DIRICHLET AND POINCARÉ SERIES

by A. GOOD*

Dedicated to Professor Robert A. Rankin on the occasion of his 70th birthday

1. Introduction. The study of modular forms has been deeply influenced by famous conjectures and hypotheses concerning

$$\Delta(z)=\sum_{n=1}^{\infty}\tau(n)e^{2\pi inz},\qquad \operatorname{Im} z>0,$$

where $\tau(n)$ denotes Ramanujan's function. The fundamental discriminant Δ is a cusp form of weight 12 with respect to the modular group. Its associated Dirichlet series

$$L_{\Delta}(s)=\sum_{n=1}^{\infty}\tau(n)n^{-s},\qquad s=\sigma+it,\qquad \sigma>\tfrac{13}{2},$$

defines an entire function of s and satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_{\Delta}(s)=(2\pi)^{s-12}\Gamma(12-s)L_{\Delta}(12-s).$$

The most penetrating statements that have been made on $\tau(n)$ and $L_{\Delta}(s)$ are:

A1. RAMANUJAN'S CONJECTURE: *For every* $\varepsilon>0$, $\tau(n)\ll n^{11/2+\varepsilon}$, $n\to\infty$.

A2. LEHMER'S CONJECTURE: $\tau(n)\neq 0$ *for all n*.

B1. LINDELÖF HYPOTHESIS FOR L_{Δ}: *For every* $\varepsilon>0$, $L_{\Delta}(6+it)\ll t^{\varepsilon}$, $t\to\infty$.

B2. RIEMANN HYPOTHESIS FOR L_{Δ}: $L_{\Delta}(s)\neq 0$ *for all s in* $\sigma>6$.

Of these four problems only A1 has been established so far. This was done by Deligne [**1**] using methods from algebraic geometry and number theory. While B1 trivially holds with $\varepsilon>1/2$, it was established in [**2**] for every $\varepsilon>1/3$. Serre [**12**] proved A2 for a positive proportion of the integers and Hafner [**5**] showed that L_{Δ} has a positive proportion of its non-trivial zeros on the line $\sigma=6$. The proofs of the last three results are largely analytic in nature.

At present, analytic methods do not seem to be powerful enough to settle any one of those problems completely. However, they often give partial answers also in cases where other techniques are no longer applicable. For instance, an estimate with the same loss in the exponent as for B1 [**3**] holds for the generalization of A1 to arbitrary finitely

* Supported in part by NSF grant MCS-8108814(A03).

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

generated Fuchsian groups of the first kind. The proof starts out from the Rankin–Selberg convolution method. An alternative approach to A1 (cf. [**11**]) consists in analyzing the Fourier coefficients of the Poincaré series introduced by Petersson [**10**]. Moreover, A2 can equivalently be expressed by the non-identical vanishing of such Poincaré series. The purpose of this paper is to introduce and study Poincare series which are related to the problems B in the same way as Petersson's series are related to the problems A. In particular, there will be an equivalent formulation of B2 by the non-identical vanishing of our Poincaré series (cf. Corollary and Remarks after Theorem 1). Thus we add further evidence for a close analogy between A1 and B1 as well as between A2 and B2.

In Section 2 we associate Poincaré series to any pair ξ of points which are cusps with respect to a finitely generated Fuchsian group of the first kind. These series define holomorphic cusp forms and they depend holomorphically on a parameter s. Their inner product with a cusp form F is essentially given by the value at s of the Dirichlet series attached to F and ξ (Theorem 1).

In Section 3 we investigate Mellin transforms of our Poincaré series in analogy with the Fourier series expansions of Petersson's Poincaré series. Their Fourier coefficients are expressible as a sum of Kloosterman sums plus a term involving the Kronecker symbol by the use of a Bruhat decomposition for $SL_2(\mathbb{R})$. On the other hand, we need an explicit A-A double coset decomposition (Lemmas 1, 2), where A denotes the subgroup of diagonal matrices. It enables us to express the Mellin transforms as a sum of three different types of terms. The bulk (${}_\xi P_x$ in Theorem 2) is a sum of the finite exponential sums (15) resembling the Kloosterman sums. As counterpart to the Kronecker symbol in Petersson's case there is an infinite sum (${}_\xi d_x$ in Theorem 2) of terms frequently given by the Riemann zeta-function. Beyond that there is a finite number (Lemma 3) of terms (${}_\xi p_x$ in Theorem 2) arising from the transversal intersections of geodesics going from one cusp to another. In comparison to Petersson's case it is more delicate here to handle the interchange of summation and integration. For we have absolute convergence only after deleting some terms from our Poincaré series (Lemma 4). Finally, the basic identities of Theorem 2 result from expanding the Poincaré series with respect to an orthonormal base for the cusp forms.

Since Δ is positive on the positive imaginary axis, $L_\Delta(s)$ is positive for positive s. In Section 4 we go beyond this simple observation in twofold respects. For, on average over the space of cusp forms, positive lower bounds are obtained for the Dirichlet series on parts of the real line provided only that the third type of terms does not occur (Theorem 3). These lower bounds are not only easy to calculate but they can also be quite close to the actual value (Corollary and Remark to Theorem 3). The positive lower bounds imply that the corresponding Poincaré series do not vanish identically. There is no counterpart for this in Petersson's case since it really amounts to looking at the constant term. In some important cases, we also show that the exponential sums (15) factor and are given by divisor functions (Theorem 4). In particular, we obtain a new representation of $|L_\Delta(s)|^2$ by a series which converges absolutely in the critical strip of L_Δ and whose arithmetical part is much simpler than that of Ramanujan's function. Finally, we determine the value of certain series explicitly by restricting variables to integer values (Corollary to Theorem 4).

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

2. Convergence and inner products. Let $G = \mathrm{SL}_2(\mathbb{R})$ act on the upper half-plane $H = \{z = x + iy \mid x \text{ real}, y > 0\}$ by

$$(M, z) \mapsto M(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We often write $z_M = x_M + iy_M$ for $M(z)$ and z'_M for $\dfrac{dM(z)}{dz}$. The hyperbolic measure

$$d\omega(z) = \frac{dx\,dy}{y^2}$$

is invariant under this action of G on H. If Γ is a discrete subgroup of G, this measure projects to the orbit space $\Gamma\backslash H$ in a natural way. Right now we assume on Γ only that the volume

$$\omega(\Gamma\backslash H) = \int_{\Gamma\backslash H} d\omega(z)$$

is finite.

Let $S_k(\Gamma)$ denote the space of cusp forms of weight k with respect to Γ. Here k will always be an even integer greater than 2. Thus F belongs to $S_k(\Gamma)$ if F is a holomorphic function on H such that

$$F(z_M)(z'_M)^{k/2} = F(z) \quad \text{for } M \text{ in } \Gamma \tag{1}$$

and

$$z \mapsto y^k\,|F(z)|^2 \quad \text{is bounded on } H. \tag{2}$$

We write $\langle F, G\rangle$ for the Petersson inner product

$$\int_{\Gamma\backslash H} y^k F(z)\bar{G}(z)\,d\omega(z)$$

on $S_k(\Gamma)$, where the bar denotes complex conjugation.

Let $\xi = (\theta_1, \theta_2)$ be a pair of different cusps for Γ. We denote the stabilizer of θ_ι in Γ by Γ_ι, $\iota = 1, 2$. Then there are a matrix M_ξ in G and a positive number λ_ξ such that, for $\iota = 1, 2$, $M_\xi(\theta_\iota) = S^\iota(\infty)$ and $\pm M_\xi \Gamma_\iota M_\xi^{-1}$ is generated by

$$\pm S^\iota \begin{bmatrix} 1 & \lambda_\xi \\ 0 & 1 \end{bmatrix} S^\iota, \quad \text{where} \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3}$$

These requirements determine λ_ξ uniquely and M_ξ up to a factor ± 1. If we set $\xi^* = (\theta_2, \theta_1)$, we easily verify that

$$\lambda_\xi^* = \lambda_\xi \quad \text{and} \quad M_{\xi^*} = \pm SM_\xi. \tag{4}$$

