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SOME REMARKS ON MODULARITY OF THE CONGRUENCE LATTICE OF REGULAR ω-SEMIGROUPS

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Abstract

In this paper conditions of M-symmetry, strong, semimodularity and θ-modularity for the congruence lattice $L(S)$ of a regular ω-semigroup S are studied. They are proved to be equivalent to modularity. Moreover it is proved that the kernel relation is a congruence on $L(S)$ if and only if $L(S)$ is modular, generalizing an analogous result stated by Petrich for bisimple ω-semigroups.

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Introduction

In [3] and [4] the authors stated conditions under which the lattice of congruences $L(S)$ of a regular ω-semigroup S is respectively modular or semimodular. Two other conditions, M-symmetry and strong semimodularity, are usually examined for the congruence lattice of semigroups. So it is natural to ask for conditions implying M-symmetry and strong semimodularity for the congruence lattice of regular ω-semigroups.

We remark that we use terminology consistent with that used by Mitsch in [6]. Following other authors, semimodularity and strong semimodularity are called, respectively, double covering property and semimodularity. It is well known that in every lattice modularity implies M-symmetry, M-symmetry implies strong semimodularity, which in turn implies semimodularity. Thus we begin with conditions for strong semimodularity. Firstly, we give a characterization of regular ω-semigroups

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whose congruence lattice is strongly semimodular. Then we prove that this condition is equivalent to the modularity conditions given in [3]. Hence this paper becomes a revisit of regular ω-semigroups with modular lattice of congruences.

Petrich introduced two relations on the congruence lattice of a regular semigroup: the trace relation and the kernel relation. In general the latter one is not a congruence on the congruence lattice, so he studied conditions in order that the kernel relation $\mathscr{K}$ is a congruence and in [8, 9] observed a strict connection between this fact and the modularity of the congruence lattice for classes of regular semigroups. We study the kernel relation for a regular ω-semigroup and prove that $L(S)$ is modular if and only if $\mathscr{K}$ is a congruence on $L(S)$.

Finally we prove that the modularity of $L(S)$ is equivalent to θ-modularity.

Notation and terminology will be as in [3] and [4], and a knowledge of these papers is useful in the following. As usual σ denotes the least group congruence, $\mathscr{H}$ and $\mathscr{D}$ the Green's relations, ι the identity congruence on S, and $\mathbb{N}$ the set of non-negative integers.

Section 1

For sake of completeness, we begin by recalling the main results on the structure of regular ω-semigroups and on the description of some their congruences.

DEFINITION 1.1 (see, for example, [10]). A *regular ω-semigroup S* is a regular semigroup whose set of idempotents $E(S)$, or shortly E, forms an ω-chain

$$e_0 > e_1 > \cdots > e_n > \cdots$$

under the natural order defined on E by the rule $e \geq f$ if and only if $ef = f = fe$.

For a regular ω-semigroup, Munn [7] proved the following result:

THEOREM A. *Let S be a regular ω-semigroup.*

(i) *If S has no kernel, then it is the union of an ω-chain of groups.*
(ii) *If the kernel of S coincides with S, then S is a simple regular ω-semigroup.*
(iii) *If S has a proper kernel, then S is a (retract) ideal extension of a simple regular ω-semigroup K by a semigroup H^0, where H is a finite chain of groups and H^0 is obtained from H by adjoining a zero. Moreover this extension is determined by means of a homomorphism of H into the group of units of K.*

Thus, the following theorem completely determines the structure of regular ω-semigroups:

THEOREM B (Kocin [5] and Munn [7]). *Let d be a positive integer and let $\{G_i \mid i = 0, \dots, d-1\}$ be a set of d pairwise disjoint groups. Let γ_{d-1} be a homomorphism of G_{d-1} into G_0 and, if $d > 1$, let γ_i be a homomorphism of G_i into G_{i+1} ($i = 0, \dots, d-2$). For every $n \in \mathbb{N}$ let $\gamma_n = \gamma_{n \pmod d}$ where $n \pmod d$ denotes the integer equivalent to n modulo d, belonging to N and less than d. For $m, n \in \mathbb{N}$ and $m < n$ write*

$$\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$$

and for all $n \in N$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{n \pmod d}$.

Let S be the set of ordered triples (m, a_i, n), where $m, n \in \mathbb{N}$, $0 \le i \le d-1$ and $a_i \in G_i$. Define a multiplication in S by the rule

$$(m, a_i, n)(p, b_j, q) = (m + p - r, (a_i \alpha_{u,w})(b_j \alpha_{v,w}), n + q - r)$$

where $r = \min\{n, p\}$, $u = nd + i$, $v = pd + j$ and $w = \max\{u, v\}$. Denote the so formed groupoid by $S(d, G_i, \gamma_i)$. Then $S(d, G_i, \gamma_i)$ is a simple regular ω-semigroup with exactly d $\mathscr{D}$-classes and any regular ω-semigroup is isomorphic to a semigroup $S(d, G_i, \gamma_i)$.

