Can. J. Math., Vol. XXIII, No. 2, 1971, pp. 247–256

SOLUTION TO A PROBLEM OF SPECTOR

A. H. LACHLAN

Introduction. In [**6**, p. 586] Spector asked whether given a number e there exists a unary partial function ϵ from the natural numbers into $\{0, 1\}$ with coinfinite domain such that for any function f into $\{0, 1\}$ extending ϵ it is the case that

$$[g \text{ is recursive in } f \text{ with Gödel number } e] \\ \rightarrow [g \text{ is recursive or } f \text{ is recursive in } g].$$

We answer this question affirmatively in Corollary 1 below and show that ϵ can be made partial recursive (p.r.) with recursive domain. The reader who is familiar with Spector's paper [**6**] will find the new trick that is required in the first paragraph of the proof of Lemma 2 below.

From one point of view, this is a theorem about trees which branch twice at every node. We shall formulate a generalization which applies to trees which branch n times at every node. This generalization was inspired by Thomason's paper [**7**]. The generalization is combined with some ideas developed in [**2**] to yield a proof that any countable upper semilattice which can be represented as an initial segment of the many-one degrees can be simultaneously represented as an initial segment of the degrees. We also indicate another application, again inspired by [**7**], to the problem of embedding finite lattices as initial segments of the degrees, and we partially solve this problem here. However, recently, Lerman completely solved the problem (see [**4**]), when he showed that every finite lattice can be represented as an initial segment of the degrees.

1. Preliminaries. Our notation and terminology is in the style of Shoenfield [**5**]. By a *string* we shall mean a finite, possibly empty, sequence of zeroes and ones. Lower case Greek letters will be used to denote strings and partial functions from N, the set of natural numbers, into itself. The number of elements in a string σ is called its *length* and is denoted $\mathrm{lh}(\sigma)$. A string σ of length l will be regarded as identical with the finite function σ defined by

$$\sigma(x) = \begin{cases} (x+1)\text{st member of } \sigma \text{ if } x < l, \\ \text{undefined otherwise.} \end{cases}$$

The string whose members are $i_0, i_1, \ldots, i_{l-1}$ in that order will be denoted by $\langle i_0, i_1, \ldots, i_{l-1}\rangle$. If σ and τ are strings, then $\sigma * \tau$ denotes the string

Received May 8, 1970.

formed by juxtaposing τ to the right of σ. The strings σ and τ are said to be *adjacent* written $\mathrm{Adj}(\sigma, \tau)$ if σ and τ differ on just one argument x, say, and further, $\sigma(x) = 0$. Note that $\mathrm{Adj}(\sigma, \tau)$ implies that σ and τ have the same length. The empty string is denoted by $\emptyset$.

A *tree* is a mapping T of the set of all strings into itself such that for all σ, $T(\sigma * \langle 0\rangle)$ and $T(\sigma * \langle 1\rangle)$ are incompatible extensions of $T(\sigma)$. Since strings can be coded by natural numbers in an effective way, the notion of a *recursive tree* is clear. If τ extends σ, written $\sigma \subseteq \tau$, then $\tau - \sigma$ denotes the string ν such that $\tau = \sigma * \nu$. The tree T is said to be a 1-*tree* if for all σ:

(i) for $i = 0, 1$ the string $T(\sigma * \langle i\rangle) - T(\sigma)$ depends only on i and the length of σ, and

(ii) $T(\sigma * \langle 0\rangle)$ and $T(\sigma * \langle 1\rangle)$ are adjacent.

The reason for this nomenclature is that if T is a 1-tree, $A \subseteq N$ has characteristic function f, and $B \subseteq N$ is the set whose characteristic function extends $T(\langle f(0), \ldots, f(n-1)\rangle)$ for all n, then A is uniformly one-to-one reducible to B, and conversely B is uniformly the disjoint union of a recursive set and a set one-to-one reducible to A.

The domain and range of a map M are abbreviated to dom M and rng M, respectively. A set $A \subseteq N$ is said to be *on* the tree T if every initial segment of the characteristic function of A has an extension in rng T.

For any unary function f, let $\{e\}^f$ denote the eth partial function p.r. in f. If A is a set, then $\{e\}^A$ denotes $\{e\}^f$, where f is the characteristic function of A. In the usual way we can regard $\{e\}^\sigma$ as being defined and in fact as being a finite function uniformly recursive in e and σ. We say that σ and τ *split for e* if for some n, $\{e\}^\sigma(n)$ and $\{e\}^\tau(n)$ are both defined and different. T is called *e-regular* if for every σ, $T(\sigma * \langle 0\rangle)$ and $T(\sigma * \langle 1\rangle)$ split for e.

2. Solution of Spector's problem. Given e we show how to construct a recursive 1-tree T such that either $\{e\}^A$ is not total for any A on T, or $\{e\}^A$ is recursive for every A on T, or A is recursive in $\{e\}^A$ for every A on T.