If F is in $S_k(\Gamma)$ then

$$F_\xi : z \mapsto F(M_\xi^{-1}(z))\left(\frac{dM_\xi^{-1}(z)}{dz}\right)^{k/2} \tag{5}$$

belongs to $S_k(M_\xi\Gamma M_\xi^{-1})$ by (1) and (2). Thus by (3) it has a Fourier series expansion of the form

$$F_\xi(z)=\sum_{n=1}^{\infty}a_\xi(n)e\left(\frac{nz}{\lambda_\xi}\right), \tag{6}$$

where $e(z)=e^{2\pi iz}$. The associated Dirichlet series

$$L_\xi(s)=\sum_{n=1}^{\infty}a_\xi(n)n^{-s}, \qquad s=\sigma+it, \tag{7}$$

converges absolutely in the half-plane $\sigma>\frac{k+1}{2}$ by a well-known bound for the Fourier coefficients $a_\xi(n)$ (cf. [**11**]). It follows from (3)–(5) that

$$F_{\xi^*}(z)=F\left(M_\xi^{-1}\left(-\frac{1}{z}\right)\right)\left(\frac{dM_\xi^{-1}\left(-\frac{1}{z}\right)}{dz}\right)^{k/2}=F_\xi\left(-\frac{1}{z}\right)z^{-k}. \tag{8}$$

Thus a familiar argument (cf. [**9**, p. I-5]) extends $L_\xi(s)$ to an entire function satisfying the functional equation

$$\left(\frac{2\pi}{\lambda_\xi}\right)^{-s}\Gamma(s)L_\xi(s)=(-1)^{k/2}\left(\frac{2\pi}{\lambda_{\xi^*}}\right)^{k-s}\Gamma(k-s)L_{\xi^*}(k-s), \tag{9}$$

where $\Gamma(s)$ denotes the gamma function. Moreover F_ξ can be recovered from $L_\xi(s)$ by the formula

$$F_\xi(z)=\frac{1}{2\pi i}\int_{(\rho)}\left(\frac{2\pi}{\lambda_\xi}\right)^{-s}\Gamma(s)L_\xi(s)\left(\frac{z}{i}\right)^{-s}ds. \tag{10}$$

Here and later on $\int_{(\rho)}$ denotes integration along $\mathrm{Re}(s)=\rho$ in the direction of increasing imaginary parts and $\left(\frac{z}{i}\right)^s$, z in H, is defined by $\exp\left(s\log\frac{z}{i}\right)$ with the principal branch of the logarithm.

We now introduce a family of Poincaré series depending on a complex parameter s by

$$P_\xi(z,s)=\sum_{M\in\Gamma/Z}(z_{M_\xi M}/i)^{-s}(z'_{M_\xi M})^{k/2}, \tag{11}$$

where Z denotes the center of Γ.

THEOREM 1. *The defining series of P_ξ converges absolutely and locally uniformly for z in H and s in the strip $1<\sigma<k-1$. Then $P_\xi(\cdot,s)$ belongs to $S_k(\Gamma)$ and, if F and L_ξ are related by* (5)–(7), *the inner product formula*

$$\langle P_\xi(\cdot,s),F\rangle=\frac{4\pi\Gamma(k-1)}{2^k\Gamma(s)}\left(\frac{2\pi}{\lambda_\xi}\right)^{s-k}\overline{L_\xi(k-\bar{s})}$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

holds. Moreover, P_ξ and P_{ξ^*} are connected by the functional equation

$$P_\xi(z,s)=(-1)^{k/2}P_{\xi^*}(z,k-s).$$

Proof. It follows from (1), (5), (6), the G-invariance of $d\omega$ and the chain rule that

$$\int_{\Gamma\backslash H} y^k \sum_{M\in\Gamma/Z} |z_{M_\xi M}|^{-\sigma}\,|z'_{M_\xi M}|^{k/2}\,|F(z)|\,d\omega(z)=\int_H y^k\,|z_{M_\xi}|^{-\sigma}\,|z'_{M_\xi}|^{k/2}\,|F(z)|\,d\omega(z)$$

$$=\int_H y^k\,|z|^{-\sigma}\,|F_\xi(z)|\,d\omega(z)=\int_0^\infty y^{k-2}\int_{-\lambda_\xi/2}^{\lambda_\xi/2}\sum_{n=-\infty}^{\infty}|z+n\lambda_\xi|^{-\sigma}\,|F_\xi(z)|\,dx\,dy.$$

If $\sigma>1$ the last sum is $|z|^{-\sigma}+O(1)$ for $|x|\le\lambda_\xi/2$ while

$$F_\xi(z)\ll\begin{cases}\exp(-2\pi y/\lambda_\xi) & \text{for } y\ge 1,\\ y^{-k/2} & \text{for } y\le 1,\\ |z|^{-k}\exp\left(-\dfrac{2\pi y}{|z|^2\lambda_\xi}\right) & \text{for } \dfrac{y}{|z|^2}\ge 1,\end{cases}$$

by (6), (2) and (8) respectively. Thus the above integrals are

$$\ll\int_1^\infty y^{k-2}e^{-2\pi y/\lambda_\xi}\,dy+\int_0^1 y^{k-2}\left(\int_0^{\sqrt{y(1-y)}}|z|^{-\sigma-k}e^{-2\pi y/|z|^2\lambda_\xi}\,dx+y^{-k/2}\int_{\sqrt{y(1-y)}}^{\lambda_\xi/2}|z|^{-\sigma}\,dx\right)dy$$

$$\ll 1+\int_0^1 y^{(k-\sigma-3)/2}\,dy<\infty,$$

provided that $1<\sigma<k-1$. Therefore our assertions on convergence follow from familiar theorems in function theory. They also imply that $P_\xi(z,s)$ is holomorphic for z in H and s in the strip $1<\sigma<k-1$. Consequently, it is a cusp form if (2) holds with $F=P_\xi(\cdot,s)$.

Now let n be an integer, $\chi=(\theta'_1,\theta'_2)$ an arbitrary pair of different cusps for Γ and let Γ_∞ denote the stabilizer of ∞ in $M_\chi\Gamma M_\chi^{-1}$. Then the preceding considerations justify the interchange of summation and integration so that we obtain, from (3), (5) and (11),

$$\int_0^{\lambda_\chi}P_\xi(M_\chi^{-1}(z),s)\left(\frac{dM_\chi^{-1}(z)}{dz}\right)^{k/2}e\left(-\frac{nz}{\lambda_\chi}\right)dx=\sum_{M\in M_\xi\Gamma M_\chi^{-1}/\Gamma_\infty}\int_{-\infty}^{\infty}\left(\frac{z_M}{i}\right)^{-s}(z'_M)^{k/2}e\left(-\frac{nz}{\lambda_\chi}\right)dx.$$

For $n\le 0$ and $1<\sigma<k-1$, all the latter integrals tend to zero if their line of integration is pushed arbitrarily high up in the upper half-plane. This means that $P_\xi(z,s)$ decays exponentially near θ'_2 or, equivalently, that it satisfies (2). Thus indeed $P_\xi(\cdot,s)$ is a cusp form for all s in $1<\sigma<k-1$.

Similarly, as at the beginning of the proof, (5), (6) and (11) yield

$$\langle P_\xi(\cdot,s),F\rangle=\int_H y^k\left(\frac{z_{M_\xi}}{i}\right)^{-s}(z'_{M_\xi})^{k/2}\bar F(z)\,d\omega(z)$$

$$=\int_H y^k\left(\frac{z}{i}\right)^{-s}\bar F_\xi(z)\,d\omega(z)$$

$$=\int_0^\infty y^{k-2}\int_0^{\lambda_\xi}\sum_{n=-\infty}^{\infty}\left(\frac{z+n\lambda_\xi}{i}\right)^{-s}\bar F_\xi(z)\,dx\,dy.$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

The preceding inner integral equals

$$\left(\frac{2\pi}{\lambda_\xi}\right)^s \frac{\lambda_\xi}{\Gamma(s)} \sum_{n=1}^{\infty} \bar{a}_\xi(n) n^{s-1} e^{-4\pi n y/\lambda_\xi}$$

by (6), Parseval's identity and the Lipschitz formula (cf. [**6**, p. 65])

$$\sum_{n=-\infty}^{\infty} \left(\frac{z+n\lambda}{i}\right)^{-s} = \frac{(2\pi/\lambda)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e\left(\frac{nz}{\lambda}\right) \tag{12}$$

valid for z in $H, \lambda > 0$ and for s in $\sigma > 1$. Thus we conclude from (7) that

$$\langle P_\xi(\cdot, s), F\rangle = \frac{4\pi\Gamma(k-1)}{2^k\Gamma(s)} \left(\frac{2\pi}{\lambda_\xi}\right)^{s-k} \overline{L_\xi(k-\bar{s})}$$

for s in $1 < \sigma < \frac{k-1}{2}$. By analytic continuation this identity also holds in the strip $1 < \sigma < k-1$. Finally, absolute convergence, (3), (4) and (11) show that

$$P_\xi(z, s) = \sum_{M \in \Gamma/Z} (z_{M_{\xi^*}M}/i)^s (z_{M_{\xi^*}M})^k (z'_{M_{\xi^*}M})^{k/2} = (-1)^{k/2} P_{\xi^*}(z, k-s)$$

and the theorem is proved.