For $n \in \mathbb{N}$ and $i = 0, \dots, d-1$ write $e_i^n = (n, e_i, n)$ where e_i is the identity of the group G_i. The elements e_i^n are the indempotents of $S(d, G_i, \gamma_i)$ and we have

$$e_0^0 > e_1^0 > \cdots > e_{d-1}^0 > e_0^1 > \cdots > e_{d-1}^1 > e_0^2 > \cdots .$$

Hence in the remainder of the paper we will denote a simple regular ω-semigroup by $S(d, G_i, \gamma_i)$.

The congruences on $S(d, G_i, \gamma_i)$ belonging to the interval $[\iota, \sigma \vee \mathscr{H}]$ were completely described by Baird [1, 2] and it was proved that all congruences on S which are not in $[\iota, \sigma \vee \mathscr{H}]$ are group congruences (see [1, Remark p.164]).

We need the following definitions:

DEFINITION 1.2 (see [1, 2 and 3], [2, 2]). Let $S = S(d, G_i, \gamma_i)$. A *congruence* μ on the set $E(S)$ of indempotents is called *uniform* if $(e_i^n, e_j^m) \in \mu$ implies that $(e_i^{n+p}, e_j^{m+p}) \in \mu$ for all integers $p \ge -\min\{m, n\}$.

Put $G = G_0 \times G_1 \times \cdots \times G_{d-1}$, the cartesian product of the G_i. A subset A of G will be called *γ-admissible* if

(i) $A = A_0 \times \cdots \times A_{d-1}$, for some $A_i \subseteq G_i$, $i = 0, 1, \dots, d-1$.
(ii) $A_i \trianglelefteq G_i$, for $i = 0, 1, \dots, d-1$.
(iii) $A_{d-1}\gamma_{d-1} \subseteq A_0$ and $A_i \gamma_i \subseteq A_{i+1}$, for $i = 0, 1, \dots, d-2$.

If $A = A_0 \times \cdots \times A_{d-1}$ and $B = B_0 \times \cdots \times B_{d-1}$ are γ-admissible subsets of G we define $A \cdot B = A_0 \cdot B_0 \times \cdots \times A_{d-1} \cdot B_{d-1}$. Obviously $A \cdot B$ is a γ-abmissible subset of G. Let μ be a uniform congruence on $E(S)$, A a γ-admissible subset of G

and i an integer with $0 \leq i < d$. Put $\mu - \operatorname{rad} A_i = \{a_i \in G_i | a_i \alpha_{nd+i, md+j} \in A_j$ for some positive integers n, m and for some j with $0 \leq j < d$ such that $(e_i^n, e_j^m) \in \mu$ and $e_j^m \leq e_i^n\}$, and $\mu - \operatorname{rad} A = \mu - \operatorname{rad} A_0 \times \cdots \times \mu - \operatorname{rad} A_{d-1}$.

If $\mu - \operatorname{rad} A = A$, μ and A are called *linked*.

We remark that the set of the γ-admissible subsets of G forms a lattice $\Gamma(S)$, with respect to the union gived by the previously defined product and set theoretical intersection (see [1, p. 164]). Moreover, for every uniform congruence μ, the set $\Gamma_\mu(S)$ of μ-linked subsets of G is a subsemilattice of $\Gamma(S)$, with respect to intersection. This is a lattice because the μ-linked subsets of G form a partially ordered set isomorphic to the set of congruences in $[\iota, \sigma \vee \mathscr{H}]$ having trace μ (see [10]).

Let $S = S(d, G_i, \gamma_i)$. Subsequently we will use the following notation:

(i) we let 1 be the γ-admissible subset $\{e_0\} \times \cdots \times \{e_{d-1}\}$ of G;

(ii) for any congruence τ on S, let $A^\tau = A_0^\tau \times \cdots \times A_{d-1}^\tau$ where $A_i^\tau = \{a_i \in G_i | (0, a_i, 0) \tau e_i^0\}$, $i = 0, \ldots, d-1$.

The following definition is well known (see [10]).

DEFINITION 1.3. Let ρ be a congruence on an inverse semigroup S. The *trace* of ρ, denoted by $\operatorname{tr} \rho$, is the restriction of ρ to the set $E(S)$.