LEMMA 1. *For each e there exists a recursive* 1-*tree T such that either $\{e\}^A$ is total for all A on T, or there exists x such that $\{e\}^A(x)$ is not defined for any A on T.*

Proof. We attempt to construct a 1-tree T such that $\{e\}^A$ is total for all A on T as follows. Define $T(\emptyset) = \emptyset$. For induction purposes, suppose that T has been defined on all strings of length $\leqq l$ such that for any σ of length l and any $x < l$, $\{e\}^{T(\sigma)}(x)$ is defined. Let $\tau_0, \ldots, \tau_m$ be all the strings of length l. Let Σ be the set of strings σ such that $\{e\}^\sigma(l)$ is defined. Choose strings $\sigma_0, \sigma_0', \ldots, \sigma_m, \sigma_m'$ in that order such that each is an extension of the one before and such that for each $i \leqq m$, $T(\tau_i) * \langle 0\rangle * \sigma_i$ and $T(\tau_i) * \langle 1\rangle * \sigma_i'$ are both in Σ. If one of the choices cannot be made, say that of σ_j, $j > 0$, then $T(\tau_j) * \langle 0\rangle * \sigma_{j-1}'$ has no extension in Σ, whence T' defined by

$$T'(\sigma) = T(\tau_j) * \langle 0\rangle * \sigma_{j-1}' * \sigma$$

establishes the lemma. Otherwise, T can be appropriately defined on sequences of length $l + 1$ by setting

$$T(\tau_i * \langle j \rangle) = T(\tau_i) * \langle j \rangle * \sigma_m'$$

for each $i \leqq m$ and $j \leqq 1$. Since $\sigma_0, \sigma_0', \ldots, \sigma_m, \sigma_m'$ can be chosen effectively if at all, T can be made recursive.

LEMMA 2. *Let $e_0, e_1, \ldots, e_n$ be such that for each $i \leqq n$ and each set A, $\{e_i\}^A$ is total. Either there exist σ, σ' such that* Adj(σ, σ') *and such that for each $i \leqq n$, σ and σ' split for e_i, or for some $i \leqq n$ there exists a recursive* 1*-tree T such that $\{e_i\}^A$ is independent of A for A on T.*

Proof. We first treat the case $n = 0$, and for brevity e_0 is replaced by e. If no pair τ, τ' splits for e, then I, the identity tree, is a 1-tree such that $\{e\}^A$ is independent of A for A on I, and so the lemma is true. Suppose that τ and τ' split for e and let x be a number such that $\{e\}^\tau(x)$, $\{e\}^{\tau'}(x)$ are both defined and distinct. Without loss of generality, τ and τ' have the same length. Choose a sequence $\tau_0, \tau_1, \ldots, \tau_m$ such that $\tau_0 = \tau$, $\tau_m = \tau'$ and such that Adj(τ_i, τ_{i+1}) for each $i < m$. Choose ν_0 such that $\{e\}^\nu(x)$ is defined when $\nu = \tau_0 * \nu_0$, then choose ν_1 such that $\{e\}^\nu(x)$ is defined when $\nu = \tau_1 * \nu_0 * \nu_1$, and so on. For $i \leqq m$ define $\omega_i = \tau_i * \nu_0 * \nu_1 * \ldots * \nu_m$. Then Adj$(\omega_i, \omega_{i+1})$ for each $i < m$, and $\{e\}^{\omega_i}(x)$ is defined for each $i \leqq m$. Since $\{e\}^{\omega_0}(x) \neq \{e\}^{\omega_m}(x)$, there exists $q < m$ such that $\{e\}^{\omega_q}(x) \neq \{e\}^{\omega_{q+1}}(x)$. The conclusion of the lemma is satisfied by taking τ, τ' to be ω_q, ω_{q+1}, respectively.

For the case $n = k + 1$ assume, for induction purposes, that the lemma holds for $n = k$. For *reductio ad absurdum* let $e_0, e_1, \ldots, e_k, e$ constitute a counterexample for $n = k + 1$. By the lemma for $n = k$ there exist σ_0, σ_0' such that Adj(σ_0, σ_0') and such that σ_0, σ_0' split for each e_i, $i \leqq k$. Suppose that σ_j and σ_j' have been defined for each $j \leqq l$ such that $\sigma_0 * \sigma_1 * \ldots * \sigma_j$, $\sigma_0 * \sigma_1 * \ldots * \sigma_{j-1} * \sigma_j'$ split for each e_i, $i \leqq k$. For each $i \leqq k$, choose e_i' such that for all ξ, $\{e_i'\}^\xi = \{e_i\}^\eta$, where $\eta = \sigma_0 * \sigma_1 \ldots * \sigma_e * \xi$. Now apply the lemma for $n = k$ to $e_0', \ldots, e_k'$. If there were a 1-tree T' and $i \leqq k$ such that $\{e_i'\}^A$ was independent of A for A on T', defining T by $T(\xi) = \sigma_0 * \ldots * \sigma_l * T'(\xi)$, then $\{e_i\}^A$ would be independent of A for A on T, contradicting the assumption that $e_0, e_1, \ldots, e_k, e$ constitute a counterexample for $n = k + 1$. Hence there exist $\sigma_{l+1}, \sigma_{l+1}'$ such that Adj$(\sigma_{l+1}, \sigma_{l+1}')$ and such that for each $i \leqq k$, σ_{l+1} and σ_{l+1}' split for e_i'. Thus σ_j and σ_j' can be found for all j and further may clearly be found effectively. Consider the unique recursive 1-tree U whose range consists of all strings of the form $\xi_1 * \ldots * \xi_n$, where n runs through the natural numbers and ξ_i is σ_i or σ_i'. Since the lemma fails for $e_0, \ldots, e_k, e$, $\{e\}^A$ is not independent of A for A on U. Thus for some x there exist λ and λ' in rng U such that $\{e\}^\lambda(x)$ and $\{e\}^{\lambda'}(x)$ are defined and distinct. Since $\{e\}^A(x)$ is defined for all A, we may suppose that λ and λ' have the same length l and that $\{e\}^\sigma(x)$ is defined for every σ of length l on U. Without loss of generality, choose λ such that $U^{-1}(\lambda)$ is a

