1

FUNDAMENTAL CONCEPTS

1.1 Introduction

The object of this chapter is to lay out the principal ideas and nomenclature of group theory in preparation for the physical applications discussed in later chapters. We shall look at what group theory deals with, we shall define the mathematical meaning of a group, we shall show examples of several groups, and we shall discuss the key subject of matrix representations of groups (with an example). A review of matrix algebra and definitions of some special matrices concludes the chapter.

As a student of science, you spent several years studying calculus, differential equations, and the properties of important mathematical functions (trigonometric functions, exponentials, Bessel functions, etc.). You used these tools to solve problems in Newtonian mechanics, electromagnetism, and maybe even problems in quantum mechanics.

At its heart, group theory is very different from calculus. It is more abstract and more fundamental, with little reliance on explicit mathematical functions. As we shall see in this text, group theory, though abstract, nevertheless has great power in dealing with a wide range of physical phenomena. One example is the angular momentum ("spin") of an electron that is experimentally a dimensionless "point" particle with no analogue in Newtonian mechanics.

In physical applications, group theory calculates numerical results by using mathematical functions in the group's representation matrices.

More profoundly, group theory can give deeper insight into subjects you may have already studied; for instance, the conservation of energy and the structure of hydrogen-atom wave functions in quantum mechanics. Newton invented calculus to explain how forces acting on an object determine its motion. In modern high-energy particle physics the forces are not well known, yet group theory in its abstract generality provides predictive schemes for classifying "strange" particles.

P. W. Anderson (1923–2020, Nobel laureate in physics 1977) wrote "It is only slightly overstating the case to say that physics is the study of symmetry."

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1.2 Operations

Group theory deals with *operations*, also called *transformations*. In this book the symbols for operations are written in bold. We use the convention that an operation operates on the object (the *operand*) to its right.

Consider the simple example of a transformation **T** that operates on a variable x (the operand) to change its sign to $-x$. This is written symbolically as

$$\mathbf{T}x = -x.$$

If **T** operates on the function $ax + b$, where a and b are constants, **T** operates only on x and has no effect on constants. Hence

$$\begin{aligned}\mathbf{T}(ax + b) &= a\mathbf{T}x + b \\ &= -ax + b.\end{aligned}$$

If **T** operates twice in succession, the sequence **TT** can be written as $\mathbf{T}^2$:

$$\begin{aligned}\mathbf{T}^2 x &= \mathbf{T}\mathbf{T}x \\ &= \mathbf{T}(\mathbf{T}x) \\ &= \mathbf{T}(-x) \\ &= x.\end{aligned}$$

An even simpler operator is the *identity* operator, which produces no change in the operand. The identity operator in group theory is conventionally given the symbol **E**, from the German *Einheit*, unity or, literally, oneness:

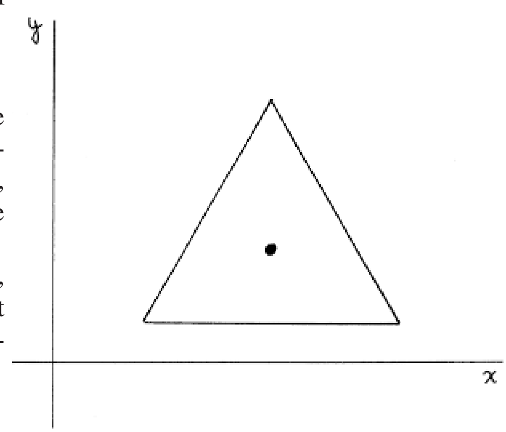
$$\mathbf{E}x = x.$$

These simple examples illustrate the abstract nature of group theory. The operators are not expressed in terms of explicit mathematical functions; instead, operators are defined in terms of their effect on the operand.

1.2.1 Symmetry Operations

The figure shows an equilateral triangle in the x-y plane. The dot marks the location of the triangle's geometric center, the point equally distant from all three apexes.

Consider now three operations, **E**, **A**, and **B**, that rotate the triangle about its geometric center through the specified angles.



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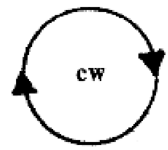
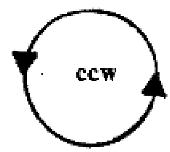
E: $rotate\,by\; 0°\ (equivalently,\ rotate\,by\; 360°)$
A: $rotate\,by\; 120°$
B: $rotate\,by\; 240°$

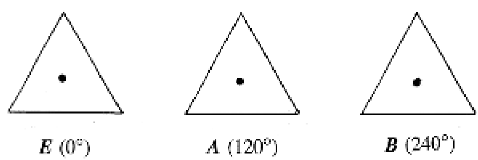
As the notation implies, operation **E** (rotation by 0°) clearly plays the role of the identity operation.

In this digital age, clocks with hands are no longer common but the terms *counterclockwise* and *clockwise* for the sense of a rotation are firmly entrenched. If a rotation when seen looking down the rotation axis toward the origin turns in the same sense as the hands of a clock, it is termed a clockwise rotation (cw), and if the rotation is in the opposite sense, it is counterclockwise (ccw) as illustrated by the sketches. This text follows the usual convention that counterclockwise rotations are positive.