THEOREM C (see [1, Theorems 4.2, 5.1, 5.2, 5.3]). *Let $S = S(d, G_i, \gamma_i)$, let μ be a uniform congruence on E, let A be a γ-admissible subset of G, and suppose that μ and A are linked. Then*

$$\tau = \{((m, a_i, n), (p, b_j, q)) \in S \times S | (a_i \alpha_{u,w})(b_j^{-1} \alpha_{v,w}) \in A_{w (\operatorname{mod} d)}, \quad \text{where} \\ u = nd + i,\ v = qd + j,\ w = \max\{u, v\};\ m - n = p - q;\ (e_i^m, e_j^p) \in \mu\}$$

is a congruence on S contained in $[\iota, \sigma \vee \mathscr{H}]$ such that $\operatorname{tr} \tau = \mu$ *and* $A^\tau = A$.

Conversely, let τ be a congruence on $S = S(d, G_i, \gamma_i)$ contained in $[\iota, \sigma \vee \mathscr{H}]$. Then τ is of the above form with $\mu = \operatorname{tr} \tau$ and $A = A^\tau$. Moreover, let ρ, λ be congruences on $S = S(d, G_i, \gamma_i)$ contained in $[\iota, \sigma \vee \mathscr{H}]$. Then

(i) $\rho \leq \lambda$ *if and only if* $\operatorname{tr} \rho \leq \operatorname{tr} \lambda$ *and* $A^\rho \subseteq A^\lambda$,

(ii) $\operatorname{tr}(\rho \vee \lambda) = \operatorname{tr} \rho \vee \operatorname{tr} \lambda$, $\operatorname{tr}(\rho \wedge \lambda) = \operatorname{tr} \rho \wedge \operatorname{tr} \lambda$,

(iii) $A^{\rho \vee \lambda} = (\operatorname{tr} \rho \vee \operatorname{tr} \lambda) - \operatorname{rad} A^\rho \cdot A^\lambda$, $A^{\rho \wedge \lambda} = A^\rho \cap A^\lambda$.

We recall the following well-known definitions.

DEFINITION 1.4. Let L be a lattice and let $a, b \in L$. We say that *a covers b*, denoted $a \succ b$, if $a > b$ and there is no element $c \in L$ such that $a > c > b$.

DEFINITION 1.5 (see [7]). A lattice L is called *strongly semimodular* if, for every $a, b \in L$, $a \succ a \cap b$ implies $a \cup b \succ b$.

We prove the following.

PROPOSITION 1.6. *Let $S = S(d, G_i, \gamma_i)$. The following are equivalent.*

(i) *Let A, B be γ-admissible subsets of G. If $B \supseteq A$ and A is μ-linked, then B is μ-linked, for each uniform conqruence μ.*

(ii) *The interval $[\iota, \sigma \vee \mathcal{H}]$ of $L(S)$ is strongly semimodular.*

(iii) *The set Γ_μ of μ-linked subsets of G is a filter of the lattice Γ of γ-admissible subsets of G, for each uniform congruence μ.*

(iv) *The map $f : [\iota, \sigma \vee \mathcal{H}] \to \Gamma$ defined by $f(\rho) = A^\rho$ is a homomorphism of lattices.*

PROOF. (i) *implies* (ii). Let λ, ρ, be two congruences in $[\iota, \sigma \vee \mathscr{H}]$ such that $\lambda \succ \lambda \wedge \rho$. Denote by ν and μ the traces of λ and ρ respectively. We distinguish the following cases:

CASE 1: ν and μ are not comparable, or $\mu < \nu$. Then $\mathrm{tr}(\lambda \wedge \rho) \neq \nu$ and from [4, Lemma 2.6 (ii)], it follows that ν covers $\mathrm{tr}(\lambda \wedge \rho)$ and $A^{\lambda \wedge \rho} = A^\lambda$. Hence $A^\rho \supseteq A^\lambda$. Since A^λ is ν-linked, by hypothesis A^ρ is ν-linked; thus A^ρ is $(\nu \vee \rho)$-linked by [3, Lemma 1.3]. For the congruence $\lambda \vee \rho$, we have $A^{\lambda \vee \rho} = (\nu \vee \mu) - \mathrm{rad}\, A^\rho \cdot A^\lambda = (\nu \vee \mu) - \mathrm{rad}\, A^\rho = A^\rho$ and $\mathrm{tr}(\lambda \vee \rho) = \nu \vee \mu$. It is well known that uniform congruences form a modular lattice; hence $\nu \vee \mu$ covers μ, and so $\lambda \vee \rho$ covers ρ by [4, Lemma 2.5. (ii)].