https://doi.org/10.4153/CJM-1971-024-7 Published online by Cambridge University Press

sequence all of whose members are zero. Choose λ' amongst all the sequences of length l on U such that $\{e\}^{\lambda'}(x) \neq \{e\}^{\lambda}(x)$ and such that $\mu y[U^{-1}(\lambda', y) \neq 0]$ is as large as possible. Let ν and ν' be defined by $\nu' = U^{-1}(\lambda')$,

$$\nu(x) = \begin{cases} 0 & \text{if } x \leqq m, \\ U^{-1}(\lambda', x) & \text{if } m < x < l, \end{cases}$$

where $m = \mu y[U^{-1}(\lambda', y) \neq 0]$. Then $U(\nu)$ and $U(\nu')$ extend $\sigma_0 * \ldots * \sigma_m$ and $\sigma_0 * \ldots * \sigma_{m-1} * \sigma_m'$, respectively, and hence split for e_i for each $i \leqq k$. From the definition of ν, either $U(\nu) = \lambda$ which happens if $l = m + 1$, or $\mu y[\nu(y) \neq 0]$ if it exists is $>m$ which happens if $l > m + 1$. In the latter case, $\{e\}^{\lambda}(x) = \{e\}^{U(\nu)}(x)$ by choice of λ' while in the former case this is obvious from $\lambda = U(\nu)$. Hence $U(\nu)$ and $U(\nu')$ also split for e. But $\mathrm{Adj}(U(\nu), U(\nu'))$ by inspection and since the conclusion is satisfied for $e_0, \ldots, e_k, e$ by letting σ and σ' be $U(\nu)$ and $U(\nu')$ respectively; the lemma is proved.

THEOREM 1. *For each e there exists a recursive 1-tree T such that either for some x, $\{e\}^A(x)$ is not defined for any A on T, or $\{e\}^A$ is a fixed recursive function for A on T, or T is e-regular and $\{e\}^A$ is total for every A on T.*

Proof. From Lemma 1 we obtain either the conclusion of the theorem immediately or a 1-tree T_1 such that $\{e\}^A$ is total for all A on T_1. Replacing e by e' such that $\{e'\}^\sigma = \{e\}^{T_1(\sigma)}$ for all σ, suppose that T' is a 1-tree satisfying the conclusion of the theorem for e'; then $T_1 \circ T'$, the composite function, is clearly a 1-tree satisfying the conclusion for e. Thus we may assume that $\{e\}^A$ is total for all A. We attempt to construct a recursive 1-tree which is e-regular. Define $T(\emptyset) = \emptyset$, and suppose, for induction purposes, that T has been defined on all strings of length $\leqq l$. Let $\tau_0, \ldots, \tau_n$ be all the strings of length l. For each $i \leqq n$ choose e_i such that $\{e_i\}^\sigma = \{e\}^{T(\tau_i)*\sigma}$ for all σ. From Lemma 2, either there exists $i \leqq n$ and a recursive 1-tree T_2 such that $\{e_i\}^A$ is independent of A for A on T_2, or there exist σ and σ' which split for each e_i, $i \leqq n$, and such that $\mathrm{Adj}(\sigma, \sigma')$. In the former case, T_2' defined by $T_2'(\sigma) = T(\tau_i) * T_2(\sigma)$ is a 1-tree, establishing the theorem. In the latter case, σ and σ' can be found effectively, and T may be suitably extended to sequences of length $l + 1$ by letting $T(\tau_i * \langle 0 \rangle) = T(\tau_i) * \sigma$, $T(\tau_i * \langle 1 \rangle) = T(\tau_i) * \sigma'$ for each $i \leqq n$. This completes the proof.