From (9) and Theorem 1 we obtain the following corollary.

COROLLARY. *Let s_0 be a fixed complex number with $1 < \mathrm{Re}(s_0) < k-1$. Then the functions F orthogonal to $P_\xi(\cdot, s_0)$ in $S_k(\Gamma)$ are precisely those for which the corresponding $L_{\xi^*}(s)$ vanishes at $s = \bar{s}_0$.*

REMARKS. (i) Note that the functional equation (9) can alternatively be deduced from the last part of Theorem 1.
(ii) If $S_k(\Gamma)$ is one-dimensional, the corollary characterizes the zeros of $L_\xi(s)$ in $1 < \sigma < k-1$ by the identical vanishing of the cusp form $P_{\xi^*}(\cdot, \bar{s})$. In particular, a 'Riemann hypothesis' holds for $L_\xi(s)$ if and only if $P_{\xi^*}(\cdot, s)\Gamma(k-s) \neq 0$ for all s in $\sigma > \frac{k}{2}$. For higher dimensional $S_k(\Gamma)$, the latter condition is necessary if a Riemann hypothesis should hold for at least one $L_\xi(s)$ attached to the forms in $S_k(\Gamma)$.

3. Expansions of $P_\xi(z, s)$. Let F_j, $j = 1, 2, \ldots, J = \dim S_k(\Gamma)$ constitute an orthonormal basis for $S_k(\Gamma)$. Let $L_{j\xi}(s)$ denote the Dirichlet series attached to $F = F_j$ and ξ as in (6) and (7). Hence Theorem 1 yields the expansion

$$P_\xi(z, s) = \sum_{j=1}^{J} \langle P_\xi(\cdot, s), F_j\rangle F_j(z) = \frac{4\pi\Gamma(k-1)}{2^k\Gamma(s)} \left(\frac{2\pi}{\lambda_\xi}\right)^{s-1} \sum_{j=1}^{J} \overline{L_{j\xi}(k-\bar{s})} F_j(z). \tag{13}$$

In this section we compute Mellin transforms of both sides in (13). In order to handle the terms on the right of (11) individually, we use the following double coset decomposition of

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

G:

Let $A(\tau)$, $K(\tau)$ and $N(\tau)$ denote the matrices

$$\begin{bmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{bmatrix}, \quad \begin{bmatrix} \cos\dfrac{\tau}{2} & \sin\dfrac{\tau}{2} \\ -\sin\dfrac{\tau}{2} & \cos\dfrac{\tau}{2} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$$

respectively. We write $\mathcal{S}$ (respectively $\mathfrak{z}$) for the set of matrices M in G with $|ad|+|bc|=1$ (respectively $abcd=0$), where $a, \ldots, d$ denote the entries of M as at the beginning of Section 2.

LEMMA 1. *If* $\Lambda^l(M)=\frac{1}{2}\log\left|\dfrac{ab}{cd}\right|$ *and* $\Lambda^r(M)=\frac{1}{2}\log\left|\dfrac{ac}{bd}\right|$, *we have:*

(i) *Every* M *in* $G-\mathcal{S}$ *is of the form*

$$M=\pm A(\Lambda^l(M)K(\pi[\delta(M)-\tfrac{1}{2}])A(2\log\nu(M))K(\pi[\delta'(M)-\tfrac{1}{2}])A(\Lambda^r(M)),$$

where $\delta(M)$, $\delta'(M)$ *equal* 0 *or* 1 *so that* $(-1)^{\delta(M)}ac>0$, $(-1)^{\delta'(M)}dc<0$ *and, where* $\nu(M)=|ad|^{1/2}+|bc|^{1/2}$;

(ii) *Every* M *in* $\mathcal{S}-\mathfrak{z}$ *is of the form*

$$M=\pm A(\Lambda^l(M))K(\nu)A(\Lambda^r(M))$$

with ν *uniquely determined by*

$$\operatorname{tg}\frac{\nu}{2}=\left|\frac{bc}{ad}\right|^{1/2}\cdot\frac{bd}{|bd|}, \qquad |\nu-\pi|<\pi.$$

Proof. Except for the explicit form of δ, δ' and ν, this is Lemma 1 of [**4**] in the case $\xi=\eta$ and $\chi=\eta'$. For then $\Lambda^l(M)$ was defined by $\frac{1}{2}\log|M(0)M(\infty)|$ and $\Lambda^r(M)$ by $-\Lambda^l(M^{-1})$. Now let

$$L=A(-\Lambda^l(M))MA(-\Lambda^r(M)).$$

Thus the above representations with unspecified δ, δ' and ν yield

$$0<M(\infty)L(\infty)=\frac{a}{c}(-1)^{\delta}\left(\frac{\nu^2-1}{\nu^2+1}\right)^{(-1)^{\delta+\delta'}}, \qquad 0<M^{-1}(\infty)L^{-1}(\infty)=-\frac{d}{c}(-1)^{\delta'}\left(\frac{\nu^2-1}{\nu^2+1}\right)^{(-1)^{\delta+\delta'}},$$

$$\frac{bc}{ad}=\frac{M(0)}{M(\infty)}=\frac{L(0)}{L(\infty)}=\left(\frac{\nu^2+1}{\nu^2-1}\right)^{2(-1)^{\delta+\delta'}}$$

in case (i) and

$$0<M(0)L(0)=\frac{b}{d}\operatorname{tg}\frac{\nu}{2}, \qquad \frac{bc}{ad}=\frac{M(0)}{M(\infty)}=\frac{L(0)}{L(\infty)}=-(\operatorname{tg}\nu/2)^2$$

in case (ii). Since by [**4**, Lemma 1] we have $\nu>1$ in (i) and $0<|\nu-\pi|<\pi$ in (ii) the proof is easily completed.

Next we choose a matrix M_θ for every cusp θ such that $M_\theta(\theta)=\infty$ and $\pm M_\theta^{-1}N(1)M_\theta$ generate the stabilizer of θ in Γ. If θ and θ' are Γ-equivalent we may further assume that $M_\theta^{-1}M_{\theta'}$ belongs to Γ (cf. [**4**, p. 40]).

LEMMA 2. *Let $\xi=(\theta_1,\theta_2)$ and $\chi=(\theta_1',\theta_2')$ be two pairs of different cusps for Γ. Set $b_\iota(\xi)=-M_{\theta_\iota}(\theta_{3-\iota})$ for $\iota=1,2$. Then*

$$M_\xi=S^\iota A(\log\lambda_\xi)N(b_\iota(\xi))M_{\theta_\iota}$$

for $\iota=1,2$ and every M in $M_\xi\Gamma M_\chi^{-1}\cap \mathscr{A}$ is of the form

$$M=\pm S^\iota A(\log\lambda_\xi)N(b_\iota(\xi)-b_{\iota'}(\chi)+n)A(-\log\lambda_\chi)S^{\iota'},$$

where ι,ι' equal 1 or 2 and n is an integer. Moreover, M is represented by more than one triple (ι,ι',n) if and only if $M=S^j$ with $j=0,1$. This happens with $j=0$ (respectively $j=1$) precisely if $b_\iota(\xi)-b_\iota(\chi)\equiv 0 \pmod 1$ (respectively $b_\iota(\xi)-b_{3-\iota}(\chi)\equiv 0 \pmod 1$) or precisely if an element in Γ maps θ_ι to θ_ι' (respectively to $\theta_{3-\iota}'$) simultaneously for $\iota=1$ and 2.

Proof. If we denote $S^\iota A(\log\lambda_\xi)N(b_\iota(\xi))M_{\theta_\iota}$ by L_ι for $\iota=1,2$ we observe that $L_\iota(\theta_\iota)=S^\iota(\infty)$, $L_\iota(\theta_{3-\iota})=S^\iota(0)$ and that $\pm S^\iota N(\lambda_\xi)S^\iota$ generates $\pm L_\iota\Gamma_\iota L_\iota^{-1}$. Since these three conditions characterize L_ι up to a factor ±1 they imply that $L_\iota=\pm M_\xi$.