CASE 2: $\nu < \mu$. Since $\mathrm{tr}(\lambda \wedge \rho) = \mathrm{tr}\, \lambda$, A^λ covers $A^\rho \cap A^\lambda$ in the semilattice Γ_ν of ν-linked subsets of G (by [4, Lemma 2.5 (1)]). Now let C be a γ-admissible subset of G containing $A^\rho \cap A^\lambda$. Then condition (i) implies that C is ν-linked; hence A^λ covers $A^\rho \cap A^\lambda$ in the modular lattice Γ of γ-admissible subsets of G, and so $A^\rho \cdot A^\lambda$ covers A^ρ in Γ. Moreover, A^ρ is μ-linked, and by (i) $A^\rho \cdot A^\lambda$ is μ-linked; hence we have $A^{\rho \vee \lambda} = (\nu \vee \mu) - \mathrm{rad}\, A^\rho \cdot A^\lambda = A^\rho \cdot A^\lambda$. Thus, since $\mathrm{tr}(\rho \vee \lambda) = \mathrm{tr}\, \rho$, it follows that $\lambda \vee \rho$ covers ρ, using [4, Lemma 2.5.(i)].

CASE 3: $\mu = \nu$. The statement immediately follows from the fact that congruences having the same trace form a modular lattice (see [10, Corollary III.2.7]).

(i) *implies* (iv). Let λ and ρ be two congruences in $[\iota, \sigma \vee \mathscr{H}]$ having traces ν and μ respectively. From (i) it follows that $A^\rho \cdot A^\lambda$ is ν-linked and μ-linked, so it is $(\nu \vee \mu)$-linked and $A^{\rho \vee \lambda} = (\nu \vee \mu) - \mathrm{rad}\, A^\rho \cdot A^\lambda = A^\rho \cdot A^\lambda$.

(iv) *implies* (iii). For all μ-linked subsets A, B of G, it is well known that $A \cap B$ is μ-linked. Now, let λ be the congruence with trace μ and $A^\lambda = A$, and let ρ be the congruenec with trace ι and $A^\rho = B$. Note that λ and ρ are in $[\iota, \sigma \vee \mathscr{H}]$. Hence $A^{\rho \vee \lambda} = (\mu \vee \iota) - \mathrm{rad}\, A^\rho \cdot A^\lambda = \mu - \mathrm{rad}\, A \cdot B$, but (iv) implies that $A^{\rho \vee \lambda} = A \cdot B$ and so $A \cdot B$ is μ-linked.

(iii) *implies* (i). If A is μ-linked and B is a γ-admissible subset containing A, then $A \cdot B = B$ is μ-linked by condition (iii).

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(ii) *implies* (i). Let A be a μ-linked subset and let B be a γ-admissible subset containing A. Now B is ν-linked for some uniform congruence $\nu \leq \mu$. Suppose that ν is a maximal uniform congruence with $\nu \leq \mu$, such that B is ν-linked, suppose $\nu < \mu$, and consider a finite chain of uniform congruences $\mu \succ \nu_1 \succ \nu_2 \succ \nu_3 \succ \cdots \succ \nu_k = \nu$. Then A is ν_i-linked for every $1 \leq i \leq k$. The congruence ρ having trace ν_{k-1} and satisfying $A^\rho = A$, and the congruence λ with $\operatorname{tr}\lambda = \nu$ satisfying $A^\lambda = B$, are in $[\iota, \sigma \vee \mathscr{H}]$. The congruence $\rho \wedge \lambda$ has trace ν and $A^{\rho\wedge\lambda} = A$, and so $\rho \succ \rho \wedge \lambda$. Hence $\lambda \vee \rho \succ \lambda$ and $\operatorname{tr}(\lambda \vee \rho) \neq \operatorname{tr}\lambda$ implies $A^{\rho\vee\lambda} = \nu_{k-1} - \operatorname{rad} B = A^\lambda = B$. Then by [4, Remark 2.4], B is ν_{k-1}-linked, contradicting the assumption of maximality of ν. Hence $\nu = \mu$.

Section 2

In this section we prove the equivalence between the conditions of modularity and strong semimodularity for the congruence lattice of a simple regular ω-semigroup.

We recall that a lattice L is called *modular* if for every $a, b, c \in L$ with $a \leq b$, if $a \cup c = b \cup c$ and $a \cap c = b \cap c$, then $a = b$.

LEMMA 2.1. *Let* $S = S(d, G_i, \gamma_i)$. *If* $[\iota, \sigma \vee \mathscr{H}]$ *is strongly semimodular, then* $L(S)$ *is modular.*

PROOF. Since $[\iota, \sigma \vee \mathscr{H}]$ is strongly semimodular, condition (i) of Proposition 1.6 holds. Let A be a γ-admissible subset of G and let μ be a uniform congruence on the idempotents $E(S)$. It is well known that $\mu - \operatorname{rad} 1 \subseteq A\cdot\mu - \operatorname{rad} 1$. Since $A\cdot\mu - \operatorname{rad} 1$ is a γ-admissible subset and $\mu - \operatorname{rad} 1$ is μ-linked; condition (i) of Proposition 1.6 implies that $A\cdot\mu - \operatorname{rad} 1$ is μ-linked, whence $A\cdot\mu - \operatorname{rad} 1 = \mu - \operatorname{rad}(A\cdot\mu - \operatorname{rad} 1) \supseteq \mu - \operatorname{rad} A$. Thus $A \cdot \mu - \operatorname{rad} 1 = \mu - \operatorname{rad} A$ and $L(S)$ is modular by [3, Theorem 1.8].