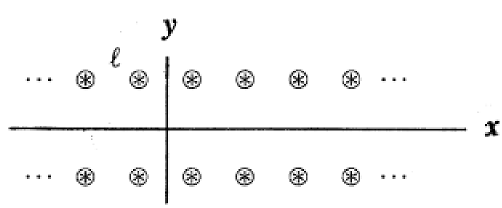


The sketch shows the effect of the operations **E**, **A**, and **B** on the triangle. The operations have left the appearance of the triangle unchanged, which is the essence of the concept of symmetry. Frank Wilczek (Nobel laureate in physics 2004) coined a pithy phrase to describe the connection between operations and symmetry: "change without change." With reference to the triangle example, we have made a change – an operation was performed on the triangle by rotating it – but the triangle still looks the same.



More generally, if an operation on an object leaves it unchanged, or *invariant*, the object must have a symmetry property. In the triangle example a 3-fold symmetry is revealed by rotation through the particular angles 0°, 120°, and 240°.

The symmetry of an equilateral triangle under certain rotations is an example of a *rotation* symmetry. There are many other examples of geometric symmetry. Consider the repeated pattern in the sketch, which could be a decorative frieze along the edge of a building. The two rows are parallel to the x-axis and are equidistant from the x-axis. The columns are all equally spaced along x by a distance ℓ, and the dots signify that the pattern extends far to the left and far to the right.



If the pattern is translated parallel to the x-axis by an integer multiple of ℓ, its appearance remains the same. This is an example of *translation* symmetry, important for the discussion of crystal lattices in Chapter 4.

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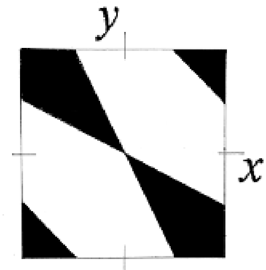
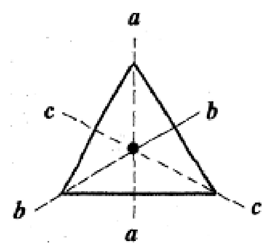
If the pattern is folded along the x-axis, the two rows coincide. Each row is a mirror image of the other, an example of *reflection* symmetry, also called *mirror* symmetry.

Symmetry is appealing and has long played a role in art and architecture, from ancient rock carvings to mosaics in ancient Rome to ephemeral foam patterns on coffee drinks.

For the black-and-white geometric pattern in the figure, the z-axis is normal to the page and passes through the origin. Rotations about z by 0° and 180° and reflections about the diagonals are symmetry operations. Rotations about z by 90° and 240° and reflections about the x- and y-axes are not symmetry operations.



The equilateral triangle has additional symmetries revealed by no longer requiring the triangle to lie in the x-y plane. The figure shows three new axes *aa*, *bb*, and *cc*. Each axis passes through an apex and is perpendicular to the opposite edge. It follows by geometry that the axes intersect at the geometric center of the triangle.



Suppose now that the triangle is "flipped" 180° about axis *aa*. The front becomes the back and vice versa; the appearance of the triangle is unchanged, so this is a symmetry operation on the triangle, and similarly for flips about axes *bb* and *cc*.

The three rotation operations in the plane and the three flip operations identify six symmetry operations for the equilateral triangle. These operations are easily demonstrated with a cardboard triangle.

Inversion symmetry is abstract and cannot be shown pictorially or demonstrated by a physical model. Space inversion reverses the signs of the coordinates so that x is replaced by $-x$, y by $-y$, and z by $-z$. These replacements are conveniently expressed by the symbol $\mapsto$, which means "maps to" or "is replaced by." Thus, inversion can be written $x \mapsto -x$, $y \mapsto -y$, and $z \mapsto -z$.

Consider a sphere of radius R, which can be described algebraically by the equation $x^2 + y^2 + z^2 = R^2$. Upon applying the space inversion operation to the coordinates, the equation is unchanged; the sphere is invariant under space inversion. We shall see important examples of inversion when symmetry and the quantum theory of atoms are discussed in Chapter 5.

But the use of symmetry in decorative arts and the description of geometric figures barely scratches the surface of its deep importance. Steven Weinberg (1933–2021, Nobel laureate in physics 1979) has written that symmetry is the "key to nature's secrets," which is why the application of symmetry principles to physical problems is the subject of this text.

1.2.2 Products of Operations

Consider again the set of three operations $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ from the triangle example discussed in Section 1.2.1. The *product* of two operations is the result of applying first one operation followed by a second. If, for example, **A** is applied first, followed by **B**, the product is written symbolically as **BA**. The operation on the right, here **A**, is considered to be applied first. Note that although the product **BA** has the appearance of "multiplication" of **B** times **A**, abstract group theory puts no restrictions on the method by which operations are actually combined. Some books on group theory use the term "multiplication" where we use "product." Such terms are symbolic only, with no reference to ordinary arithmetic.

