple products may be found in the author’s doctoral dissertation.
Assume that \((\hat{R}, +, \circ)\) is a ring, where \(m \circ r = mr\) for all \(m \in \hat{R}, r \in R\). Let \(\Lambda = \text{Hom}_R(\hat{R}, \hat{R})\). We first prove a standard embedding lemma.

**LEMMA 1.** \((\hat{R}, +, \circ)\) is isomorphic to a subring of \(\Lambda\).

**Proof.** Define a map from \(\hat{R}\) to \(\Lambda\) by \(m \mapsto \bar{m}\), where \(\bar{m}(x) = m \circ x\) for all \(m, x \in \hat{R}\). For all \(m, n, x \in \hat{R}\),

\[
(\bar{m} + \bar{n})(x) = (m + n) \circ x = m \circ x + n \circ x = \bar{m}(x) + \bar{n}(x) = (\bar{m} + \bar{n})(x),
\]

\[
(\bar{m} \circ n)(x) = (m \circ n) \circ x = m \circ (n \circ x) = \bar{m}(n(x)) = (\bar{m} \bar{n})(x),
\]

so this map is a ring homomorphism. If \(\bar{m} = 0\)

\[
0 = \bar{m}(1) = m \circ 1 = m 1 = m,
\]

so the map is one-to-one.

We will denote the image of \(R\) under this map by \(\mathcal{R}\), and the image of \(\hat{R}\) by \(\hat{\mathcal{R}}\).

**LEMMA 2.** \(\Lambda\) is a unital \(\hat{\mathcal{R}}\) module.

**Proof.** Let \(e\) be the identity of \(\Lambda\). \(\overline{1} \overline{1} = \overline{1 \circ 1} = \overline{1}\), so \(\overline{1}\) is an idempotent of \(\Lambda\). Hence \(e - \overline{1}\) is also idempotent, and \((e - \overline{1})(r) = r - r = 0\) for all \(r \in R\).

Since \(\Lambda\) is the endomorphism ring of the injective module \(\hat{R}_R\), the Jacobson radical of \(\Lambda\) consists precisely of those elements of \(\Lambda\) which annihilate an essential submodule of \(\hat{R}_R\) (see [11], Lemma 8). Then \((e - \overline{1})\) is an idempotent in the Jacobson radical, so \(e - \overline{1} = 0\). Thus, \(e\) actually belongs to \(\hat{\mathcal{R}}\).

We wish to show that \(\mathcal{R}^\perp\) is an injective module. To do so we need some more information about the structure of \(\Lambda\). Let \(\mathcal{R}^\perp = \{ \lambda \in \mathcal{R} \mid \lambda(1) = 0 \}\).

**LEMMA 3.** \(\mathcal{R} = \hat{\mathcal{R}} \oplus \mathcal{R}^\perp\).

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Proof. Let $\lambda, \mu \in R, r \in R$. $(\lambda \pm \mu)(1) = 0$ so $\lambda \pm \mu \in R^\perp$. $\lambda F(1) = \lambda(r) = 0$ so $\lambda F \in R^\perp$. Thus $R^\perp$ is an $R$ module.

Let $\lambda \in \Lambda$, $m = \lambda(1)$. Then $(\lambda - m)(1) = m - m = 0$, so $\lambda - m \in R^\perp$ and $\Lambda = R + R^\perp$. If $x \in \hat{R} \cap R^\perp$, let $m \in \hat{R}$ be such that $x = \hat{m}$. Then $0 = x(1) = m(1) = m$, so $0 = \hat{m} = x$. Thus the sum is direct.

We are now ready to prove the theorem.

THEOREM. $\Lambda$ is an injective $\hat{R}$ module.

Proof. Since $\Lambda$ is unital by Lemma 2, to prove that $\Lambda$ is injective we need only show that every $f \in \text{Hom}_\Lambda(\mathcal{U}, \Lambda)$, for $\mathcal{U}$ a right ideal of $\hat{R}$, is induced by some element $\theta \in \Lambda$.

For $\lambda \in \Lambda$, let $\Pi \lambda$ be the projection of $\lambda$ onto $\hat{R}$ with respect to $R^\perp$. Then $\Pi \in \text{Hom}_\Lambda(\Lambda, \hat{R})$. Let $\mathcal{U}$ be a right ideal of $\hat{R}$, $f \in \text{Hom}_\Lambda(\mathcal{U}, \Lambda)$. Then $\Pi f \in \text{Hom}_\Lambda(\mathcal{U}, \hat{R})$. Since $\hat{R}$ is injective, by the isomorphism of Lemma 1, $\hat{R}$ is injective. Then there exists $\theta \in \text{Hom}_R(\hat{R}, \hat{R})$ such that $\hat{f}$ restricted to $\mathcal{U}$ is $\Pi f$. For all $m \in \hat{R}$, define $\theta \in \Lambda$ by $\theta(m) = [\theta(m)](1)$. Then

$$\theta(mr) = [\theta(m)(r)](1) = [\theta(m)f](1) = [\theta(\hat{m})](1) = \theta(m)(r) = \theta(m)r,$$

so $\theta$ is indeed an $R$ homomorphism.

For all $x \in \mathcal{U}$,

$$(f(x) - \theta \hat{x})(1) = f(\hat{x})(1) - \theta(x) = f(\hat{x})(1) - [\theta(\hat{x})](1)$$

$$= [f(\hat{x}) - \Pi f(\hat{x})](1) = 0.$$

Hence $f(\hat{x}) - \theta \hat{x} = u_x \in R^\perp$.

Let $m$ be any element of $\hat{R}$.

$$u_x \hat{m} = (f(x) - \theta \hat{x})\hat{m} = f(\hat{x})\hat{m} - (\theta \hat{x})\hat{m} = f(\hat{x})\hat{m} - \theta \hat{x}\hat{m} = u_{x \hat{m}} \in R^\perp.$$
Hence \( u_x \bar{m}(1) = u_x(m) = 0 \), so \( u_x = 0 \). Thus \( f(\bar{x}) = 0 \bar{x} \) for all \( \bar{x} \in \mathcal{X} \) and \( \Lambda \bar{x} \) is injective.

**COROLLARY.** Let \( R \) be a ring with 1 such that the injective hull \( \hat{R} \) of \( R \) is a rational extension of \( R \). Then \( \hat{R}_R \) is injective.

**Proof.** In this case, \( \hat{R} \) is Utumi's ring of quotients, and it is a ring isomorphic to \( \Lambda \). Then \( \hat{R}_R = \Lambda \hat{R} \) is injective by the theorem.

This corollary is just \((2) \Rightarrow (6)\) in the proposition of section 5 of Lambek [8].

The author does not know whether \( \hat{R}_R \) must always be injective if \( \hat{R} \) may be made into a ring. In the example of section 3, we do get a self injective ring. For there is only one irreducible left \( \hat{R} \) module and one irreducible right \( \hat{R} \) module, and they are the duals of each other. Hence \( \hat{R}_R \) is injective (see [4], section 58). Similarly, one may show that \( \Lambda \Lambda \) is not injective in this example.

**REFERENCES**

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