THEOREM 2.2. *Let* $S = S(d, G_i, \gamma_i)$.*The congruence lattice* $L(S)$ *of* S *is modular if and only if it is strongly semimodular.*

PROOF. It is well known that $L(S)$ being modular implies that $L(S)$ is strongly semimodular.

Suppose that $L(S)$ is strongly semimodular. Then $[\iota, \sigma \vee \mathscr{H}]$ is strongly semimodular because it is a convex sublattice of $L(S)$ (see for instance [11, Lemma 2.2]), and the statement follows from Lemma 2.1.

We recall the following

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DEFINITION 2.3. A lattice L is called *M-symmetric* if the modularity relation M, defined by aMb if and only if for every $x \in [a \cap b, b]$ we have $x = (x \cup a) \cap b$, is symmetric.

From the above result we obtain the following

COROLLARY 2.4. *Let $S = S(d, G_i, \gamma_i)$. The following are equivalent.*

(i) *$L(S)$ is modular.*
(ii) *$L(S)$ is M-symmetric.*
(iii) *$L(S)$ is strongly semimodular.*

PROOF. This result immediately follows from the well known fact that every modular lattice is M-symmetric and every M-symmetric lattice is strongly semimodular.

REMARK 2.5. Among the conditions similar to modularity quoted in [6, p.6] the only one strictly weaker than modularity for the congruence lattice of a regular ω-semigroup is semimodularity.

We recall that a lattice L is called *semimodular* if, for every $a, b \in L$ such that a and b both cover $a \cap b$, then $a \cup b$ covers a and b. The authors studied this condition in [4].

Section 3

When we establish modularity conditions for the congruence lattice of a simple regular ω-semigroup S, we need consider only suitable sublattices of the congruence lattice of S.

First we prove the following.

LEMMA 3.1. *Let $S = S(d, G_i, \gamma)$. The following are equivalent.*

(i) *Let A, B be γ-admissible subsets of G and let μ be a uniform congruence μ covering ι. If $B \supseteq A$ and A is μ-linked, then B is μ-linked.*
(ii) *Let A, B be γ-admissible subsets of G and let μ be a uniform congruence. If $B \supseteq A$ and A is μ-linked, then B is μ-linked.*

PROOF. It is immediate that (ii) *implies* (i).

(i) *implies* (ii). Suppose ν is a uniform congruence which does not cover ι; then $\nu = \nu_1 \vee \nu_2 \vee \cdots \vee \nu_h$ with ν uniform congruences covering ι. Now let A, B be two γ-admissible subsets of G, and suppose that $B \supseteq A$ and that A is ν-linked. Then A is ν_i-linked for every $1 \leq i \leq h$, and so B is ν_i-linked from the hypothesis and by [3, Lemma 1.3] it follows that B is $(\nu_1 \vee \nu_2 \vee \cdots \vee \nu_h)$-linked.

THEOREM 3.2. *Let μ be any uniform congruence of simple regular ω-semigroup S covering ι in the lattice of uniform congruences. Let σ_μ be the least congruence having trace μ. Then $L(S)$ is modular if and only if $[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$ is strongly semimodular.*

PROOF. $[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$ is a sublattice of $L(S)$, so if $L(S)$ is modular, it is also modular and obviously strongly semimodular. Now let $[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$ be a strongly semimodular lattice. In view of Lemma 2.1 it is enough to prove that $[\iota, \sigma \vee \mathscr{H}]$ is strongly semimodular. Suppose that μ is a uniform congruence covering ι, that A and B are γ-admissible subsets with $B \supseteq A$, and that A is μ-linked. Let λ be the congruence having trace ι and $A^\lambda = B$, and let ρ be the congruence having trace μ and $A^\mu = B$. Both the congruences are in $[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$. In fact $\mathscr{H} \wedge \sigma_\mu$ has trace ι, $A^{\mathscr{H} \wedge \sigma_\mu} = \text{rad}\,1$, $\mathscr{H} \vee \sigma_\mu$ has trace μ and $A^{\mathscr{H} \vee \sigma_\mu} = G$. Thus $\lambda \wedge \rho$ has trace ι, $A^{\lambda \wedge \rho} = A$ and ρ covers $\lambda \wedge \rho$. From the hypothesis it follows that $\lambda \vee \rho$ cover λ. But $\lambda \vee \rho$ has trace μ and $A^{\lambda \vee \rho} = \mu - \text{rad}\,B$, so $\mu - \text{rad}\,B = B$ [4, Remark 2.4], and the statement follows from the previous lemma and Proposition 1.6.