In the example, **B** is applied to **A** "from the left." Alternatively, an operation can be applied "from the right" to give, in this case, **AB**. These same ideas are also used with equations relating operations. Equations involving operations conform to the usual rule from algebra that both sides are to be treated equally. Consider, for example, the product of two operations $\mathbf{T_1}$ and $\mathbf{T_2}$ to give a third operation $\mathbf{T_3}$:

$$\mathbf{T_2T_1} = \mathbf{T_3}.$$

Now apply an operation **C** from the left; **C** must act on both sides of the relation to maintain the equality.

$$\mathbf{CT_2T_1} = \mathbf{CT_3}$$

Applying **C** from the right gives

$$\mathbf{T_2T_1C} = \mathbf{T_3}C.$$

The distinction between operations from the left and from the right is important. The reason is that for many group operations the order of combination makes a difference, unlike the multiplication of numbers or algebraic functions. If two operations $\mathbf{T_1}$ and $\mathbf{T_2}$ are combined, the two possible products $\mathbf{T_1T_2}$ and $\mathbf{T_2T_1}$ may not necessarily be equal. However, if $\mathbf{T_1T_2} = \mathbf{T_2T_1}$, then $\mathbf{T_1}$ and $\mathbf{T_2}$ are said to *commute*.

1.2.3 Product Tables

Rotation symmetry operations on the equilateral triangle are rotations through defined angles about defined axes, so it is easy to determine the product of any two operations. Consider, for example, the product **BA**. First applying **A** produces an initial rotation of 120°. The second operation **B** causes an additional rotation through 240°, for a net result of 360° (equivalently 0°). This is the same result as using operation **E** alone, so the product is written

$$\mathbf{BA} = \mathbf{E}.$$

Table 1.1 Products of **E**, **A**, and **B**

	E	**A**	**B**
E	**EE** = **E**	**EA** = **A**	**EB** = **B**
A	**AE** = **A**	**AA** = **B**	**AB** = **E**
B	**BE** = **B**	**BA** = **E**	**BB** = **A**

Table 1.2 Products of **E**, **A**, and **B**

	E	**A**	**B**
E	**E**	**A**	**B**
A	**A**	**B**	**E**
B	**B**	**E**	**A**

The same reasoning can be used to evaluate all of the nine possible products of **E**, **A**, and **B**, being sure as a general rule to maintain the order of the operations. The products are conveniently displayed in the form of a *product table*, where an operation in the top horizontal row is applied first followed by an operation from the left-hand vertical column. For clarity in this first illustration, both the product and the net result are given in Table 1.1, but after this a table will show only net results, as in Table 1.2.

The tables show that $\mathbf{AA} = \mathbf{A}^2 = \mathbf{B}$; geometrically, two successive counterclockwise rotations by 120° give the same result as a single counterclockwise rotation by 240°. All the members of this set are powers of a single member **A**. The triangle rotation operations **E**, **A**, and **B** are *cyclic* because they can all be written as powers of **A**: $\mathbf{E} = \mathbf{A}^0$, $\mathbf{A} = \mathbf{A}^1$, and $\mathbf{B} = \mathbf{A}^2$.

Table 1.2 shows that in this particular example the operations $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ all commute with one another, for instance, $\mathbf{AB} = \mathbf{BA}$. The identity operation **E** always commutes with any operation **T** because $\mathbf{ET} = \mathbf{TE} = \mathbf{T}$.

1.2.4 The Inverse of an Operation

For any operation **T** there may be an *inverse* operation, symbolized $\mathbf{T}^{-1}$, that undoes the effect of **T** on the operand. Because the identity operation **E** always signifies no change, it follows that $\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{E}$. An operation always commutes with its inverse.

In the triangle example, **A** is a counterclockwise rotation through 120°, so one way to undo the effect of **A** is by a further counterclockwise rotation through an additional 240°, to give a net rotation of 360° = 0°. In the set $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ the inverse of

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A is identified as $\mathbf{A}^{-1} = \mathbf{B}$. By similar reasoning, $\mathbf{B}^{-1} = \mathbf{A}$. These results can also be read from Table 1.2. The entries $\mathbf{AB} = \mathbf{BA} = \mathbf{E}$ show, for example, that $\mathbf{B} = \mathbf{A}^{-1}$.

Another operation that undoes the effect of **A** is to rotate clockwise through 120° to bring the triangle back to the starting point. This clockwise rotation is equivalent to a counterclockwise rotation through −120°. This is a new operation and not a member of the operations **E**, **A**, and **B**, which are defined here only for counterclockwise rotations.

Here is an example involving inverses and a product table. Consider the set {**E**, **A**} with the following partial product tables:

	E	**A**
E	**E**	**A**
A	**A**	$\mathbf{A}^2$

What is the unidentified member $\mathbf{A}^2$? Try $\mathbf{A}^2 = \mathbf{A}$. Multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}\mathbf{A} \\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{A} &= \mathbf{E} \\ \mathbf{EA} &= \mathbf{E} \\ \mathbf{A} &= \mathbf{E} \end{aligned}$$

The result $\mathbf{A} = \mathbf{E}$ gives the dull and useless Table 1.3.

The alternative possibility $\mathbf{A}^2 = \mathbf{E}$ gives the more useful product Table 1.4 that has two distinctly different members.

In the product table for a set of operations, a given symmetry operation appears only once in each column as seen in the example. As a proof consider a set of distinctly different symmetry operations **A**, **B**, **C**, and **D**. Suppose that **A** occurs twice in the column headed by **B**, so that $\mathbf{BC} = \mathbf{A}$ and $\mathbf{BD} = \mathbf{A}$. Then $\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}$ and $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}$ so that $\mathbf{C} = \mathbf{D}$, a contradiction because the operations are assumed to be different. Similarly, each operation occurs only once in a given row.