Now M belongs to $M_\xi\Gamma M_\chi^{-1}\cap\mathscr{A}$ precisely if there is a pair (ι,ι') such that $\theta_{\iota'}'$ is mapped to θ_ι by $M_\xi^{-1}MM_\chi$ in Γ. In the latter case $L=M_\xi^{-1}MM_\chi\,.\,M_{\theta_{\iota'}'}^{-1}M_{\theta_\iota}$ belongs to the stabilizer of θ_ι in Γ by our choice of the M_θ. Therefore

$$\pm M_{\theta_\iota}M_\xi^{-1}MM_\chi M_{\theta_{\iota'}'}^{-1}=\pm M_{\theta_\iota}LM_{\theta_\iota}^{-1}=\pm N(n)$$

with a suitable integer n. Thus the desired representation of M follows from the already proven part of Lemma 2. By comparing such ways of representing M we readily obtain the following: If two different triples (ι,ι',n) yield the same M they must necessarily be of the form (ι,ι',n) and $(3-\iota,3-\iota',m)$, where $b_\iota(\xi)-b_{\iota'}(\chi)+n=0$, $b_{3-\iota}(\xi)-b_{3-\iota'}(\chi)+m=0$ and $\lambda_\xi=\lambda_\chi$. Thus indeed $M=S^j$ with $j=0$ or 1. Moreover $M_\xi^{-1}MM_\chi$ then belongs to Γ and maps $\theta_{\iota'}'$ to θ_ι, where $\iota'=1,2$ and $\iota\equiv\iota'+j \pmod 2$. Since the conclusions in the last three sentences can easily be inverted Lemma 2 is established.

Next we introduce the sets

$${}_\xi\gamma_\chi=(M_\xi\Gamma M_\chi^{-1}\cap\mathscr{S}-\mathscr{A})/Z,\qquad {}_\xi\Gamma_\chi=(M_\xi\Gamma M_\chi^{-1}\cap G-\mathscr{S})/Z$$

and define the exponential sums

$${}_\xi s_\chi(s,w,\nu)=\sum\left|\frac{ab}{cd}\right|^{s/2}\left|\frac{ac}{bd}\right|^{w/2}, \tag{14}$$

$${}_\xi^\delta S_\chi^{\delta'}(s,w,\nu)=\sum\left|\frac{ab}{cd}\right|^{s/2}\left|\frac{ac}{bd}\right|^{w/2}, \tag{15}$$

where the summation in (14) is over those M in ${}_\xi\gamma_\chi$ with

$$\operatorname{tg}\frac{\nu}{2}=\left|\frac{bc}{ad}\right|^{1/2}\frac{bd}{|bd|},\qquad |\nu-\pi|<\pi,$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

and in (15) over those M in ${}_{\varepsilon}\Gamma_{\chi}$ with

$$(-1)^{\delta}ac>0, \qquad (-1)^{\delta'}dc<0, \qquad \nu=|ad|^{1/2}+|bc|^{1/2}.$$

Here Z denotes the center of Γ and $a,\ldots,d$ the entries of M as in Section 2. If the summation is over the empty set the above sums are understood to be zero. From an analytical point of view there is a striking resemblance between ${}^{\delta}_{\varepsilon}S^{\delta'}_{\chi}$ and the classical Kloosterman sums. We shall not elaborate on this here, since the crude upper bounds in the following lemmata suffice for this paper. In (14) and (15) only finite sums occur as we may conclude from the next lemma.

LEMMA 3. *There is a positive number C such that*

$$e^{|\Lambda^{\mathrm{l}}(M)|}\le C, \qquad e^{|\Lambda^{\mathrm{r}}(M)|}\le C$$

for every M in ${}_{\varepsilon}\gamma_{\chi}$ and

$$e^{|\Lambda^{\mathrm{l}}(M)|}\le C\nu^2(M), \qquad e^{|\Lambda^{\mathrm{r}}(M)|}\le C\nu^2(M)$$

for every M in ${}_{\varepsilon}\Gamma_{\chi}$. Moreover, ${}_{\varepsilon}\gamma_{\chi}$ is a finite set and for every positive ν_0 there are only finitely many M in ${}_{\varepsilon}\Gamma_{\chi}$ with $\nu(M)\le\nu_0$.

Proof. A well-known property of the groups Γ under consideration says the following (cf. [**13**, Lemma 1.26 and Lemma 1.27]): If C is large enough then a point on the geodesic from θ'_1 to θ'_2 is mapped by L in Γ to a point in the sets

$$M_{\varepsilon}^{-1}S^{\iota}(\{z \mid y>C\}), \qquad \iota=1 \quad\text{or}\quad 2,$$

if and only if $L(\theta'_{\iota'})=\theta_{\iota}$ for $i=1$ or 2 and $\iota'=1$ or 2. By the definition of $\mathscr{s}$ this may equivalently be expressed as follows: For sufficiently large C the image of the positive imaginary axis under a transformation M is $M_{\varepsilon}\Gamma M_{\chi}^{-1}$ hits $S^{\iota}\{z \mid y>C\}$, $\iota=1$ or 2, if and only if M neither belongs to ${}_{\varepsilon}\gamma_{\chi}$ nor ${}_{\varepsilon}\Gamma_{\chi}$. On the other hand Lemma 1 yields

$$M(ie^{-\Lambda^{\mathrm{r}}(M)})=ie^{\Lambda^{\mathrm{l}}(M)}$$

for M in ${}_{\varepsilon}\gamma_{\chi}$ and

$$M(ie^{-\Lambda^{\mathrm{r}}(M)})=e^{\Lambda^{\mathrm{l}}(M)}K(\pi[\delta-1/2])(i\nu^2(M))=e^{\Lambda^{\mathrm{l}}(M)}\frac{i\nu^2(M)-(-1)^{\delta}}{i(-1)^{\delta}\nu^2(M)+1}$$

for M in ${}_{\varepsilon}\Gamma_{\chi}$. We conclude that

$$C\ge \operatorname{Im} S^{\iota}M(ie^{-\Lambda^{\mathrm{r}}(M)})=\begin{cases} e^{(-1)^{\iota}\Lambda^{\mathrm{l}}(M)}, & \text{if } M \text{ is in } {}_{\varepsilon}\gamma_{\chi},\\ e^{(-1)^{\iota}\Lambda^{\mathrm{l}}(M)}\dfrac{2\nu^2(M)}{\nu^4(M)+1}, & \text{if } M \text{ is in } {}_{\varepsilon}\Gamma_{\chi},\end{cases}$$

for $\iota=1$ and 2. These are the desired inequalities for $\Lambda^{\mathrm{l}}(M)$. We obtain those for $\Lambda^{\mathrm{r}}(M)$ by interchanging the role of θ_{ι} and $\theta'_{\iota'}$ and by considering M^{-1} instead of M.

The set of M in $\mathscr{S}-\mathscr{s}$ with

$$e^{|\Lambda^{\mathrm{l}}(M)|}\le C, \qquad e^{|\Lambda^{\mathrm{r}}(M)|}\le C$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

and that of M in $G-\mathscr{S}$ with

$$e^{|\Lambda^{\iota}(M)|}\le C\nu^2(M),\qquad e^{|\Lambda^{\iota}(M)|}\le C\nu^2(M),\qquad \nu(M)\le\nu_0$$

obviously have compact closure in G. Therefore their intersections with $M_\xi\Gamma M_\chi^{-1}$ are finite sets since Γ is discrete. Thus Lemma 3 is proved.