Section 4

The following definition is well-known (see [10]).

DEFINITION 4.1. Let S be an inverse semigroup. A full inverse subsemigroup K of S is called a *kernel* in S if it satisfies $ab \in K$ implies $aKb \subseteq K$ $(a, b \in S)$. Denote by $K(S)$ the set of kernels in S ordered by inclusion. The *kernel map* is defined by $\kappa : \rho \to \ker \rho$. The map κ is a complete $\cap$-homomorphism of $L(S)$ onto $K(S)$. The equivalence relation $\mathscr{K}$ on $L(S)$ induced by κ is called the *kernel relation*.

The relation $\mathscr{K}$ is a congruence with respect to intersection but in general it is not a congruence on $L(S)$ (see either [10, Ex.III.4.11], or [8], or [9])

We prove the following

THEOREM 4.2. *Let $S = S(d, G_i, \gamma_i)$. $L(S)$ is modular if and only if $\mathscr{K}$ is a congruence on $L(S)$.*

PROOF. Suppose that $L(S)$ is modular and let λ, ρ be two congruence on S such that $\lambda \mathscr{K} \rho$. Let τ be a congruence on S. We have to prove that $\lambda \vee \tau \mathscr{K} \rho \vee \tau$. Firstly we prove that $\ker(\rho \vee \tau) = \ker \rho \vee \ker \tau$ for every congruence ρ, τ on S. It is immediate that $\ker(\rho \vee \tau) \supseteq \ker \rho \vee \ker \tau$. In order to prove that $\ker(\rho \vee \tau) \subseteq \ker \rho \vee \ker \tau$ we distinguish two cases:

CASE 1. Both the congruences ρ, τ are in $[\iota, \sigma \vee \mathscr{H}]$. Firstly remark that from the definition of A^ρ, it follows immediately that $\ker \rho = \{(m, a, m) | m \in \mathbb{N}, a \in A^\rho\}$

for every congruence $\rho \in [\iota, \sigma \vee \mathscr{H}]$. Hence $\rho \vee \tau \in [\iota, \sigma \vee \mathscr{H}]$ gives $\ker(\rho \vee \tau) = \{(n, c, n) | n \in \mathbb{N}, c \in A^{\rho \vee \tau}\}$. Moreover Proposition 1.6 implies that that $A^{\rho \vee \tau} = A^{\rho} \cdot A^{\tau}$. Since $\ker \rho \vee \ker \tau$ contains the set $\{(m, ab, m) | m \in \mathbb{N}, a \in A^{\rho}, b \in A^{\tau}\}$ we immediately deduce that $\ker(\rho \vee \tau) \subseteq \ker \rho \vee \ker \tau$.

CASE 2. At least one of the congruences ρ, τ is not in $[\iota, \sigma \vee \mathscr{H}]$. Let $\tau \notin [\iota, \sigma \vee \mathscr{H}]$. Then τ is a group congruence. Since $\ker(\sigma \vee \mathscr{H}) \not\supseteq \ker \rho \vee \ker \tau$, $\ker \rho \vee \ker \tau$ is a kernel of a congruence $\theta \notin [\iota, \sigma \vee \mathscr{H}]$. Hence θ is a group congruence greater than ρ and τ, so $\theta \geq \rho \vee \tau$ and $\ker \theta \supseteq \ker(\rho \vee \tau)$.

Now it immediately follows that $\mathscr{K}$ is also a congruence with respect to the join.

Conversely let $\mathscr{K}$ be a congruence. Let B and A be γ-admissible subgroups such that $B \supseteq A$ and let A be μ-linked for some uniform congruence μ. Let ρ, λ, τ be three congruences in $[\iota, \sigma \vee \mathscr{H}]$ such that $\operatorname{tr} \lambda = \iota$, $\operatorname{tr} \rho = \mu$, $\operatorname{tr} \tau = \iota$ and $A^{\lambda} = A$, $A^{\rho} = A$, $A^{\tau} = B$. Thus we have $\lambda \mathscr{K} \rho$. Hence $\lambda \vee \tau \mathscr{K} \rho \vee \tau$. Since both $\lambda \vee \tau$ and $\rho \vee \tau$ are in $[\iota, \sigma \vee \mathscr{H}]$, we have $\ker(\lambda \vee \tau) = \{(m, a, m) | m \in \mathbb{N}, a \in A^{\lambda \vee \tau}\}$ and $\ker(\lambda \vee \tau) = \{(m, a, m) | m \in \mathbb{N}, a \in A^{\rho \vee \tau}\}$, but $A^{\lambda \vee \tau} = A \cdot B = B$ and $A^{\rho \vee \tau} = \mu - \operatorname{rad} A \cdot B = \mu - \operatorname{rad} B$. Thus $\mu - \operatorname{rad} B = B$ and $L(S)$ is modular from Proposition 1.6 and Lemma 2.1.