Table 1.3 $\mathbf{A} = \mathbf{E}$

	E	**E**
E	**E**	**E**
E	**E**	**E**

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Table 1.4 $\mathbf{A}^2 = \mathbf{E}$

	E	**A**
E	**E**	**A**
A	**A**	**E**

1.3 What Is a Group?

With a solid foundation on the nature of operations, their products, and their inverses, it is time to take up the heart of the matter: the definition of a group. The definition is summarized in the following five axioms (i) to (v). They may seem a little dry, but they are needed because if a set of operations can be shown to form a group, a raft of useful theorems are then immediately applicable.

To illustrate the axioms, we shall show that the set of triangle rotation operations $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ form a group.

(i) A group consists of a set of operations called *members* of the group. We shall show that the set $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ are members of a group.

(ii) The product of any two members of a group is also a member of the group; products do not take us to new operations outside the set of group members. Table 1.2 shows that the products of **E**, **A**, and **B** are all members of the same set. Contrariwise, clockwise rotations of the triangle do not appear in Table 1.2 and are therefore not members of this group.

(iii) The group contains an identity member **E** that produces no change when combined with any group member. Table 1.2 for the triangle rotations show that $\mathbf{EE} = \mathbf{E}$, $\mathbf{EA} = \mathbf{AE} = \mathbf{A}$, and $\mathbf{EB} = \mathbf{BE} = \mathbf{B}$, showing that the notation is justified; **E** is truly the identity member in the set.

(iv) For every member **T** of a group, there is also a member $\mathbf{T}^{-1}$ in the group that is the inverse of **T**, such that $\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{E}$. As shown in Section 1.2.4 and also in Table 1.2, $\mathbf{E}^{-1} = \mathbf{E}$, $\mathbf{A}^{-1} = \mathbf{B}$, and $\mathbf{B}^{-1} = \mathbf{A}$.

(v) An additional axiom is that the products of operations are *associative* so that $\mathbf{T_1}(\mathbf{T_2T_3}) = (\mathbf{T_1T_2})\mathbf{T_3}$, where the products in parentheses are evaluated first, then combined with the remaining operation. This axiom will be satisfied automatically by the operations in applications.

Let $\boldsymbol{\Gamma}$ be the symbol for the group $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$. The number of members in a group is called the *order* of the group: $\boldsymbol{\Gamma}$ is of order 3.

Note that **E** is always a member of any group and satisfies the group definition axioms. Therefore **E** is itself a group (of order 1). If a subset of group members are themselves a group, the subset is called a *subgroup*. **E** is a trivial subgroup of every group. The whole group itself is also a trivial subgroup of the group.

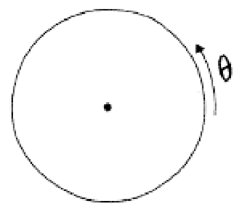
In the group Γ, the set $\{\mathbf{E}, \mathbf{A}\}$ is not a subgroup, because the product $\mathbf{AA} = \mathbf{B}$, an operation not included in the set. In a subgroup, just as in a group, the product of two operations in the subgroup must also be a member of the subgroup.

The product table for a set of operations can be checked to see whether the group axioms are satisfied. The product table tells all.

1.3.1 Discrete and Continuous Groups

Groups with a finite (countable) number of members are called *discrete* or *finite* groups. The triangle rotation group $\Gamma = \{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ has a finite number of members and is an example of a discrete group.

Consider now a flat circular disk with a perpendicular axis through its center, as suggested by the sketch. Rotation of the disk by an arbitrary counterclockwise angle θ leaves the disk invariant, so this leads us to suspect that there is a group involving rotations. The rotations form a group: rotation by 0° is the identity, two successive rotations by θ_1 and θ_2 give the same result as a single rotation by $\theta_1 + \theta_2$, and to every rotation θ there corresponds an inverse rotation $360° - \theta$.



Because θ can be any angle, this group has an "infinite" (uncountable) number of members; it is an example of a *continuous group*. A continuous group depends on a continuous parameter, in this example the angle θ. Continuous groups are important in physics, for example, in the quantum-mechanical wave function of a hydrogen atom, which depends on two continuous parameters: the polar angle θ and the azimuthal angle ϕ.

1.4 Examples of Groups

1.4.1 Abelian Groups

A group in which all of the members commute is called an *Abelian* group, after the Norwegian mathematician Niels Henrik Abel (1802–29). The triangle rotation group composed of the set $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ is an Abelian group. This group is also a cyclic group and can be written as $\{\mathbf{E}, \mathbf{A}, \mathbf{A}^2\}$ as shown in Section 1.2.3.

All groups of order less than 6 are Abelian.

1.4.2 The 32 Group

Table 1.5 is the product table for a group of order 6, a popular example in textbooks on group theory. It is termed the **32** ("three-two") group.

