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# THE SPECTRAL RESOLUTION OF SOME NON-SELFADJOINT PARTIAL DIFFERENTIAL OPERATORS

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Let $\mathscr{H}$ and $\mathscr{H}_1$ be Hilbert spaces, $L(\mathscr{H}, \mathscr{H}_1)$ the set of all densely defined linear operators from $\mathscr{H}$ to $\mathscr{H}_1$, and $\mathscr{B}(\mathscr{H}, \mathscr{H}_1)$ its subset of bounded ones. Let $T^*$, $\mathscr{R}(T)$, $\mathscr{D}(T)$, $[T]$ and $\sigma(T)$ denote the adjoint, range, domain, closure and spectrum of $T$ respectively. $R_i(z)$ will denote the resolvent $(z - T_i)^{-1}$. In [**5**] the following result was obtained:

THEOREM. *Let $T_0 \in L(\mathscr{H},\mathscr{H})$ be a spectral operator of scalar type with spectral resolution $E_0(\Delta)$ such that $\sigma(T_0)$ is contained in a finite collection of smooth curves $\Gamma$. Let $A \in L(\mathscr{H}_1, \mathscr{H})$ and $B \in L(\mathscr{H}, \mathscr{H}_1)$ satisfy the following assumptions.*
$A_1 : \mathscr{D}(B) \supset \mathscr{D}(T_0),\ \mathscr{D}(A^*) \supset \mathscr{D}(T_0^*) = \mathscr{D}(T_0),\ \mathscr{R}(B) \subset \mathscr{D}(A).$
$A_2 : T_1 = T_0 + AB$ *defined on* $\mathscr{D}(T_0)$ *is closed.*
$A_3$ : *For $Z \notin \sigma(T_0)$, $BR_0(z) \in \mathscr{B}(\mathscr{H}, \mathscr{H}_1)$ and $A^*R_0(z)^* \in \mathscr{B}(\mathscr{H}, \mathscr{H}_1)$ are analytic in $z$ and assume boundary values as $z \to \Gamma$ in the following sense. There is a real number $r$ and a subset $\mathscr{E}$ of $\Gamma$, such that, for any interval $\Delta$ of $\Gamma$ whose closure is disjoint from $\mathscr{E}$, for every $f \in \mathscr{H}$ and $\epsilon > 0$ sufficiently small,*

$$(1 + |\lambda|)^r BR_0(\lambda \pm \epsilon\eta)f \text{ and } (1 + |\lambda|)^{-r} A^*R_0(\lambda \pm \epsilon\eta)^*f$$

*belong to $L_2(\Delta; \mathscr{H}_1)$ and have strong limits as $\epsilon \to 0+$. Here $\eta = \eta(\lambda)$ is a direction (i.e. a complex number of absolute value 1) non-tangential to $\Gamma$ at the point $\lambda$, piece-wise constant in $\lambda$ and such that in any integral over $\Gamma$, the angle between the direction of integration and $\eta(\lambda)$ is between 0 and $\pi$.*
$A_4$ : *$BR_0(z)A$ has a bounded closure $Q_0(z)$ for $z \notin \sigma(T_0)$. For all $f \in \mathscr{H}_1$ and almost all $\lambda \in \Gamma$, $Q_0(\lambda \pm \epsilon\eta)f$ converges strongly as $\epsilon \to 0+$, the limit being denoted by $Q_0(\lambda\pm)f$. With $\Delta$ as in $A_3$, $I - Q_0(\lambda\pm)$ is invertible for almost all $\lambda \in \Delta$ and for each $\Delta$,*

$$\operatorname*{ess\,sup}_{\lambda\in\Delta} ||(I - Q_0(\lambda\pm))^{-1}|| \leqq C$$

*for some constant $C$. For $z \notin \sigma(T_0)$, $(I - Q_0(z))^{-1}$ maps $\mathscr{D}(A)$ into itself.*

*Then the operator $T_1$ has spectral resolution $E_1(\Delta)$ for any Borel set $\Delta$ of $\Gamma$ whose closure is disjoint from $\mathscr{E}$. Moreover $T_1$ restricted to the subspace $E_1(\Delta)\mathscr{H}$ is similar to $T_0$ restricted to $E_0(\Delta)\mathscr{H}$.*

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*Remark.* In [**5**] $\mathscr{E}$ was, quite unnecessarily, taken to be finite.