LEMMA 4. *If* $1<\sigma, u<k-1$ *then*

$$\int_0^\infty \sum_{M\in_\xi\Gamma_\chi} |(iy)_M|^{-\sigma}\,|(iy)'_M|^{k/2}\,y^{u-1}\,dy<\infty.$$

Proof. First we note that

$$|z_M|^{-\sigma}\,|z'_M|^{k/2}=y^{-k/2}y_M^{(k-\sigma)/2}y_{SM}^{\sigma/2}.$$

It follows from (4) that SM belongs to ${}_{\xi^*}\Gamma_\chi$ and MS to ${}_\xi\Gamma_{\chi^*}$ whenever M is in ${}_\xi\Gamma_\chi$. Therefore

$$\sum_{M\in_\xi\Gamma_\chi} |z_M|^{-\sigma}\,|z'_M|^{k/2}=y^{-k/2}\Bigg(\sum_{\substack{M\in_\xi\Gamma_\chi\\ |z_M|\ge1}} y_M^{(k-\sigma)/2}y_{SM}^{\sigma/2}+\sum_{\substack{M\in_{\xi^*}\Gamma_\chi\\ |z_M|>1}} y_M^{\sigma/2}y_{SM}^{(k-\sigma)/2}\Bigg).\tag{16}$$

By (3) the first sum on the right of (16) equals

$$\sum_{M\in M_\xi\Gamma_2M_\xi^{-1}\backslash_\xi\Gamma_\chi} y_M^{(k-\sigma)/2}\sum_{\substack{n=-\infty\\ |z_M+n\lambda_\xi|\ge1}}^{\infty}\left(\frac{y_M}{|z_M+n\lambda_\xi|^2}\right)^{\sigma/2}\ll\sum_{M\in M_\xi\Gamma_2M_\xi^{-1}\backslash_\xi\Gamma_\chi} y_M^{k/2}$$

provided that $\sigma>1$. If we express M in ${}_\xi\Gamma_\chi$ as $M_\xi LM_\chi^{-1}$ we infer from Lemma 2 and the definition of ${}_\xi\Gamma_\chi$ that L belongs to $\Gamma-\Gamma_2$ and that

$$y_M=\lambda_\xi\,\mathrm{Im}\,M_{\theta_2}LM_\chi^{-1}(z).$$

Thus by (3) and [**7**, p. 11] the last sum is

$$\ll E_2\left(M_\chi^{-1}(z),\frac{k}{2}\right)-y_{M_{\theta_2}M_\chi^{-1}}^{k/2},$$

where $E_\iota(z,s)$ denotes the Eisenstein series attached to the cusps θ_ι. If y tends to ∞ then $M_\chi^{-1}S^\iota(iy)$ tends to θ'_ι for $\iota=1,2$. Therefore the Fourier expansions of $E_2(z,s)$ in those cusps (cf. [**7**, pp. 14–16]) reveal that the first sum on the right of (16) is

$$\ll y^{1-k/2}\quad\text{for}\quad y\to\infty\quad\text{and}\quad\ll\left(\frac{y}{|z|^2}\right)^{1-k/2}\quad\text{for}\quad\frac{y}{|z|^2}\to\infty$$

if $\sigma>1$. Under the condition $\sigma<k-1$ we obtain the same bounds for the second sum on the right of (16) if, in the estimates above, we replace σ,Γ_2 and $E_2(z,s)$ by $k-\sigma,\Gamma_1$ and $E_1(z,s)$ respectively. Hence the lemma follows. For, if $1<\sigma<k-1$, the integral under investigation converges absolutely on the interval $[1,\infty)$ provided that $u<k-1$ and on the interval $(0,1]$ provided that $u>1$.

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

In order to state our next theorem we set

$$\delta(\theta, \theta') = \begin{cases} 1, & \text{if } \theta' = M(\theta) \quad \text{with } M \text{ in } \Gamma, \\ 0, & \text{otherwise,} \end{cases}$$

$$\zeta(s; b) = \sum_{n=1}^{\infty} n^{s-1} e(nb), \qquad \text{if } b \text{ is real and } \sigma > 1,$$

$$j_k(\nu, s, w) = \frac{1}{i} \int_0^{i\infty} \left(\frac{z \cos\frac{\nu}{2} + \sin\frac{\nu}{2}}{-z \sin\frac{\nu}{2} + \cos\frac{\nu}{2}} \Big/ i \right)^{-s} \frac{(z/i)^{w-1}}{\left(-z \sin\frac{\nu}{2} + \cos\frac{\nu}{2}\right)^k} dz, \qquad \text{if } 0 < |\nu - \pi| < \pi,$$

and

$$J_k(\nu, s, w) = \frac{1}{i} \int_0^{i\infty} \left(\frac{z(\nu - 1/\nu) - (\nu + 1/\nu)}{z(\nu + 1/\nu) - (\nu - 1/\nu)} \Big/ i \right)^{-s} \frac{2^k (z/i)^{w-1}}{(z(\nu + 1/\nu) - (\nu - 1/\nu))^k} dz, \qquad \text{if } \nu > 1.$$

The last two integrals converge absolutely for $0 < u = \operatorname{Re} w < k$.

THEOREM 2. *In the notations introduced so far, let*

$$_\xi d_\chi(s, w) = \sum_{\iota, \iota'=1}^{2} \delta(\theta_\iota, \theta'_{\iota'})(-1)^{(\iota+\iota')\frac{k}{2}} \left(\frac{2\pi}{\lambda_\xi}\right)^{(-1)^\iota (s - \frac{k}{2})} \left(\frac{2\pi}{\lambda_\chi}\right)^{(-1)^{\iota'}(\frac{k}{2} - w)} \frac{\Gamma\left(\frac{k}{2} + (-1)^{\iota'}\left(w - \frac{k}{2}\right)\right)}{\Gamma\left(\frac{k}{2} + (-1)^{\iota}\left(s - \frac{k}{2}\right)\right)}$$

$$\cdot\, \zeta\left(1 + (-1)^\iota\left(\frac{k}{2} - s\right) - (-1)^{\iota'}\left(\frac{k}{2} - w\right); b_\iota(\xi) - b_{\iota'}(\chi)\right),$$

$$_\xi p_\chi(s, w) = \sum_\nu {}_\xi s_\chi\left(\frac{k}{2} - s, \frac{k}{2} - w, \nu\right) j_k(\nu, s, w)$$

and

$$_\xi P_\chi(s, w) = \sum_{\delta, \delta'=0}^{1} (-1)^{(\delta+\delta')\frac{k}{2}} \sum_\nu {}_\xi^\delta S_\chi^{\delta'} \left(\frac{k}{2} - s, \frac{k}{2} - w, \nu\right) J_k\left(\nu, \frac{k}{2} + (-1)^\delta\left(s - \frac{k}{2}\right), \frac{k}{2} + (-1)^{\delta'}\left(w - \frac{k}{2}\right)\right).$$

Then $_\xi p_\chi$ *is given by a finite sum and* $_\xi P_\chi$ *by an infinite series converging absolutely in* $1 < \sigma$, $u < k - 1$. *In this domain* $_\xi d_\chi$, $_\xi p_\chi$ *and* $_\xi P_\chi$ *are analytic functions of s and w. They are related to our Poincaré series by*

$$P_\xi(M_\chi^{-1}(z), s) \left(\frac{dM_\chi^{-1}(z)}{dz}\right)^{k/2} = \frac{1}{2\pi i} \int_{(u)} \{ {}_\xi d_\chi(s, w) + {}_\xi p_\chi(s, w) + {}_\xi P_\chi(s, w)\} \left(\frac{z}{i}\right)^{-w} dw$$

or, equivalently, to the Dirichlet series attached to cusp forms by

$$\frac{4\pi \Gamma(k-1)\Gamma(w)}{2^k \Gamma(s)} \left(\frac{2\pi}{\lambda_\xi}\right)^{s-k} \left(\frac{2\pi}{\lambda_\chi}\right)^{-w} \sum_{j=1}^{J} \overline{L_{j\xi}(k - \bar{s})} L_{j\chi}(w) = {}_\xi d_\chi(s, w) + {}_\xi p_\chi(s, w) + {}_\xi P_\chi(s, w).$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

Proof. If $L = A(\rho)MA(\tau)$ then

$$\frac{1}{i}\int_0^{i\infty}\left(\frac{z_L}{i}\right)^{-2}(z_L')^{k/2}\left(\frac{z}{i}\right)^{w-1}dz = e^{\rho(\frac{k}{2}-s)+\tau(\frac{k}{2}-w)}\frac{1}{i}\int_0^{i\infty}\left(\frac{z_M}{i}\right)^{-s}(z_M')^{k/2}\left(\frac{z}{i}\right)^{w-1}dz \tag{17}$$

by the definition of $A(\rho)$. Thus we obtain from Lemma 1, (14) and the definition of j_k

$$\frac{1}{i}\int_0^{i\infty}\sum_{M\in_\xi\gamma_x}\left(\frac{z_M}{i}\right)^{-s}(z_M')^{k/2}\left(\frac{z}{i}\right)^{w-1}dz = \sum_\nu {}_\xi s_x\left(\frac{k}{2}-s, \frac{k}{2}-w, \nu\right)j_k(\nu, s, w) = {}_\xi p_x(s, w). \tag{18}$$