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Table 1.5 The **32** group (order 6)

	E	**A**	**B**	**C**	**D**	**F**
E	**E**	**A**	**B**	**C**	**D**	**F**
A	**A**	**E**	**F**	**D**	**C**	**B**
B	**B**	**D**	**E**	**F**	**A**	**C**
C	**C**	**F**	**D**	**E**	**B**	**A**
D	**D**	**B**	**C**	**A**	**F**	**E**
F	**F**	**C**	**A**	**B**	**E**	**D**

The product table shows that the group axioms are satisfied:

(i) Only members from the set appear in the product table.

(ii) The product of two members is a member of the set.

(iii) There is an identity member identified as **E** that obeys the properties of the identity operation such as $\mathbf{EA} = \mathbf{AE} = \mathbf{A}$.

(iv) Every member of a group has an inverse in the set as shown by products such as $\mathbf{DF} = \mathbf{E}$ so that $\mathbf{F} = \mathbf{D}^{-1}$.

Members of a given group may or may not commute. For example, $\mathbf{AB} = \mathbf{F}$ and $\mathbf{BA} = \mathbf{D}$. **A** and **B** do not commute, so **32** is not an Abelian group. It is the smallest group that is nonAbelian, accounting for its popularity as a teaching tool.

Table 1.5 shows that the **32** group has three nontrivial subgroups of order 2, namely $\{\mathbf{E}, \mathbf{A}\}$, $\{\mathbf{E}, \mathbf{B}\}$, and $\{\mathbf{E}, \mathbf{C}\}$ and also a subgroup of order 3 $\{\mathbf{E}, \mathbf{D}, \mathbf{F}\}$, but no others. A theorem in group theory states that for a group of order n each of its subgroups has an order that is a factor of n. The example of the **32** group demonstrates this theorem because $6 = 2 \cdot 3$ for the nontrivial subgroups of orders 2 and 3. $6 = 6 \cdot 1$ is satisfied by the trivial subgroup $\{\mathbf{E}\}$ of order 1 and by the group itself of order 6.

It follows from this theorem that if the order of a group is a prime number, the group has no nontrivial subgroups and must therefore be a cyclic group. The group $\mathbf{\Gamma}$ of order 3 is an example.

1.4.3 The Permutation (Symmetric) Group

This section discusses the apparently different example of the *permutation group*. Here is the permutation group of order 6.

$$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix} \quad \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$$

$$\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3 \quad \mathbf{P}_4 \quad \mathbf{P}_5 \quad \mathbf{P}_6$$

A permutation rearranges a set of numbers. A permutation group of degree n has n different numbers, so there are n choices for the first number in the permutation, $n-1$ for the second ... hence $n! = n \cdot (n-1) \dots 1$ different permutations. This example is a permutation group of degree 3: there are 3 numbers and $3! = 3 \cdot 2 \cdot 1 = 6$ permutations as shown.

A permutation can be viewed as a one-to-one mapping of n different numbers into a possibly different order. The permutation $\mathbf{P}_1$ does not cause a reordering, so it is evidently the identity operation $\mathbf{E}$. Using the $\mapsto$ "maps to" symbol, $j \mapsto k$ means that the number j is replaced by k. For example, the permutation $\mathbf{P}_5$ is the reordering $1 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$. As another example, consider a function of three variables $f(x_1, x_2, x_3) = {x_3}^2 + x_1 x_2$. If permutation $\mathbf{P}_5$ is applied to f, it becomes ${x_2}^2 + x_3 x_1$.

A permutation is classed as *even* if the bottom row of its matrix requires an even number of interchanges (*transpositions*) to bring it to standard ascending numerical order, and as *odd* if it involves an odd number. For example, $\mathbf{P}_2$ is odd, because in the bottom row only one interchange $1 \leftrightarrow 2$ brings the bottom row to the numerical order 1 2 3. $\mathbf{P}_5$ is even, because two interchanges $1 \leftrightarrow 3$ followed by $3 \leftrightarrow 2$ are required.

The permutation group of degree n is also called the *symmetric* group of degree n. The reason for the name "symmetric" is that permuting the labels of identical objects (such as electrons) does not alter the symmetry – another example of "change without change." The symmetric group of degree n has been given a variety of symbols, and this text uses $\mathbf{S}_n$. The factorial $n!$ increases more rapidly with n than either powers x^n or exponentials e^{nx}, so $\mathbf{S}_n$ can be a very large group even for relatively small n; for example, $\mathbf{S}_{10}$ has $3,628,800$ group elements.

Products of Permutations

To show how to find the product of two permutations, walk through the evaluation of $\mathbf{P}_4\mathbf{P}_5$ as an example. Consider an operand x_{123} with three subscripts *123*. First apply the permutation $\mathbf{P}_5$, written as $\mathbf{P}_5\, x_{123} = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)x_{123}$. $\mathbf{P}_5$ says $1 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$. The result is therefore $\mathbf{P}_5\, x_{123} = x_{312}$. Now apply $\mathbf{P}_4$ to this result, $\mathbf{P}_4\, x_{312} = \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)x_{312}$. $\mathbf{P}_4$ says $1 \mapsto 3$, $2 \mapsto 2$, and $3 \mapsto 1$ to give the final product $\mathbf{P}_4\mathbf{P}_5\, x_{123} = x_{132}$. This is the same permutation as $\mathbf{P}_3\, x_{123} = x_{132}$, so $\mathbf{P}_4\mathbf{P}_5 = \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right) = \mathbf{P}_3$.