COROLLARY 1. *For every e there exists a p.r. function ϵ in $(2^N)^*$ whose domain is recursive and coinfinite such that either*

(i) *$\{e\}^f$ is not total for any $f \supseteq \epsilon$, $f \in 2^N$, or*

(ii) *for every $f \supseteq \epsilon$, $f \in 2^N$, $\{e\}^f$ is recursive, or*

(iii) *for every $f \supseteq \epsilon$, $f \in 2^N$, $\{e\}^f$ is total and has the same degree as f.*

Proof. Applying Theorem 1 to e we obtain a recursive 1-tree T satisfying the conclusion of the theorem. From the definition of 1-tree, it is clear that the length of $T(\sigma)$ depends only on the length of σ. For σ of length i denote the length of $T(\sigma)$ by l_i and let $T(\sigma * \langle 0 \rangle) - T(\sigma)$ and $T(\sigma * \langle 1 \rangle) - T(\sigma)$ be

https://doi.org/10.4153/CJM-1971-024-7 Published online by Cambridge University Press

denoted by σ_i^0 and σ_i^1. Let x_i be the unique x such that $\sigma_i^0(x)$ and $\sigma_i^1(x)$ are defined and distinct. Then f is the characteristic function of a set on T if and only if f extends $T(\emptyset)$ and for every i, f extends either $\lambda x \sigma_i^0(x - l_i)$ or $\lambda x \sigma_i^1(x - l_i)$, where $x - l_i$ is regarded as undefined if $x < l_i$. Let $\epsilon \in (2^N)^*$ be defined by

$$\epsilon(x) = \begin{cases} T(\emptyset)(x) & \text{if } x < l_0, \\ \sigma_i^0(x - l_i) & \text{if } \sigma_i^0(x - l_i) = \sigma_i^1(x - l_i) \text{ and } l_i \leqq x < l_{i+1}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then ϵ is a p.r. function and the complement of its domain is $\{x_0, x_1, \ldots\}$. Hence the domain of ϵ is recursive and coinfinite. Clearly, f is the characteristic function of a set on T if and only if $f \in 2^N$ and $\epsilon \subseteq f$. The conclusion of the corollary is now immediate from the conclusion of the theorem and [**5**, Lemma 3]. The referee has pointed out to me that the hyperarithmetic analogue of Corollary 1 is also true.

COROLLARY 2. *There exists a set A such that the degree of A is minimal and such that the m-degree of A is minimal.*

Proof. Define a sequence $\langle \epsilon_i \rangle_{i<\omega}$ of members of $(2^N)^*$ as follows. Let ϵ_0 be the completely undefined function. For induction purposes, suppose that ϵ_e has been defined, is p.r., and has domain which is recursive and coinfinite. Let $\langle a_i \rangle_{i<\omega}$ be a recursive enumeration without repetitions of $N - \text{dom } \epsilon_e$. Let $A \subseteq N$ have characteristic function $f \supseteq \epsilon_e$, define $H(A) = \{x \mid f(b_x) = 1\}$. Choose e' such that for all A, $\{e'\}^{H(A)} = \{e\}^A$. Now applying Corollary 1 we obtain $\epsilon' \in (2^N)^*$ such that either for all $f' \in 2^N$, $f' \supseteq \epsilon'$, $\{e'\}^{A'}$ is recursive, or for all $f' \in 2^N, f' \supseteq \epsilon'$, $\{e'\}^{A'}$ has the same degree as A', the set of which f' is the characteristic function. Now define

$$\epsilon_{e+1}(x) = \begin{cases} \epsilon_e(x) & \text{if } x \in \text{dom } \epsilon_e, \\ \epsilon'(y) & \text{if } x = a_y \text{ for some } y, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

If the characteristic function of A extends ϵ_{e+1}, then the characteristic function of $H(A)$ extends ϵ'. Hence either for all such A, $\{e\}^A$ is recursive, or for all such A, $\{e\}^A$ has the same degree as A. Let $f \in 2^N$ be a function extending ϵ_e for every e; then it may easily be seen that the set A of which f is the characteristic function has minimal degree.

To see that the m-degree of A is minimal, suppose that h is a unary recursive function and that for all B, $\{e\}^B$ is $g \circ h$, where g is the characteristic function of B. Then we have $\text{rng } h - \text{dom } \epsilon_{e+1}$ finite, or $\text{rng } h \cup \text{dom } \epsilon_{e+1}$ cofinite. Otherwise we can construct a set B with characteristic function $\supseteq \epsilon_{e+1}$ such that $\{e\}^B$ is not recursive yet has degree strictly less than that of B. Since A has characteristic function $\supseteq \epsilon_{e+1}$, if $\{e\}^A$ is not recursive, then A is m-reducible to $\{e\}^A$.

3. A generalization. We shall now state a generalization of Corollary 1 which pertains to trees which branch n times at each node rather than just twice, where n is an arbitrary integer $\geqq 2$.