In (18) the interchange of summation and integration is no problem since ${}_\xi\gamma_x$ is a finite set by Lemma 3. Now

$$\left(\frac{z_{SM}}{i}\right)^{-s}(z_{SM}')^{k/2} = (-1)^{k/2}\left(\frac{z_M}{i}\right)^{s-k}(z_M')^{k/2}$$

and (19)

$$(z_{MS}')^{k/2}\left(\frac{z}{i}\right)^{w-1} = (-1)^{k/2}((z_S)_M')^{k/2}\left(\frac{z_S}{i}\right)^{k-w+1}.$$

Therefore we deduce from Lemma 1, (15), (17) and the definition of J_k

$$\frac{1}{i}\int_0^{i\infty}\sum_{M\in_\xi\Gamma_x}\left(\frac{z_M}{i}\right)^{-s}(z_M')^{k/2}\left(\frac{z}{i}\right)^{w-1}dz$$

$$= \sum_{\delta,\delta'=0}^{1}(-1)^{k(\delta+\delta')/2}\sum_\nu {}_\xi^\delta S_x^{\delta'}\left(\frac{k}{2}-s, \frac{k}{2}-w, \nu\right)J_k\left(\nu, \frac{k}{2}+(-1)^\delta\left(s-\frac{k}{2}\right), \frac{k}{2}+(-1)^{\delta'}\left(w-\frac{k}{2}\right)\right)$$

$$= {}_\xi P_x(s, w). \tag{20}$$

In (20) the interchange of summation and integration is justified by Lemma 4 provided that $1<\sigma, u<k-1$. Furthermore it follows that the series defining ${}_\xi P_x$ converges absolutely and locally uniformly in that domain.

Now let

$$d(z, b, s) = {\sum_{n=-\infty}^{\infty}}'\left(\frac{z+b+n}{i}\right)^{-s},$$

where $\sum'$ means that we omit the term $(z/i)^{-s}$ in the case $b \equiv 0 (\text{mod } 1)$ and $|z|<1$. If $\delta_b = 1$ when b is an integer and $\delta_b = 0$ otherwise, we obtain, from the Lipschitz formula (12),

$$\int_0^\infty d(iy, b, s)y^{w-1}\,dy = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty\sum_{n=1}^\infty n^{s-1}e(n[b+iy])y^{w-1}\,dy - \delta_b\int_0^1 y^{w-s-1}\,dy$$

$$= \frac{(2\pi)^{s-w}\Gamma(w)}{\Gamma(s)}\zeta(1+w-s; b) - \frac{\delta_b}{w-s}$$

for $u>\sigma>1$. We also note that

$$\int_0^\infty |d(iy, b, s)|y^{u-1}\,dy \ll \int_0^1\sum_{n=1}^\infty |n+iy|^{-\sigma}y^{u-1}\,dy + \int_1^\infty\sum_{n=1}^\infty n^{\sigma-1}e(iny)y^{u-1}\,dy < \infty$$

for $\sigma, u>1$. If ${}_\xi\sigma_\chi = M_\xi \Gamma M_\chi^{-1} \cap \mathcal{s}$, we thus conclude from Lemma 2, (17) and (19) that

$$\frac{1}{i}\int_0^{i\infty} \sum_{M\in{}_\xi\sigma_\chi} \left(\frac{z_M}{i}\right)^{-s} (z'_M)^{k/2}\left(\frac{z}{i}\right)^{w-1} dz$$

$$= \sum_{\iota,\iota'=1}^{2} \delta(\theta_\iota, \theta'_{\iota'})(-1)^{k(\iota+\iota')/2}\lambda_\xi^{(-1)^\iota(\frac{k}{2}-s)}\lambda_\chi^{(-1)^{\iota'}(w-\frac{k}{2})}\int_0^\infty d\left(iy, b_\iota(\xi)-b_{\iota'}(\chi), \frac{k}{2}+(-1)^\iota\left(s-\frac{k}{2}\right)\right)$$

$$\times y^{\frac{k}{2}+(-1)^{\iota'}(w-\frac{k}{2})-1}\,dy = {}_\xi d_\chi(s, w) \quad (21)$$

for $1<\sigma, u<k-1$. Since the left hand sides of (18), (20) and (21) add up to

$$\frac{1}{i}\int_0^{i\infty} P_\xi(M_\chi^{-1}(z), s)\left(\frac{dM_\chi^{-1}(z)}{dz}\right)^{k/2}\left(\frac{z}{i}\right)^{w-1} dz$$

$$= \frac{4\pi\Gamma(k-1)\Gamma(w)}{2^k\Gamma(k-s)}\left(\frac{2\pi}{\lambda_\xi}\right)^{s-k}\left(\frac{2\pi}{\lambda_\chi}\right)^{-w}\sum_{j=1}^{J}\overline{L_{j\xi}(k-\bar{s})}L_{j\chi}(w)$$

by (11), (13), (10) and Mellin inversion, the theorem now follows.

Remark. The functions j_k and J_k can be expressed in terms of the hypergeometric function $F(a, b; c; z)$, e.g.,

$$J_k(\nu, s, w) = \frac{\Gamma(w)\Gamma(k-w)}{\Gamma(k)}\left(\frac{2}{\nu-1/\nu}\right)^k\left(\frac{(\nu-1/\nu)e^{i\pi/2}}{\nu+1/\nu}\right)^{s+w}F\left(s, w; k; \frac{4}{(\nu+1/\nu)^2}\right).$$

This incidentally shows that also the hypergeometric function in its most general form naturally comes up in the analysis of SL_2.

4. Lower bounds and special cases.

Theorem 3. *Let $\xi=(\theta_1, \theta_2)$ be a pair of different cusps such that the orbit of the geodesic from θ_1 to θ_2 in $\Gamma\backslash H$ has no transversal intersections. If k or λ_ξ are sufficiently large and $k\equiv 0 \pmod 4$ then the Dirichlet series $L_{j\xi}(s)$, $j=1,\ldots,J$, have no common zero on $\frac{k}{2}\le s\le\frac{k+1}{2}$. More specifically, if $h(s)=(2\pi/\lambda_\xi)^{k-2s}\frac{\Gamma(s)}{\Gamma(k-s)}$ then*

$$4\pi\Gamma(k-1)\left(\frac{\lambda_\xi}{4\pi}\right)^k h(s)\sum_{j=1}^{J}|L_{j\xi}(s)|^2$$

$$>\tfrac{1}{2}(h(s)+1/h(s))+\frac{1}{2s-k}(h(s)-1/h(s))$$

$$-2\log 2(\delta(\theta_1,\theta_2)-\delta(\xi,\xi^*))+\delta(\xi,\xi^*)\left(2\gamma+\frac{h'}{h}(s)\right)$$

for $\frac{k}{2}\le s\le\frac{k+1}{2}$, where γ denotes Euler's constant and

$$\delta(\xi,\xi^*)=\begin{cases}1, & \text{if there is } M \text{ in } \Gamma \text{ with } M(\theta_\iota)=\theta_{3-\iota} \text{ for } \iota=1 \text{ and } 2,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. We just have to prove the above inequality since its right hand side clearly becomes positive if k or λ_ξ gets large.

If ${}_\xi\gamma_\xi$ is not empty then by Lemma 1(ii) there is an M in Γ such that $M_\xi MM_\xi^{-1}(\infty)$ and $M_\xi MM_\xi^{-1}(0)$ are real numbers of different sign. Hence the geodesic from $M(\theta_1)$ to $M(\theta_2)$ intersects that from θ_1 to θ_2 transversally. Since these two geodesics have the same projection in $\Gamma\backslash H$ it follows that ${}_\xi\gamma_\xi$ is empty under the assumptions of Theorem 3.