Here is a simple way to find the product of permutations. Consider again the product $\mathbf{P}_4\mathbf{P}_5 = \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$. Rearrange the entries in $\mathbf{P}_4$ so that the top row of the $\mathbf{P}_4$ matrix duplicates the bottom row of $\mathbf{P}_5$: $\left(\begin{smallmatrix}3&1&2\\1&3&2\end{smallmatrix}\right)\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$. The result is $\mathbf{P}_3$. It is as if the top row of the second permutation "cancels" the bottom row of the first.

Proceeding in this fashion develops the product Table 1.6 for the permutation group of degree 3. Study of the product table verifies that the six permutations form a group because there is an identity $\mathbf{P}_1$, each permutation occurs only once in each row and column, and each permutation has an inverse, as shown in the table, wherever a product is $\mathbf{E}$.

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Cycle Notation

Another way of representing a permutation is by *cycles*. A permutation cycle is developed by tracing the permutation from one number to another until the cycle begins to repeat. Consider permutation $\mathbf{P_2}$: starting with 1 in the top row, $1 \mapsto 2$ and $2 \mapsto 1$, ending a cycle of length 2. Continuing to the next unused number 3, $3 \mapsto 3$ is a cycle of length 1. Combining, $\mathbf{P_2}$ is written in cycle notation as (1 2)(3). Similarly,

$$\begin{array}{lll} \mathbf{P}_1\ (1)(2)(3) & \mathbf{P}_2\ (1\,2)(3) & \mathbf{P}_3\ (1)(2\,3) \\ \mathbf{P}_4\ (1\,3)(2) & \mathbf{P}_5\ (1\,3\,2) & \mathbf{P}_6\ (1\,2\,3). \end{array}$$

A cycle of length 1 means that the permutation does not change that number. Cycles of length 1 are sometimes omitted in cycle notation.

To read a cycle, note that any number in the cycle maps to the number at its right, and the number at the right-hand end of the cycle is considered to return to the start of the cycle at the left-hand end. Take as an example the cycle (3 5 2 6 1 4). Starting from number 3, the permutation maps 3 to the number 5 at its right: $3 \mapsto 5$, etc. The last element 4 cycles around to the beginning so that $4 \mapsto 3$.

1.5 Matrix Representations of Groups

Mathematicians have proved a great number of theorems on the properties of abstract groups, but for work in physics it is a help to be able to relate abstract groups to calculable mathematics. For applications of group theory, the mathematical objects of choice are *matrices*, and the rule of combination (product) is matrix multiplication. For applications in physics, group theory is much less the theory of abstract groups and much more the theory of matrix representations, as this text amply demonstrates.

1.5.1 Isomorphism

Comparing the product Table 1.5 for **32** and Table 1.6 for $\mathbf{S}_3$ shows they have exactly the same form except for the names of the members. When every member of one

Table 1.6 Permutation group of degree 3

	$\mathbf{E}$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$
$\mathbf{E}$	$\mathbf{E}$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$
$\mathbf{P}_2$	$\mathbf{P}_2$	$\mathbf{E}$	$\mathbf{P}_6$	$\mathbf{P}_5$	$\mathbf{P}_4$	$\mathbf{P}_3$
$\mathbf{P}_3$	$\mathbf{P}_3$	$\mathbf{P}_5$	$\mathbf{E}$	$\mathbf{P}_6$	$\mathbf{P}_2$	$\mathbf{P}_4$
$\mathbf{P}_4$	$\mathbf{P}_4$	$\mathbf{P}_6$	$\mathbf{P}_5$	$\mathbf{E}$	$\mathbf{P}_3$	$\mathbf{P}_3$
$\mathbf{P}_5$	$\mathbf{P}_5$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_2$	$\mathbf{P}_6$	$\mathbf{E}$
$\mathbf{P}_6$	$\mathbf{P}_6$	$\mathbf{P}_4$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{E}$	$\mathbf{P}_5$

group corresponds to one and only one member of another and vice versa, the groups are said to be *isomorphic* (from Greek "identical form"). In the example of **32** and $\mathbf{S}_3$ the correspondence between the two sets of members is $\mathbf{E} \leftrightarrow \mathbf{P}_1$, $\mathbf{A} \leftrightarrow \mathbf{P}_2$, $\mathbf{B} \leftrightarrow \mathbf{P}_3$, $\mathbf{C} \leftrightarrow \mathbf{P}_4$, $\mathbf{D} \leftrightarrow \mathbf{P}_5$, and $\mathbf{F} \leftrightarrow \mathbf{P}_6$.

$$\underset{\mathbf{E}}{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}} \quad \underset{\mathbf{B}}{\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}} \quad \underset{\mathbf{C}}{\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}} \quad \underset{\mathbf{D}}{\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}} \quad \underset{\mathbf{F}}{\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}}$$

Incidentally, these groups are also isomorphic to the group of six symmetry operations (rotations and flips) of the equilateral triangle.

If a set of matrices is isomorphic to the members of an abstract group, the matrices are called a *matrix representation* of the group.