The purpose of this note is to apply this result to a class of non-self adjoint partial differential operators.

For the unperturbed operator we take an operator with constant coefficients. More precisely, with the usual multi-index notation, let $\mathscr{H} = L^2(R^n)$, $p$ the polynomial $p(x) = \sum_{|i| \leqq m} b_i x^i$ and define $T_0$ by $T_0 f(x) = p(D)f(x)$ on the domain $\mathscr{D}(T_0) = \{f \in \mathscr{H};\ (1 + p)\mathscr{F}f \in \mathscr{H}\}$ where $\mathscr{F}f$(or $\hat{f}$) denotes the Fourier transform of $f$. We assume that the coefficients of $p$ are real so that $T_0$ is a self-adjoint operator in $\mathscr{H}$ with spectrum $\sigma(T_0) = \Gamma = \{\lambda;\ \lambda = p(x)$ for some $x \in R^n\}$ and with spectral resolution $E_0(\Delta) = \mathscr{F}^{-1}\mathscr{X}_{p^{-1}(\Delta)}\mathscr{F}$, where $\mathscr{X}_{p^{-1}(\Delta)}$ is the operator of multiplication by the characteristic function of $p^{-1}(\Delta)$, the preimage in $R^n$ of the subset $\Delta$ of the complex plane $C$ under $p$. Let $P = \sum_{|i| \leqq k} a_i D^i$, where the $a_i$ are functions of $x$. $P$ can be written as a product $P = AB$ by ordering all partial derivatives up to order $k$ in some manner and letting $A$ be the operator from $\mathscr{H}_1$, a product of $N = (1 + n + n^2 + \ldots + n^k)$ copies of $L^2(R^n)$, to $\mathscr{H}$ and $B$ be the operator from $\mathscr{H}$ to $\mathscr{H}_1$ given by

$$A[f_1, \ldots, f_N](x) = \sum_{i=1}^{N} \alpha_i(x) f_i(x)$$

$$[Bf]_i(x) = \beta_i(x) D^i f \quad i = 1, \ldots, N,$$

where $\alpha_i(x)\beta_i(x) = a_i(x)$ for all $x$.

Taking

$$\alpha_i(x) = \min\{|a_i(x)|^{1/2}, 1\}$$

$$\beta_i(x) = \begin{cases} a_i(x) \text{ if } |a_i(x)| > 1 \\ a_i(x)|a_i(x)|^{-1/2} \text{ if } 0 < |a_i(x)| \leqq 1 \\ 0 \quad\quad \text{if } a_i(x) = 0 \end{cases}$$

we obtain the following lemma.

LEMMA 1. *The operators $A$, $B$ and $T_0$ satisfy conditions $A_1$ and $A_2$ if*

(i) *$T_0$ is elliptic*

(ii) $k < m - n/2$

(iii) $a_i(x) \in L^1(R^n) \cap L^2(R^n)$.

*Proof.* Theorem 4.5 of [**9**], Chapter 6 is applicable. In fact $P$ and $B$ are $T_0$-compact and since $\alpha_i \in L^2(R^n) \cap L^\infty(R^n)$, $A$ is bounded.

It is clear that, for $z \notin \sigma(T_0)$, $BR_0(z)$ and $A^*R_0(z)^*$ belong to $\mathscr{B}(\mathscr{H}, \mathscr{H}_1)$ and that $BR_0(z)A = Q_0(z)$ is closed and bounded, in fact compact. All these operators are analytic in $z$ for $z \notin \sigma(T_0)$ and $(I - Q_0(z))^{-1}$ exists if and only if $z \notin \sigma(T_1)$.

Assumptions $A_3$ and $A_4$ concern the behaviour of these operators near the spectrum $\Gamma$ which is on the real line.