Let E be an equivalence relation on N, then E is called *strongly recursive* if there is a strongly recursively enumerable sequence $\langle F_i \rangle$ of finite sets such that F_i is the equivalence class of i with respect to E. The set of equivalence classes is denoted by N/E. Let $n \geqq 2$; then by $n^{N/E}$ we mean the class of functions f mapping N into $\{0, 1, \ldots, n-1\}$ such that $E(x, y)$ implies $f(x) = f(y)$. Let $(n^{N/E})^*$ denote the corresponding set of partial functions. Let F be an equivalence relation on $\{0, 1, \ldots, n-1\}$ and $f \in n^{N/E}$; then f/F is the function defined by

$$(f/F)(x) = \mu y[y \in \{0, 1, \ldots, n-1\} \ \& \ F(y, f(x))].$$

Let $\{e\}^f$ denote the eth partial function p.r. in f. With this notation we have the following result.

THEOREM 2. *For every e there exists a strongly recursive equivalence relation E and a p.r. function $\epsilon \in (n^{N/E})^*$, with domain which is recursive and coinfinite such that either*

(i) *$\{e\}^f$ is not total for any $f \supseteq \epsilon$, $f \in n^{N/E}$, or*

(ii) *there exists an equivalence relation F on n such that for each $f \supseteq \epsilon$, $f \in n^{N/E}$, $\{e\}^f$ is total and has the same degree as f/F.*

The case $n = 2$ is Corollary 1 except that we have (in Corollary 1) the additional information that E can be taken to be the equality relation. In general, one cannot take E to be equality. To see this, let $n = 3$, let "string" now mean a finite sequence all of whose members are in $\{0, 1, 2\}$ and let the definition of "tree" be modified accordingly. Define a map Ψ from the set of strings into $\{0, 1\}$ by: $\Psi(\emptyset) = 0$, and for all σ, $\Psi(\sigma * \langle 0 \rangle) = \Psi(\sigma * \langle 1 \rangle) = 1 - \Psi(\sigma)$ and $\Psi(\sigma * \langle 2 \rangle) = \Psi(\sigma)$. Now choose e such that for all $f \in 3^N$,

$$\{e\}^f(x) = \begin{cases} f(x) & \text{if } \Psi(\langle f(0), \ldots, f(x-1) \rangle) = 0, \\ 0 & \text{if } \Psi(\langle f(0), \ldots, f(x-1) \rangle) = 1, \text{ and } f(x) = 0 \text{ or } 1, \\ 2 & \text{otherwise.} \end{cases}$$

Suppose for proof by contradiction that the conclusion of Theorem 2 holds for e with E being equality. Under these assumptions we have the following result.

LEMMA 3. *Let σ be a string consistent with ϵ, i.e. for all*

$$x \in \text{dom}\, \epsilon \cap \{y \mid y < \text{lh}(\sigma)\}, \qquad \epsilon(x) = \sigma(x).$$

For $j < 2$ there exists an extension τ_j of σ, consistent with ϵ, such that $\Psi(\tau_j) = j$ and $\text{lh}(\tau_j) \notin \text{dom}\, \epsilon$.

Proof. Let m and m' in that order be the first two members of $N - \text{dom}\, \epsilon$ which are $\geqq \text{lh}(\sigma)$. For $k \leqq 2$, choose an extension σ_k of σ consistent with ϵ

such that $\text{lh}(\sigma_k) = m'$, $\sigma_k(m) = k$, and otherwise $\sigma_k(x)$ is independent of k. Let σ' be the unique string of length m extended by σ_k. From the definition of Ψ we have $\Psi(\sigma' * \langle 0 \rangle) = \Psi(\sigma' * \langle 1 \rangle) \neq \Psi(\sigma' * \langle 2 \rangle)$. Further, since $\sigma_0 - \sigma' * \langle 0 \rangle = \sigma_1 - \sigma' * \langle 1 \rangle = \sigma_2 - \sigma' * \langle 2 \rangle$, it is easy to see that $\Psi(\sigma_0) = \Psi(\sigma_1) \neq \Psi(\sigma_2)$. Thus we can take τ_0, τ_1 to be σ_0, σ_2 in some order.

From the lemma with $j = 0$ we can construct a tree T such that for every string σ, $\Psi(T(\sigma)) = 0$; $T(\sigma * \langle 0 \rangle)$, $T(\sigma * \langle 1 \rangle)$, and $T(\sigma * \langle 2 \rangle)$ all have the same length and differ only at the mth place where m is the length of $T(\sigma)$. Then every function on T extends ϵ, and for all f on T, $\{e\}^f = f$. It follows easily that F in the conclusion of Theorem 2 must be equality. But arguing similarly from the lemma with $j = 1$ we can show that F must be the equivalence relation whose equivalence classes are $\{0, 1\}$ and $\{2\}$. This contradiction proves the claim that E cannot always be taken as equality.