1.5.2 Homomorphism

Sometimes the requirement is dropped that each representation has a unique 1-to-1 correspondence between members of two groups. Instead of an isomorphism, this is a *homomorphism*. The only requirement for a homomorphism is that the matrices have the same product table as the abstract group.

Here is a simple homomorphism of $\mathbf{\Gamma}$ (Table 1.2) in terms of 1-dimensional matrices.

$$\underset{\mathbf{E}}{(1)} \quad \underset{\mathbf{A}}{(1)} \quad \underset{\mathbf{B}}{(1)}$$

This set of permutations obviously obeys the product table for $\mathbf{\Gamma}$. The homomorphism where every group member is represented by the matrix (1) is called the *identity homomorphism* **1**. Every group has an identity homomorphism.

The **32** group of order 6 in Table 1.5 has, like every group, an identity homomorphism. However, it also has another distinctly different 1-dimensional homomorphism.

$$\underset{\mathbf{E}}{(1)} \quad \underset{\mathbf{A}}{(-1)} \quad \underset{\mathbf{B}}{(-1)} \quad \underset{\mathbf{C}}{(-1)} \quad \underset{\mathbf{D}}{(1)} \quad \underset{\mathbf{F}}{(1)}$$

The permutation group $\mathbf{S}_3$ in Table 1.6 is isomorphic to the group of order 6, so this homomorphism applies to it also with a change of labels.

These homomorphisms are said to be *unfaithful*, because it is possible to form a matrix product such as $\mathbf{AB} = \mathbf{D}$ that does not agree with the group's product table. Put another way, it is impossible to reconstruct the group's product table from a homomorphism. In contrast, an isomorphism is *faithful*, because there is a 1-to-1 correspondence between group members and their matrices.

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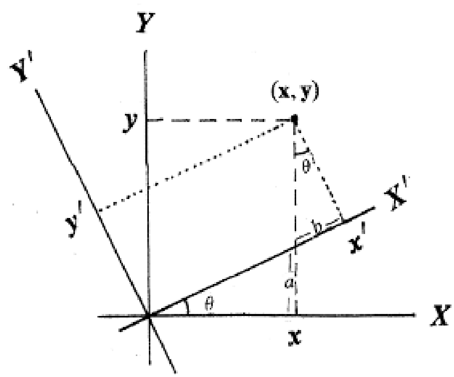
1.5.3 An Example of a Matrix Representation

This example sets up an isomorphism between the equilateral triangle group $\mathbf{\Gamma} = \{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ and a set of matrices. Because the group operations of $\mathbf{\Gamma}$ describe rotations in the plane, we start by seeing how to use matrices to express a planar rotation.

To lay the groundwork, consider two Cartesian coordinate systems in the plane. The X′-Y′ axes have the same origin as the X-Y axes, but they are rotated by angle θ as the figure shows.

A point in the plane has coordinates (x, y) with respect to the X-Y axes, as shown by the dashed lines. The same point has coordinates (x', y') with respect to the X′-Y′ axes, as shown by the dotted lines.



Express (x', y') in terms of (x, y) using trigonometry.

$$
\begin{aligned}
a &= x\,\tan\theta \qquad b = (y-a)\sin\theta \\
x' &= \frac{x}{\cos\theta} + b = \frac{x}{\cos\theta} + (y-a)\sin\theta \\
&= \frac{x}{\cos\theta} + y\,\sin\theta - \frac{x\,\sin^2\theta}{\cos\theta} = \frac{x}{\cos\theta} - \frac{x\,(1-\cos^2\theta)}{\cos\theta} + y\,\sin\theta \\
&= x\,\cos\theta + y\,\sin\theta \qquad (1.1) \\
y' &= (y-a)\cos\theta = (y - x\,\tan\theta)\,\cos\theta \\
&= -x\,\sin\theta + y\,\cos\theta \qquad (1.2)
\end{aligned}
$$

Writing Eqs. (1.1) and (1.2) in matrix form:

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
$$

One-column matrices such as $\binom{x}{y}$ are called *column vectors*. The matrix itself is square 2×2 with two rows and two columns.

The three members $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ of group $\mathbf{\Gamma}$ correspond geometrically to specific angles of rotation $0°, 120°, 240°$; evaluate the rotation matrix for each angle. Trigonometric identities such as $\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \pm \sin(\theta_1)\sin(\theta_2)$ and $\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \cos(\theta_1)\sin(\theta_2)$ can be useful for referring sines and cosines to the first quadrant. To keep the signs straight, note that $\sin\theta \geq 0$ only in the first and second quadrants, and $\cos\theta \geq 0$ only in the first and fourth quadrants.

A common notation for the matrix representation of a group member $\mathbf{T}$ is $D(\mathbf{T})$, from the German *Darstellung*, representation. Here is a matrix representation of the members of $\mathbf{\Gamma}$:

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$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$D(\mathbf{E}) \qquad D(\mathbf{A}) \qquad D(\mathbf{B}).$$

The matrix $D(\mathbf{E})$ clearly acts as the group identity because its product with any matrix in the set leaves that matrix unchanged.