The operator $R_0(\lambda)$ may, for $\lambda \notin \Gamma$, be written

$$R_0(\lambda)f(x) = \mathscr{F}^{-1}\{(\lambda - p(y))^{-1}\mathscr{F}f(y)\} = (2\pi)^{-n/2}\int_{R^n} e^{ixy}(\lambda - p(y))^{-1}\hat{f}(y)dy.$$

Each component of $BR_0(\lambda)$ is of the form

$$[BR_0(\lambda)]_j f(x) = (2\pi)^{-n/2}\int_{R^n} \beta_j(x)y^j e^{ixy}(\lambda - p(y))^{-1}\hat{f}(y)dy,$$

those of $A^*R_0(\lambda)^*$ are

$$[A^*R_0(\lambda)^*]_j f(x) = (2\pi)^{-n/2}\int_{R^n} \overline{\alpha_j(x)e}^{ixy}(\overline{\lambda - p(y)})^{-1}\hat{f}(y)dy,$$

and each entry of the matrix $BR_0(\lambda)A$ is given by

$$[BR_0(\lambda)A]_{jn}f(x) = (2\pi)^{-n/2}\int_{R^n} \beta_j(x)y^j e^{ixy}(\lambda - p(y))^{-1}\widehat{\alpha_n f}(y)dy.$$

Clearly, all of these operators are bounded and analytic in $\lambda$ for $\lambda \notin \Gamma$ but exhibit a singularity of the Cauchy type for $\lambda \in \Gamma$.

LEMMA 2. *Let $\mathscr{E}_1$ be the set of points of $\Gamma$ where the change of variables $y \in R^n \to (p, \omega) \in \Gamma \times S^n$ given by $p = p(y)$, $\omega = |y|^{-1}y$, is singular. Suppose $k \leqq m/2$ and $m \geqq 2n$, then the operators $BR_0(z)$, $A^*R_0(z)^*$ and $Q_0(z)$ may be continued onto either side of any interval $\Delta$ whose closure is distinct from $\mathscr{E}_1$ as bounded operators in the sense of assumptions $A_3$ and $A_4$ with $r = -n/m$.*

*Proof.* The change of variables $y \to (p, \omega)$ has Jacobian $J$ given by $J = |y|^n |\text{grad } p \cdot y|^{-1}$ such that $dy = J\,dp d\omega$. In general $J$ will be singular for certain values of $p$ (e.g. if $p$ is homogeneous elliptic only $p = 0$) and near $\infty$, $J = O(|p|^{(n-m)/m})$. If $f$ is restricted to $E_0(\Delta)\mathscr{H}$, $f$ has support in $p^{-1}(\Delta)$ and we may make the change of variable in the integrals so we can write:

$$[BR_0(\lambda)]_j f(x) = \beta_j(x)\int_\Gamma \frac{B_j(x, p)}{\lambda - p}dp$$

$$\overline{[A^*R_0(\lambda)^*]_j f(x)} = \alpha_j(x)\int_\Gamma \frac{A_j(x, p)}{\lambda - p}dp$$

$$[Q_0(\lambda)]_{jq} f(x) = \beta_j(x)\int_\Gamma \frac{Q_{jq}(x, p)}{\lambda - p}dp$$

with

$$B_j(x, p) = \int_{S^n} y^j e^{ixy}\hat{f}(y)\,Jd\omega$$

$$\overline{A_j(x, p)} = \int_{S^n} e^{ixy}\hat{f}(y)\,Jd\omega$$

$$Q_{jq}(x, p) = \int_{S^n} y^j e^{ixy}\widehat{\alpha_q f}(y)\,Jd\omega$$

It then is obvious that $BR_0(\lambda)$, $A^*R_0(\lambda)^*$ and $Q_0(\lambda)$ may be continued onto either side of $\Gamma$ as bounded operators whose norms are square integrable with respect to $\lambda$ over any finite interval of $\Gamma$. Over the entire interval $\Gamma$ (which includes $\infty$) we have the estimates:

$$\int_\Gamma |B_j(x,p)|^2|p|^\alpha dp = \int_\Gamma \left| \int_{S^n} y^j e^{ixy} f(y) J d\omega \right|^2 |p|^\alpha dp$$
$$\leqq K \int_\Gamma \int_{S^n} (1+|y|)^{2|j|} |f|^2 |J|^2 |p|^\alpha d\omega dp$$
$$\leqq K \int_{R^n} (1+|y|)^{2|j|+\alpha m+n-m} |f(y)|^2 dy$$
$$\leqq K \| f \|$$

provided we choose $\alpha = -n/m$ and $2|j| \leqq m$. By a well-known result of Hardy and Littlewood [**8**, Exercise 7.7.10, p. 432] it follows that

$$\int_\Gamma \left| \int_\Gamma \frac{B_j(x,p)}{\lambda - p} dp \right|^2 |\lambda|^{-n/m} d\lambda \leqq K \| f \|^2.$$