By moving the contour of integration for J_k to the negative real line and by substituting $\rho = 1/\tau$, we obtain

$$\begin{aligned} J_k(\nu, s, w) &= e^{i\pi(s+w)/2}2^k \\ &\quad \times \int_0^\infty (\rho(\nu - 1/\nu) + \nu + 1/\nu)^{-s}(\rho(\nu + 1/\nu) + \nu - 1/\nu)^{s-k}\rho^{w-1}\, d\rho \\ &= e^{i\pi(s+w)} J_k(\nu, k-s, k-w). \end{aligned} \tag{22}$$

Since $\Lambda^l(M) = -\Lambda^r(M^{-1})$, $\nu(M) = \nu(M^{-1})$ and $\delta(M) = \delta'(M^{-1})$ by Lemma 1(i), we also note that

$${}^0_\xi S^1_\xi(s, w, \nu) = {}^1_\xi S^0_\xi(-w, -s, \nu).$$

Hence the definition of ${}_\xi P_x$ yields

$$\begin{aligned} {}_\xi P_\xi(k-s, s) &= (-1)^{k/2} \sum_\nu \left({}^0_\xi S^0_\xi\left(s - \frac{k}{2}, \frac{k}{2} - s, \nu\right) + {}^1_\xi S^1_\xi\left(s - \frac{k}{2}, \frac{k}{2} - s, \nu\right)\right) |J_k(\nu, k-s, s)| \\ &\quad + 2\cos\left(\left(s - \frac{k}{2}\right)\pi\right) \sum_\nu {}^1_\xi S^0_\xi\left(s - \frac{k}{2}, \frac{k}{2} - s, \nu\right) |J_k(\nu, s, s)| \end{aligned}$$

for $1 < s < k-1$. Under the assumptions of Theorem 3, it therefore follows from Theorem 2 that

$$4\pi\Gamma(k-1)\left(\frac{\lambda_\xi}{4\pi}\right)^k h(s) \sum_{j=1}^J |L_{j\xi}(s)|^2 > {}_\xi d_\xi(k-s, s)$$

for $\frac{k}{2} \le s \le \frac{k+1}{2}$. Since $\zeta(s; 0)$ is the Riemann zeta-function $\zeta(s)$ and

$$\zeta(1; b) = -\log(1 - e(b)), \qquad \text{if } b \not\equiv 0 \pmod 1,$$

we obtain from Lemma 2

$$\begin{aligned} {}_\xi d_\xi(k-s, s) &= h(s)\zeta(1 + 2s - k) + \zeta(1 + k - 2s)/h(s) \\ &\quad - (-1)^{k/2}(\delta(\theta_1, \theta_2) - \delta(\xi, \xi^*))\log|1 - e(b_1(\xi) - b_2(\xi))|^2 \\ &\quad + (-1)^{k/2}\delta(\xi, \xi^*)\left(2\gamma + \frac{h'}{h}(s)\right). \end{aligned}$$

Thus Theorem 3 follows since

$$\zeta(s) \ge \frac{1}{s-1} + \frac{1}{2} \quad \text{for} \quad 0 \le s \le 2$$

by the well-known formula

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\int_1^\infty \frac{[\rho]-\rho+1/2}{\rho^{s+1}}\,d\rho, \qquad s>0,$$

where $[\rho]$ denotes the integral part of ρ.

COROLLARY. *Let Γ be $\Gamma_0(q)$, the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $c\equiv 0(\mathrm{mod}\ q)$ and $\xi=(\theta_1,\theta_2)$ with $\theta_1=0$, $\theta_2=\infty$. Then Theorem 3 applies and the inequality there yields positive lower bounds for $q>6$ if $k=4$, for $q>1$ if $k=8$ and for all q if $k\geq 12$. At the center of the critical strip the more precise bound*

$$2\pi\Gamma(k-1)\left(\frac{q^{1/2}}{4\pi}\right)^k\sum_{j=1}^{J}\left|L_{j\xi}\left(\frac{k}{2}\right)\right|^2>\log\frac{q^{1/2}}{2\pi}+\sum_{j=1}^{\frac{k}{2}-1}\frac{1}{j}$$

holds for $q>1$ and $L_\Delta(6)$ is greater than

$$\frac{2^8\pi^5}{45}\left(\langle\Delta,\Delta\rangle\frac{\pi}{7}\left(\tfrac{137}{60}-\log 2\pi\right)\right)^{1/2}.$$

Proof. In case of $\Gamma_0(q)$ the stabilizer at ∞ is generated by $\pm N(1)$ and that at 0 by $\pm SN(q)S$, whence $M_\xi=\pm A(\tfrac{1}{2}\log q)$, $\lambda_\xi=q^{1/2}$ and

$$M_\xi M M_\xi^{-1}=\begin{bmatrix} a & bq^{1/2}\\ cq^{-1/2} & d\end{bmatrix}\quad\text{if}\quad M=\begin{bmatrix} a & b\\ c & d\end{bmatrix}. \tag{23}$$

It follows that ${}_\xi\gamma_\xi$ is empty if $\xi=(0,\infty)$. For otherwise $|ad|+|bc|=1$ would be solvable in non-zero integers $a,\ldots,d$. Now $\delta(\xi,\xi^*)=1$ for $q=1$ while 0 and ∞ are not $\Gamma_0(q)$-equivalent for $q>1$. Therefore the inequality in Theorem 3 has a positive right hand side if the sum of its first two terms is positive, i.e.,

$$h^2(s)>\frac{1+\dfrac{k}{2}-s}{1+s-\dfrac{k}{2}},$$

and if $2\gamma+\dfrac{h'}{h}(s)>0$ for $q=1$. By Burnside's formula (cf. [**8**, p. 12]) we have

$$\log h(s)\geq(k-2s)\log\frac{2\pi e}{q^{1/2}}+(s-1/2)\log(s-1/2)-(k-s-1/2)\log(k-s-1/2)$$

for $\dfrac{k}{2}\leq s\leq\dfrac{k+1}{2}$. Since the derivative of the right hand side above is

$$-2\log\frac{4\pi}{(k-1)q^{1/2}}+\log\left(1-\left(\frac{2s-k}{k-1}\right)^2\right)\geq 1-2\log\frac{4\pi}{(k-1)q^{1/2}}-\frac{1}{1-\left(\dfrac{2s-k}{k-1}\right)^2}$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

and

$$\frac{1}{2}\frac{d}{ds}\log\left(\frac{1+\frac{k}{2}-s}{1+s-\frac{k}{2}}\right)=\frac{-1}{1-\left(s-\frac{k}{2}\right)^2}\le\frac{-1}{1-\left(\frac{2s-k}{k-1}\right)^2}$$

the first condition is satisfied for $\frac{k}{2}<s\le\frac{k+1}{2}$ provided that $q(k-1)^2\ge(4\pi)^2/e$. The second condition holds for $\frac{k}{2}\le s\le\frac{k+1}{2}$ if

$$0<2\gamma+\frac{h'}{h}\left(\frac{k+1}{2}\right)=-2\log 8\pi+4\sum_{j=1}^{\frac{k}{2}-1}\frac{1}{2j-1}+\frac{2}{k-1}$$

since $\left(\frac{\Gamma'}{\Gamma}\right)'(s)$ is monotonically decreasing in $s>0$. At the center of the critical strip, on the other side, the proof of Theorem 3 directly yields to lower bound

$${}_\xi d_\xi\left(\frac{k}{2},\frac{k}{2}\right)=(1+\delta(\xi,\xi^*))\left(2\gamma+\frac{h'}{h}\left(\frac{k}{2}\right)\right)=2(1+\delta(\xi,\xi^*))\left(\log\frac{q^{1/2}}{2\pi}+\sum_{j=1}^{\frac{k}{2}-1}\frac{1}{j}\right).$$

Since $J=1$ and $L_\Delta(s)=\langle\Delta,\Delta\rangle^{1/2}L_{1\xi}(s)$ for $k=12$ and $q=1$ the corollary now follows from a few simple numerical verifications.

Remark. Since $\dim S_k(\Gamma_0(q))=0$ for $q<5$ if $k=4$ and for $q=1$ if $k=8$ our method fails to provide non-trivial lower bounds only in the case $k=4$ and $q=5$ or 6. At the center of the critical strip our lower bound is positive for $k=4, q=6$ and negative for $k=4, q=5$. The inequality for $L_\Delta(6)$ shows that our lower bounds can be remarkably close to the actual value: One knows (cf. [**14**, p. 117]) that $L_\Delta(6)\le 0{\cdot}792123$ while the corollary yields $L_\Delta(6)\ge 0{\cdot}792047$.

In Theorem 2 certain combinations of the Dirichlet series attached to cusp forms are expressed by series which, in contrast to the Dirichlet series, converge absolutely inside the critical strip. Under the assumptions of the preceding Corollary, the arithmetical part of these series is given by divisor functions as shown in the following theorem.