The proof of Theorem 2 is straightforward. The strings now have entries from $\{0, 1, \ldots, n - 1\}$ and the corresponding trees branch n times at each node. We first show, as in Lemma 1, that we may suppose that $\{e\}^f$ is defined for every f. Next we construct the p.r. function ϵ and the equivalence relation E. It should be sufficient for us to indicate a suitable choice of F. For each string σ define

$$C(\sigma) = \{(j, k) | j, k < n \;\&\; (\exists f \in n^N)(\exists g \in n^N)\exists x \exists y[f \supseteq \sigma \;\&\; g \supseteq \sigma \;\&\; \{e\}^f(x) \neq \{e\}^g(x) \;\&\; \forall z[f(z) \neq g(z) \rightarrow y = z] \;\&\; f(y) = j \;\&\; g(y) = k]\}.$$

Choose σ_0 so that $C(\sigma_0)$ is minimal with respect to inclusion; then $C(\sigma) = C(\sigma_0)$ for every $\sigma \supseteq \sigma_0$. Let F be the relation on $\{0, 1, \ldots, n - 1\}$ defined by: $F(x, y)$ if and only if $(x, y) \notin C(\sigma_0)$. Further, ϵ is defined so as to extend σ_0.

4. Simultaneous initial segments of the degrees and the m-degrees. Using Theorem 2 we can sharpen the main theorem of [**2**] to obtain the following result.

THEOREM 3. *Let L be a countable upper semilattice with 0 which has the closure property. There exists an order-preserving map κ: $L \rightarrow L_m$, where L_m denotes the upper semilattice of m-degrees, such that κ is one-to-one onto an initial segment of the m-degrees, and such that κ_T, the map of L into the degrees induced by κ, is one-to-one onto an initial segment of the degrees.*

Proof. We shall assume that the reader is familiar with the construction of [**2**] and indicate the changes which are necessary. The principal change we make is in now defining a *recursive partition* to be an infinite class $\mathscr{R}$ of pairwise disjoint non-empty sets such that $\bigcup \mathscr{R} = N$, and such that there exist just two infinite recursive sets $U, V \in \mathscr{R}$, and such that $\mathscr{R} - \{U, V\}$ is canonically enumerable. A *quintuple* $(U, V, D, \pi, \mathscr{R})$ satisfies the same conditions as before and in addition U, V must be the two infinite members of $\mathscr{R}$, and $\pi(0)$ must be $U \cup V$.

https://doi.org/10.4153/CJM-1971-024-7 Published online by Cambridge University Press

The theorem is proved by means of the following four propositions.

PROPOSITION 1. *Let* $(U, V, D, \pi, \mathscr{R})$ *be a quintuple,* D^* *a finite distributive lattice with* $0 \neq 1$, *and* $\chi: D \rightarrow D^*$ *a map preserving unions,* 0, *and* 1. *Then there exists a quintuple* $(U, V^*, D^*, \pi^*, \mathscr{R}^*)$ *such that* $U \cap V^* = \emptyset$, $V \subseteq V^*$, $\mathscr{R}$ *is a refinement of* $\mathscr{R}^*$, *and* $\pi^*\chi(d) = \mathscr{R}^*\pi(d)$ *for all d in D.*

PROPOSITION 2. *Let* $(U, V, D, \pi, \mathscr{R})$ *be a quintuple,* d_1, d_2 *members of D such that* $d_1 \nleqq d_2$, *let e be in N, and* W_1, W_2 *sets equivalent to* $\pi(d_1)$, $\pi(d_2)$, *respectively with respect to* $\mathscr{R}$. *Then there is a quintuple* $(U^*, V^*, D, \pi^*, \mathscr{R}^*)$ *such that* $\mathscr{R}$ *is a refinement of* $\mathscr{R}^*$, $\pi^*(d) = \mathscr{R}^*\pi(d)$ *for all d in D,* $U \subseteq U^*$, $V \subseteq V^*$ *and such that for some n in* W_1 *there is no X,* $U^* \subseteq X \subseteq N - V^*$, *for which it is the case that* $\{e\}^{f_2}(n) = f_1(n)$, *where* f_1 *and* f_2 *are the characteristic functions of* $X \cap W_1$ *and* $X \cap W_2$, *respectively.*

PROPOSITION 3. *Let* $(U, V, D, \pi, \mathscr{R})$ *be a quintuple and W an infinite recursively enumerable set. There is a quintuple* $(U^*, V^*, D, \pi^*, \mathscr{R}^*)$ *such that* $\mathscr{R}$ *is a refinement of* $\mathscr{R}^*$, $\pi^*(d) = \mathscr{R}^*\pi(d)$ *for all d in D, and for some d in D,* $\pi^*(d)$ *and* $\mathscr{R}^*(W)$ *differ at most finitely.*

PROPOSITION 4. *Let* $(U, V, D, \pi, \mathscr{R})$ *be a quintuple and* $e \in N$. *There is a quintuple* $(U^*, V^*, D, \pi^*, \mathscr{R}^*)$ *and a recursive set W such that* $\mathscr{R}$ *is a refinement of* $\mathscr{R}^*$, $\pi^*(d) = \mathscr{R}^*\pi(d)$ *for all d in D,* $U \subseteq U^*$, $V \subseteq V^*$, *and* $\{e\}^X$ *has the same degree as* $X \cap W$ *for any X satisfying*

$$\{e\}^X \textit{ total } \& \ U^* \subseteq X \subseteq N - V^* \ \& \ X = \mathscr{R}^*(X).$$