We extend the previous results in the following.

THEOREM 4.3. *Let $S = S(d, G_i, \gamma_i)$. The following conditions are equivalent.*

(1) *$L(S)$ is modular.*
(2) *$L(S)$ is strongly semimodular.*
(3) *$L(S)$ is M-symmetric.*
(4) *$\mathscr{K}$ is a congruence on $L(S)$.*

Moreover all these equivalent properties are equivalent to the same properties on sublattices of $L(S)$. In fact the following theorem holds.

THEOREM 4.4. *Let $S = S(d, G_i, \gamma_i)$. Let μ be any uniform congruence covering ι and let σ_μ be the least congruence having trace μ. The following are equivalent.*

(1) *$[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$ is modular.*
(2) *$[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$ is strongly semimodular.*
(3) *$[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$ is M-svmmetric*
(4) *$\mathscr{K}$ is a congruence on $[\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$.*
(5) *$L(S)$ is modular.*

In Theorem 3.2 we proved the equivalence between (5) and (2). The other statements can be proved in an analogous way.

DEFINITION 4.5. For each congruence ρ on S, ρ^K denotes the greatest congruence having $\ker \rho$ as kernel.

For each uniform congruence μ, σ_μ denotes the least congruence having trace μ.

We can also prove the following

PROPOSITION 4.6. *Let $S = S(d, G_i, \gamma_i)$. Let μ be a uniform congruence covering ι. Then $L(S)$ is modular if and only if for every congruence ρ with $\mathscr{H} \wedge \sigma_\mu \leq \rho \leq \mathscr{H}$, ρ^K contains σ_μ.*

PROOF. Let μ be a uniform congruence covering ι. Suppose that $\mathscr{H} \wedge \sigma_\mu \leq \rho \leq \mathscr{H}$ implies $\rho^K \geq \sigma_\mu$. Let B and A be two γ-admissible subgroups with $B \supseteq A$ and let A be μ-linked. The congruence ρ has trace ι and $A^\rho = B$ contains $\mathscr{H} \wedge \sigma_\mu$. The congruence ρ^K is in $[\iota, \sigma \vee \mathscr{H}]$ because $\operatorname{tr} \rho^K = \omega_E = \operatorname{tr}(\sigma \vee \mathscr{H})$ and $\ker \rho^K = \{(m, B, m) | m \in \mathbb{N}\} \subseteq \{(m, G, m) | m \in \mathbb{N}\} = \ker(\sigma \vee \mathscr{H})$. So ρ^K has trace ν and $A^{\rho^K} = B$ is ν-linked. Since $\rho^K \geq \sigma_\mu$, $\nu \geq \mu$ and B is μ-linked by [4, Remark 2.4]. Hence $L(S)$ is modular. Conversely let $L(S)$ be modular and let ρ belong to $[\mathscr{H} \wedge \sigma_\mu, \mathscr{H}]$ for a uniform congruence μ covering ι. Then ρ has trace ι and $A^\rho = B$ with $B \supseteq \mu - \operatorname{rad} 1$. Now $\mu - \operatorname{rad} 1$ is μ-linked and by hypothesis B is also μ-linked. Let ν be the trace of ρ^K, $\nu \geq \mu$; since $\ker \rho^K = \{(m, B, m) | m \in \mathbb{N}\}$, we deduce $\rho^K \geq \sigma_\mu$.

Section 5

Now we examine the general case of regular ω-semigroups.

It is well-known that a non-simple regular ω-semigroup S is either an ω-chain of groups or the disjoint union of a finite chain of groups $H = [n, H_j, \phi_j]$ and a simple regular ω-semigroup $K = K(d, K_i, \psi_i)$ which is an ideal of S. In the latter case, as in [3, 4], we will use the notation $S = S([n, H_j, \phi_j]; K, \phi)$ where ϕ is the homomorphism which induces the retract extension of K by H^0, and which is actually a homomorphism of H into K_0.

Modularity, M-symmetry and strong semimodularity of the congruence lattice of ω-chains $[\Omega, H_j, \phi_j]$ of groups are equivalent to the triviality of ϕ_j for every $j \in \mathbb{N}$ (see, for instance [11, Th.6.5 and Th.6.6]).

For a regular ω-semigroup $S = S([n, H_j, \phi_j]; K, \phi)$ we deduce from [11, Theorem 6.6] that $L(S)$ is modular if and only if K has a modular lattice of congruences and ϕ and all ϕ_j are trivial. Moreover if $L(S)$ is strongly semimodular then ϕ and all ϕ_j are trivial from [11, Theorem 6.5] and K is strongly semimodular. Then we immediately deduce from Corollary 2.4 that M-symmetry and strong semimodularity of the congruence lattice of $S = S([n, H_j, \phi_j]; K, \phi)$ are equivalent to modularity.