Multiplying by $|\beta_j(x)|^2$, integrating with respect to $x$ and using Fubini's theorem we obtain that:

$$\int_\Gamma \int_{E^n} |[BR_0(\lambda)]_j f(x)|^2 dx |\lambda|^{-n/m} d\lambda \leqq K \|\beta_j\|^2 \| f \|^2.$$

Similarly we obtain, noting the absence of the terms $y^j$, that, if $m \geqq 2n$,

$$\int_\Gamma \int_{E^n} |[A^*R_0^*(\lambda)]_j f(x)|^2 dx |\lambda|^{+n/m} d\lambda \leqq K \|\alpha_j\|^2 \| f \|^2.$$

If $f$ is not restricted to $E_0(\Delta)\mathscr{H}$, we write $f = f_1 + f_2$ where $f_1 \in E_0(\Delta^1)\mathscr{H}$ and $f_2 \in E_0(\Gamma - \Delta^1)\mathscr{H}$ where $\Delta^1 \supset \Delta$ and the closure of $\Delta^1$ does not intersect $\mathscr{E}_1$. The integrals involving $f_1$ are handled as above, the ones involving $f_2$ are analytic in $\lambda$ and $\lambda \to \Delta$ since for $x \in p^{-1}(\Gamma - \Delta^1)$, $\lambda - p(x)$ is bounded away from zero.

LEMMA 3. *Suppose that in addition to the assumptions of Lemmas* 1 *and* 2,

$$(1 + |x|)^2 \cdot a_i(x) \in L^1(R^n)$$

*and that the point spectrum of $T_1$ has at most a finite number of limit points on* $\Gamma$. *Then there exists a closed, bounded subset $\mathscr{E}_2$ of $\Gamma$ of Lebesque measure zero such that $Q_0(z)$ satisfies assumption $A_4$ for any interval $\Delta$ whose closure is disjoint from $\mathscr{E}_2$.*

*Proof.* $Q_0(z)$ is compact analytic for $\lambda \notin \Gamma$, thus $(I - Q_0(z))^{-1}$, if it exists anywhere at all, exists everywhere for $\lambda \in \Gamma$ except perhaps at an isolated set of points with no accumulation point outside $\Gamma$. $T_1$ is not self adjoint in general and the singularities of $(I - Q_0(z))^{-1}$ outside $\Gamma$ make up the point spectrum of $T_1$ outside $\Gamma$. To restrict the singularities on $\Gamma$ itself we make use of the additional assumptions. The entries of $Q_0(z)$ are of the form:

$$\int \beta(x) \frac{Q(x, p)}{z - p} dp = \beta(x) G(x, z).$$

If $(1 + |x|)^2 a_i(x) \in L^1(R^n)$, $(1 + |x|)\alpha(x) \in L^2(R^n)$ and $D\hat{\alpha}$ exists in $L^2(R^n)$ so that $Q(x, p)$ is differentiable and $Q_0(z)$ and its continuity near $\Gamma$ can be estimated in terms of Holder-norms.

$||Q_0(\lambda)||$ is obtained in terms of $\sup_x |G(x, \lambda)|\ ||\beta||$ and $||Q_0(\lambda_1) - Q_0(\lambda_2)||$ in terms of $\sup_x |G(x, \lambda_1) - G(x, \lambda_2)|\ ||\beta||$. Letting, for $\theta > 0$, $0 \leqq \mu \leqq 1$,

$$|f|_{\theta,\mu} = \sup_{\substack{p_1, p_2 \in \Gamma \\ |p_2 - p_2| < 1}} (1 + |p_1|)^\theta \left\{ |f(p_1)| + \frac{|f(p_1) - f(p_2)|}{|p_1 - p_2|^\mu} \right\},$$

it is well-known (e.g. [**11**, Lemma 2.2]) that

$$|G(x, .)|_{\theta,\mu'} \leqq K |Q(x, .)|_{\theta,\mu}$$

where $\mu' = \mu$ if $\mu < 1$ and $\mu' < \mu$ if $\mu = 1$.