To prove Theorem 3 by means of the propositions, we construct, as in [**2**], a sequence $\langle Q_i \rangle$ of quintuples with the following properties:

(q1) $U_i \subseteq U_{i+1}$ and $V_i \subseteq V_{i+1}$ for all i,

(q2), (q3), and (q4) as in [**2**],

(q5) For $i < k$ define $\theta_{ik} = \theta_{k-1}\theta_{k-2} \ldots \theta_i$, for all i and e_1, e_2 in E_i and for each e in N there exists $k > i$ such that either $\theta_{ik}(e_1) \leqq \theta_{ik}(e_2)$ or there exists n in $\pi_i(e_1)$ such that for no U, $U_k \subseteq U \subseteq N - V_k$ is it the case that $\{e\}^{f_2}(n) = f_1(n)$, where f_1 and f_2 are the characteristic functions of $U \cap \pi_i(e_i)$ and $U \cap \pi_i(e_2)$, respectively.

(q6) as in [**2**],

(q7) for every e there exists i and a recursive set W such that if U is any set closed under $\mathscr{R}_i$ and satisfying $U_i \subseteq U \subseteq N - V_i$, then $\{e\}^N$ is either not total or has the same degree as $U \cap W$.

The sequence $\langle Q_i \rangle$ is constructed much as before: the strengthening of Proposition 3 corresponds to the strengthening of (q5) and Proposition 4 yields (q7). The map κ is constructed as in [**2**] and shown to be an order-preserving map of L^* onto an initial segment of L_m. The strengthening of (q5) tells us that the induced map κ_T of L^* into the degrees is one-to-one, and (q7) ensures that any degree $\leqq \kappa_T(1)$ is the degree of some m-degree $\leqq \kappa(1)$. This completes the proof of Theorem 3 except for the following.

Proof of Proposition 4. This is where we use Theorem 2 above. Let the number of atoms of D be m. Let $n = 2^m$ and let $\sigma_0, \ldots, \sigma_{n-1}$ be an enumeration of all the $\{0, 1\}$ strings of length m. Let α map the atoms of D one-to-one onto $\{0, 1, \ldots, m-1\}$. Let $\mathscr{A}$ denote the set of atoms of D. As in [**2**], for each $a \in \mathscr{A}$ let $\mathscr{R}[a]$ be the subclass of $\mathscr{R}$ consisting of those members of $\mathscr{R}$ which are subsets of $\pi(a)$, and let $\langle \mathscr{R}_i[a] \rangle$ be a canonical enumeration of $\mathscr{R}[a]$ without repetition. Choose e' such that for every $f \in n^N$ if

$$X = U \cup \bigcup \{\mathscr{R}_i[a] \mid \sigma_{f(i)}\alpha(a) = 1 \;\&\; a \in \mathscr{A}\}, \tag{1}$$

then $\{e'\}^f = \{e\}^X$. Now apply Theorem 2 to e' and let E, ϵ, and F be the equivalence relation on N, the p.r. function, and the equivalence relation on $\{0, 1, \ldots, n-1\}$, respectively. For each $a \in \mathscr{A}$ choose $j(a)$, $k(a)$ both $<n$ such that $\sigma_{j(a)}\alpha(a) = 1$, $\sigma_{k(a)}\alpha(a) = 0$, $\sigma_{j(a)}$ and $\sigma_{k(a)}$ differ only at $\alpha(a)$, and such that if possible $F(j(a), k(a))$ is false. Partition $(N - \text{dom } \epsilon)/E$ into infinite canonically enumerable classes $\mathscr{N}_a$ one for each $a \in \mathscr{A}$. Define

$$U^* = U \cup \bigcup \{\mathscr{R}_i[a] \mid i \in \text{dom } \epsilon \;\&\; a \in \mathscr{A} \;\&\; \sigma_{\epsilon(i)}\alpha(a) = 1\}$$
$$\cup \bigcup \{\mathscr{R}_i[b] \mid a, b \in \mathscr{A} \;\&\; b \neq a \;\&\; i \in \bigcup \mathscr{N}_a \;\&\; \sigma_{j(a)}\alpha(b) = 1\},$$
$$V^* = V \cup \bigcup \{\mathscr{R}_i[a] \mid i \in \text{dom } \epsilon \;\&\; a \in \mathscr{A} \;\&\; \sigma_{\epsilon(i)}\alpha(a) = 0\}$$
$$\cup \bigcup \{\mathscr{R}_i[b] \mid a, b \in \mathscr{A} \;\&\; b \neq a \;\&\; i \in \bigcup \mathscr{N}_a \;\&\; \sigma_{j(a)}\alpha(b) = 0\},$$

and let

$$\mathscr{R}^* = \{U^*, V^*\} \cup \{\bigcup \{R_i[a] \mid i \in Y\} \mid a \in \mathscr{A} \;\&\; Y \in \mathscr{N}_a\}.$$