As concerns the relation $\mathscr{H}$, Petrich [9] proved the following

THEOREM D. *Let* $[Y; S_\alpha, \phi_{\alpha,\beta}]$ *be a strong semilattice of simple regular semigroups.* $\mathscr{H}$ *is a congruence if and only if*

(1) $\mathscr{H}$ *is a congruence on* $L(S_\alpha)$ *for every* $\alpha \in Y$,

(2) $S_\alpha\phi_{\alpha,\beta} \subseteq E(S_\beta)$ *whenever* $\alpha > \beta$.

Then we easily deduce the following

THEOREM 5.1. *Let* S *be a regular* ω*-semigroup. The following conditions are equivalent*

(1) $L(S)$ *is modular.*

(2) $L(S)$ *is strongly semimodular.*

(3) $L(S)$ *is* M*-symmetric.*

(4) $\mathscr{H}$ *is a congruence on* $L(S)$.

PROOF. We just observed the equivalence of conditions (1), (2), (3). If (1) holds then (4) follows from Theorem D and Theorem 4.3.

Suppose that (4) holds. Then if S is simple, (1) follows from Theorem 4.3. If S is an ω-chain of groups Theorem D implies that ϕ_j are trivial for every $j \in \mathbb{N}$, whence (1) follows. If $S = S([n, H_j, \phi_j]; K, \phi)$, from Theorem D we deduce that $H_{n-1}\phi \subseteq E(K_0)$ and $H_j\phi_j \subseteq E(H_{j+1})$ for $0 \leq j \leq n-2$, whence ϕ and ϕ_j are trivial. Moreover Theorem D implies that $\mathscr{H}$ is a congruence on $L(K)$. So by Theorem 4.3, $L(K)$ is modular and $L(S)$ is modular by [3, Lemma 3.4].

Section 6

Now we compare the condition of modularity for $L(S)$, with the condition of θ-modularity of S. We recall the following

DEFINITION 6.1 ([12]). A regular semigroup S is called *θ-modular* if for all congruences λ, ρ, τ on S, conditions $\lambda \wedge \rho = \lambda \wedge \tau$, $\lambda \vee \rho = \lambda \vee \tau$, $\rho \geq \tau$ and $\operatorname{tr}\rho = \operatorname{tr}\tau$ imply $\rho = \tau$.

We can prove the following

LEMMA 6.2. *Let* S *be a regular semigroup, with a modular lattice of congruence traces. If* S *is* θ*-modular then* $L(S)$ *is modular.*

PROOF. Let S be θ-modular, and suppose that there exist three congruences λ, ρ, τ on S with $\lambda \wedge \rho = \lambda \wedge \tau$, $\lambda \vee \rho = \lambda \vee \tau$, $\rho \geq \tau$ and $\operatorname{tr}\rho > \operatorname{tr}\tau$. Since in a regular semigroup every congruence is uniquely determined by the pair of its trace and kernel, it follows that $\operatorname{tr}\lambda$ is neither comparable with $\operatorname{tr}\rho$ nor with $\operatorname{tr}\tau$. So we have three congruences with $\operatorname{tr}\rho > \operatorname{tr}\tau$, $\operatorname{tr}(\lambda \wedge \rho) = \operatorname{tr}(\lambda \wedge \tau)$, and $\operatorname{tr}(\lambda \vee \rho) = \operatorname{tr}(\lambda \vee \tau)$, which contradicts the modularity of the lattice of congruence traces.

PROPOSITION 6.3. *For a regular ω-semigroup S the following are equivalent.*

(i) *$L(S)$ is modular.*
(ii) *S is θ-modular.*

PROOF. It is obvious that $L(S)$ modular implies S is θ-modular. Now, let S be θ-modular. The traces, being a sublattice of the congruences lattice on a chain, form a modular lattice and the statement follows from the previous lemma.

REMARK 6.4. Notice that a result analogous to Theorem 4.4 can be proved for θ-modularity. Actually θ-modularity of a simple regular ω-semigroup S can be proved to be equivalent to the following condition: let μ be a uniform congruence covering ι. Then for all congruences $\lambda, \rho, \tau \in [\mathscr{H} \wedge \sigma_\mu, \mathscr{H} \vee \sigma_\mu]$, relations $\lambda \wedge \rho = \lambda \wedge \tau$, $\lambda \vee \rho = \lambda \vee \tau$, $\rho \geq \tau$ and $\operatorname{tr}\rho = \operatorname{tr}\tau$ imply $\rho = \tau$.

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