Theorem 4. *Let* Γ *be* $\Gamma_0(q)$ *and* $\xi=(0,\infty)$. *If* δ_{mn} *denotes the Kronecker symbol and*

$$\sigma(l,s)=\sum\left(\frac{a}{d}\right)^s$$

with summation over all pairs of positive integers a, d *whose product equals* l *then*

$$\frac{4\pi\Gamma(k-1)\Gamma(w)}{2^k\Gamma(s)}\left(\frac{2\pi}{q^{1/2}}\right)^{s-w-k}\sum_{j=1}^{J}\overline{L_{j\xi}(k-\bar{s})}L_{j\xi}(w)$$
$$=\left(\frac{2\pi}{q^{1/2}}\right)^{s-w}\frac{\Gamma(w)}{\Gamma(s)}\zeta(1+w-s)+\left(\frac{2\pi}{q^{1/2}}\right)^{w-s}\frac{\Gamma(k-w)}{\Gamma(k-s)}\zeta(1+s-w)$$

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

$$+\delta_{1q}(-1)^{k/2}\left(\left(\frac{2\pi}{q^{1/2}}\right)^{s+w-k}\frac{\Gamma(k-w)}{\Gamma(s)}\zeta(1+k-s-w)\right.$$

$$\left.+\left(\frac{2\pi}{q^{1/2}}\right)^{k-s-w}\frac{\Gamma(w)}{\Gamma(k-s)}\zeta(1+s+w-k)\right)$$

$$+\sum_{\delta=0}^{1}(-1)^{\delta k/2}(1+e^{-i\pi(s+(-1)^\delta w)})\sum_{n=1}^{\infty}\sigma\left(n,\frac{w-s}{2}\right)$$

$$\cdot\,\sigma\left(nq-(-1)^\delta,\frac{k-s-w}{2}\right)J_k\left((nq)^{1/2}+(nq-(-1)^\delta)^{1/2},s,\frac{k}{2}+(-1)^\delta\left(w-\frac{k}{2}\right)\right),$$

where the last series converges absolutely for $1<\sigma, u<k-1$.

Proof. By what we observed in the proof of the preceding corollary, the terms of the above identity involving the Riemann zeta-function add up to ${}_{\xi}d_{\xi}(s,w)$ and ${}_{\xi}p_{\xi}(s,w)=0$ under the present assumptions. It follows from (15) and (23) that

$${}^{\delta}_{\xi}S^{\delta'}_{\xi}(s,w,\nu)=\sum\left|\frac{a}{d}\right|^{(s+w)/2}\left|\frac{bq}{c}\right|^{(s-w)/2},$$

where the summation runs over all M in ${}_{\xi}\Gamma_{\xi}$ such that $\nu=|ad|^{1/2}+|bc|^{1/2}$, $(-1)^\delta ac>0$ and $(-1)^{\delta'}dc<0$. By picking the representative M in ${}_{\xi}\Gamma_{\xi}$ with $c>0$ we note that δ and δ' determine the sign of the remaining entries a, b, d uniquely, whence $|ad|=|bc|-(-1)^{\delta+\delta'}$. Thus we obtain

$${}^{\delta}_{\xi}S^{\delta'}_{\xi}(s,w,\nu)=\begin{cases}\sigma\left(l,\frac{s+w}{2}\right)\sigma\left(\frac{l+1}{q},\frac{s-w}{2}\right), & \text{if}\quad \delta+\delta'\equiv 0 \pmod 2 \quad\text{and}\quad l+1\equiv 0 \pmod q,\\ \sigma\left(l+1,\frac{s+w}{2}\right)\sigma\left(\frac{l}{q},\frac{s-w}{2}\right), & \text{if}\quad \delta+\delta'\equiv 1 \pmod 2 \quad\text{and}\quad l\equiv 0 \pmod q,\\ 0, & \text{otherwise},\end{cases}$$

where $\nu=l^{1/2}+(l+1)^{1/2}$. Therefore the definition of ${}_{\xi}P_{\xi}$ and (22) yield

$${}_{\xi}P_{\xi}(s,w)=\sum_{l\equiv-1\,(\mathrm{mod}\,q)}\sigma\left(l,\frac{k-s-w}{2}\right)\sigma\left(\frac{l+1}{q},\frac{w-s}{2}\right)J_k(\nu,s,w)(1+e^{-i\pi(s+w)})$$

$$+(-1)^{k/2}\sum_{l\equiv 0\,(\mathrm{mod}\,q)}\sigma\left(l+1,\frac{k-s-w}{2}\right)\sigma\left(\frac{l}{q},\frac{w-s}{2}\right)J_k(\nu,s,k-w)(1+e^{i\pi(w-s)}),$$

where ν and l are related as before. Hence Theorem 4 follows from Theorem 2.

Similarly as for zeta-functions at integer points, we can now sum certain infinite series involving divisor functions explicitly by restricting s and w to integer values. We state only a few examples in the next corollary.

https://doi.org/10.1017/S0017089500006066 Published online by Cambridge University Press

COROLLARY. *If $d(n)$ denotes the number of divisors of n then*

$$\sum_{n=1}^{\infty} d(n)\, d(n+1) J_k\left(n^{1/2}+(n+1)^{1/2}, \frac{k}{2}, \frac{k}{2}\right) = \begin{cases} \log 2\pi - 1, & \text{if } k=4, \\ \log 2\pi - \frac{11}{6}, & \text{if } k=8, \\ \log 2\pi - \frac{137}{60} + \dfrac{\Gamma(11)\,|L_\Delta(6)|^2}{4(4\pi)^{11}\langle \Delta, \Delta\rangle}, & \text{if } k=12. \end{cases}$$

Proof. Since $\dim S_k(\Gamma_0(1))=0$ for $k<12$ and $S_{12}(\Gamma_0(1))$ is generated by Δ we just have to note that

$$_{\varepsilon}d_{\varepsilon}\left(\frac{k}{2}, \frac{k}{2}\right) = 2(1+(-1)^{k/2})\left(-\gamma + \log 2\pi + \frac{\Gamma'}{\Gamma}\left(\frac{k}{2}\right)\right) = 4\left(-\log 2\pi + \sum_{j=1}^{\frac{k}{2}-1} \frac{1}{j}\right)$$

for $q=1$ and $k \equiv 0 \pmod 4$.

REMARK. Note that $J_k(\nu, s, w)$ is an elementary function of ν if s and w are integers, e.g.,

$$J_4(n^{1/2}+(n+1)^{1/2}, 2, 2) = (2n+1)\log\frac{n+1}{n} - 2.$$

REFERENCES

1. P. Deligne, La conjecture de Weil I, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307.

2. A. Good, The square mean of Dirichlet series associated to cusp forms, *Mathematika* **29** (1982), 278–295.

3. A. Good, Cusp forms and eigenfunctions of the Laplacian, *Math. Ann.* **255** (1981), 523–548.

4. A. Good, *Local analysis of Selberg's trace formula*, Lecture Notes in Mathematics **1040** (Springer, 1983).

5. J. L. Hafner, Zeros on the critical line for Dirichlet series attached to certain cusp forms, Preprint.

6. M. I. Knopp, *Modular functions in analytic number theory* (Markham, 1970).

7. T. Kubota, *Elementary theory of Eisenstein series* (Halsted Press, 1973).

8. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, 3rd edition (Springer, 1966).

9. A. Ogg, *Modular forms and Dirichlet series* (W. A. Benjamin, 1969).

10. H. Petersson, Theorie der automorphen Formen beliebiger reeller Dimension und ihre Dartellung durch eine neue Art Poincaréscher Reihen, *Math. Ann.* **103** (1930), 369–436.

11. A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Sympos. Pure Math. VIII (Amer. Math. Soc., 1965), 1–15.

12. J.-P. Serre, Quelques applications du théorème de densité de Chebotarev, *Inst. Hautes Études Sci.* Publ. Math. **54** (1981), 123–201.

13. G. Shimura, *Introduction to the arithmetic theory of automorphic functions* (Princeton University Press, 1971).

14. D. Zagier, *Modular forms whose coefficients involve zeta-functions of quadratic fields*, Lecture Notes in Mathematics **627** (Springer, 1977), 105–169.

SCHOOL OF MATHEMATICS
THE INSTITUTE FOR ADVANCED STUDY
PRINCETON
NJ 08540