Finally, define $\pi^*(a) = \pi(a) - (U^* \cup V^*)$ for each $a \in \mathscr{A}$, and $\pi^*(0) = U^* \cup V^*$. It is easy to check that $U \subseteq U^*$, $V \subseteq V^*$, $\mathscr{R}$ is a refinement of $\mathscr{R}^*$, and that $\pi^*(d) = \mathscr{R}^*\pi(d)$ for all d in D. Let W be the recursive set

$$\bigcup \{\pi^*(a) \mid a \in \mathscr{A} \;\&\; F(j(a), k(a)) \text{ is false}\}.$$

Consider a set X closed under $\mathscr{R}^*$ such that $\{e\}^X$ is total and

$$U^* \subseteq X \subseteq N - V^*.$$

Let $f \in n^{N/E}$ be the unique function satisfying (1). Since $U^* \subseteq X \subseteq N - V^*$, unless $i \in \bigcup \mathscr{N}_a$ for some $a \in \mathscr{A}$, we can compute $f(i)$ independently of X. If $i \in \bigcup \mathscr{N}_a$, there are two cases. First, if $F(j(a), k(a))$ is false, then

$$f(i) = \begin{cases} j(a) & \text{if } R_i[a] \subseteq X \cap W, \\ k(a) & \text{otherwise.} \end{cases}$$

Secondly, if $F(j(a), k(a))$ is true, then $f/F(i) = f/F(j(a)) = f/F(k(a))$. Thus f/F is computable if an oracle for $X \cap W$ is given. Conversely, if an oracle for f/F is given, then for $i \in \bigcup \mathscr{N}_a$, where $F(j(a), k(a))$ is false, we can effectively tell whether $f(i) = j(a)$ or $f(i) = k(a)$, i.e. whether $R_i[a] \subseteq X$ or not. Thus the membership of $X \cap W$ is computable from an oracle for f/F. Since $\{e\}^X = \{e'\}^f$ has the same degree as f/F, the proposition is proved.

The other propositions have proofs which are either easy in the case of Proposition 2, or straightforward adaptations of the proofs in [**2**] in the case of Propositions 1 and 3.

5. Initial segments of the degrees. It is evident that Theorem 2 also has some application to the problem of constructing finite initial segments of the degrees. For example it is almost obvious from Theorem 2 that there is an initial segment of the degrees dually isomorphic to the lattice of equivalence relations on $\{0, 1, \ldots, n-1\}$. The best result on initial segments obtainable from Theorem 2 is as follows. Let $\mathscr{F}$ be a sublattice of the lattice of all equivalence relations on $\{0, 1, \ldots, n-1\}$ having the equality relation as 0 and the universal relation as 1. A map g of $\{0, 1, \ldots, n-1\}$ into itself is said to *preserve* $\mathscr{F}$ if

$$\forall F \in \mathscr{F} \forall x < n \forall y < n[F(x, y) \rightarrow F(g(x), g(y))].$$

For each $J \subseteq \{0, 1, \ldots, n-1\}$ let F_J be the greatest member of $\mathscr{F}$ which is $\leqq$ the equivalence relation whose equivalence classes are J and $\{0, 1, \ldots, n-1\} - J$. We say that $\mathscr{F}$ is *good* if for every $J \subseteq \{0, 1, \ldots, n-1\}$ and all $x, y < n$ such that $F_J(x, y)$ is false, there exists g preserving $\mathscr{F}$ such that one of $\{g(x), g(y)\}$ is in J and the other is in $\{0, 1, \ldots, n-1\} - J$.

THEOREM 4. *If a finite lattice L is dually isomorphic to a good lattice of equivalence relations on $\{0, 1, \ldots, n-1\}$, then there is an initial segment of the degrees isomorphic to L.*

It can be shown that this result subsumes those in [**3**; **7**] on initial segments. However, quite recently Lerman [**4**] has shown that every finite lattice is isomorphic to an initial segment of the degrees by a method which is distinctly more powerful than the method used to obtain the earlier partial results. For that reason we shall not prove Theorem 4 here since the method is essentially the same as that used in [**3**; **7**].

REFERENCES

1. A. H. Lachlan, *Distributive initial segments of the degrees of unsolvability*, Z. Math. Logik Grundlagen Math. *14* (1968), 457–472.
2. ——— *Initial segments of many-one degrees*, Can. J. Math. *22* (1970), 75–85.
3. M. Lerman, *Some non-distributive lattices as initial segments of the degrees of unsolvability*, J. Symbolic Logic *34* (1969), 85–98.
4. ——— *Initial segments of the degrees of unsolvability* (to appear in Ann. of Math.).
5. J. R. Shoenfield, *A theorem on minimal degrees*, J. Symbolic Logic *31* (1966), 539–544.
6. C. Spector, *On degrees of recursive unsolvability*, Ann. of Math. (2) *64* (1956), 581–592.
7. S. K. Thomason, *Sublattices and initial segments of the degrees of unsolvability*, Can. J. Math. *22* (1970), 569–581.

Simon Fraser University,
Burnaby, British Columbia