In our case,

$$\begin{aligned} |Q(x, p)| &= \left| \int_{S^n} y^j e^{ixy} \widehat{\alpha f}(y) J d\omega \right| \\ &\leqq \text{vol } S^n (1 + |p|)^{k/m} (1 + |p|)^{(n-m)/m} |\widehat{\alpha f}| \\ &\leqq K ||\alpha||\ ||f|| (1 + |p|)^{(n+k-m)/m}, \end{aligned}$$

and for $|p_1 - p_2| < 1$

$$|Q(x, p_1) - Q(x, p_2)| \leqq K(||\alpha|| + ||D\hat{\alpha}||)\ ||f||\ (1 + |p_1|)^{(n+k-m)/m} |p_1 - p_2|.$$

Thus $|Q(x, .)|_{\theta,\mu}$ is finite for any $0 < \theta < (m - k - n)/m$ and $\mu < 1$.

It follows that $G(x, \lambda)$ and thus $Q_0(\lambda)$ can be extended continuously onto $\Gamma$ (with in general different values on opposite sides). In fact $Q_0(\lambda)$ is continuous in the uniform topology so that, also for $\lambda \in \Gamma$, $Q_0(\lambda)$ is a compact operator. Furthermore, $||Q_0(\lambda)|| \to 0$ as $|\lambda| \to \infty$ on $\Gamma$ as well as off $\Gamma$ so that $(I - Q_0(\lambda))^{-1}$ exists for $|\lambda|$ sufficiently large.

Except near the limit points on $\Gamma$ of the point spectrum of $T_1$ outside $\Gamma$, Lemma 6.2 of [**6**] applies and the conclusion of Lemma 3 follows.

Our results are collected in the following theorem:

THEOREM. *Let $T_0$ be the elliptic differential operator in $L^2(R^n)$ given by*

$$T_0 f = p(D) f,\ \mathscr{D}(T_0) = \{f \in L^2(R^n) : (1 + p)\mathscr{F} f \in L^2(R)\}$$

*where $p(x)$ is a polynomial of order $m$ with real coefficients. Let $P$ be the operator*

$$Pf(x) = \sum_{|i| \leqq k} a_i(x) D^i f(x).$$

*Suppose*

(i) $k < m/2$, $m \geqq 2n$

(ii) $a_i(x) \in L^2(R^n)$, $(1 + |x|)^2 a_i(x) \in L^1(R^n)$

(iii) $\operatorname{grad} p(y) \cdot y \neq 0$ *except perhaps for a discrete set of values of* $p$.

*Then there exists a bounded closed subset* $\mathscr{E}$ *of the real axis of Lebesque measure zero such that* $T_1 = T_0 + P$ *defined on* $\mathscr{D}(T_0)$ *has a spectral resolution* $E_1(\Delta)$ *for any Borel set* $\Delta$ *whose closure is disjoint from* $\mathscr{E}$, *provided that the point-spectrum of* $T_1$ *has at most a discrete set of limit points. In fact* $T_1$ *restricted to* $E_1(\Delta)\, L^2(R^n)$ *is similar to* $T_0$ *restricted to* $E_0(\Delta)\, L^2(R^n)$.

*Remarks.* 1. The restrictions on the dimension $n$ of the underlying Euclidian space and the order of the differential operator appear somewhat artificial but seem unavoidable in the arguments presented. Restrictions of a similar nature may be found in [**4**] or [**7**]. On the other hand, no such limitation appears e.g. in [**3**], [**10**] or [**11**]. Results there pertain mostly to perturbations of the Laplacian, self-adjoint perturbations or those whose coefficients have compact support.

2. If we assume that the coefficients $a_i$ decay exponentially at $\infty$, it follows by arguments similar to those in [**7**] that outside of a neighborhood of the points where $\operatorname{grad} p \cdot y = 0$, $Q_0(\lambda)$ may be analytically continued across $\Gamma$ so that there are only a finite number of singularities.

3. The result may be slightly extended to nonelliptic operators. The conditions on $p(x)$ and the perturbation $P$ are then given as in [**9**], chapter 5 to insure $T_0$-compactness. It would seem that the result may be further extended to $p(x)$ nonreal and $\Gamma$ a curve in the complex plane, but the appropriate analogue of the Hardy-Littlewood result on the singular integrals is not clear.

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