This representation is faithful. If $\mathbf{C}$ and $\mathbf{F}$ are members of a group, a faithful representation $D(\mathbf{C})D(\mathbf{F}) = D(\mathbf{CF})$ agrees with the product table entry $\mathbf{CF}$ because of the isomorphism between the representation matrices and the group members. As an example, consider the product $D(\mathbf{A})D(\mathbf{B})$ for the group $\boldsymbol{\Gamma}$.

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{3}{4} & \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & \frac{3}{4} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The product is $D(\mathbf{E})$ in agreement with the $\boldsymbol{\Gamma}$ product table $\mathbf{AB} = \mathbf{E}$. The result also shows that the matrix $D(\mathbf{A})$ has an inverse equal to $D(\mathbf{B})$, and vice versa.

A mathematical theorem states that a matrix has a unique inverse only if the determinant of the matrix $\neq 0$. The determinants of all three representation matrices of $\boldsymbol{\Gamma}$ are $\neq 0$ assuring that each of the matrices has an inverse.

In this text the matrix representation $D(\mathbf{A})$ of a group member $\mathbf{A}$ is often written simply A.

In summary, we found a faithful matrix representation of group $\boldsymbol{\Gamma}$ by establishing an isomorphism with the abstract group members, in this example with the help of the algebraic geometry of rotations. We showed that there is a correspondence between the matrices and the abstract group members and also showed that the matrices agree with the group's product table, hence they obey the criterion for a faithful representation.

1.6 Matrix Algebra

Matrix notation, matrix multiplication, and determinants of matrices are often studied in math courses. This section is intended as a refresher.

A general notation is needed to describe matrices of arbitrary size. Every element of a matrix is given two subscripts as, for example, a_{ij}. The first subscript (here i) signifies the row, and the second subscript (here j) signifies the column. If matrix $D(\mathbf{A})$ has n rows and n columns, it is termed a square $n \times n$ matrix and in element notation is written

$$D(\mathbf{A}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

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If a matrix has m rows and n columns, it is described as $m \times n$. Here is a 2×3 matrix:

$$\begin{pmatrix} 4 & 7 & 0 \\ 1 & 3 & -1 \end{pmatrix}.$$

1.6.1 A Constant Times a Matrix

If a constant multiplies a matrix, every element in the matrix is multiplied by that constant. Let $B = cA$ where c is a constant and A, B are matrices. In matrix element notation:

$$b_{ij} = ca_{ij}.$$

Every element of A is multiplied by c as shown:

$$c \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2c & 0 \\ c & 0 & -c \\ 0 & 0 & c \end{pmatrix}.$$

1.6.2 Addition of Matrices

Let $C = A + B$ be the sum of two matrices, all with the same number of rows and columns. In matrix element notation:

$$c_{ij} = a_{ij} + b_{ij}.$$

Corresponding elements are added. Here is an example:

$$\begin{pmatrix} 1 & 4 & 0 \\ 3 & 0 & -1 \\ 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 2 & -3 \\ -2 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 6 & -3 \\ 1 & 0 & 1 \\ 3 & 0 & 5 \end{pmatrix}.$$

1.6.3 Products of Matrices

Consider the product $C = AB$ of $n \times n$ matrices. The element c_{ik} of the product matrix is written

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

The sum runs from $j = 1$ to $j = n$ as indicated on the summation sign. Often the range of the summation is clear from the context, so that only the subscript summed over (a dummy variable) needs to be shown on the summation sign:

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad \longrightarrow \quad \sum_{j} a_{ij} b_{jk}.$$

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As an example, consider the product of two 2×2 matrices $AB = C$. Written with explicit subscripts:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} & a_{11}\,b_{12} + a_{12}\,b_{22} \\ a_{21}\,b_{11} + a_{22}\,b_{21} & a_{21}\,b_{12} + a_{22}\,b_{22} \end{pmatrix}$$
$$= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

When multiplying ordinary numbers, the result is independent of the order: $2 \times 3 = 3 \times 2$. This may not be the case with matrices, where the order can be important as the following example demonstrates:

$$\begin{aligned} AB &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ BA &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &\neq AB. \end{aligned}$$

Matrices can be multiplied even if they are not square, if the number of columns in the first matrix is equal to the number of rows in the second, so that the first matrix is $m \times s$ and the second is $s \times n$. The product matrix is then $m \times n$. Here is the product of a 2×3 matrix and a 3×1 matrix to give a 2×1 result:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31} \\ a_{21}\,b_{11} + a_{22}\,b_{21} + a_{23}\,b_{31} \end{pmatrix}.$$

1.6.4 Determinant of a Matrix

The determinant of a $n \times n$ matrix has $n!$ terms, where each term is the product of n different elements of the matrix. The determinant of a square matrix A with coefficients a_{ij} is symbolized $|A|$. Equations (1.3) show the determinants of a 2×2 matrix (2! = 2 terms) and a 3×3 matrix (3! = 6 terms).

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$$

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{21}\,(a_{12}\,a_{33} - a_{13}\,a_{32}) \end{aligned}$$

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$$
\begin{aligned}
&+ a_{31}\,(a_{12}\,a_{23} - a_{13}\,a_{22}) \\
&= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} + a_{13}\,a_{21}\,a_{32} - a_{12}\,a_{21}\,a_{33} \\
&\quad + a_{12}\,a_{23}\,a_{31} - a_{13}\,a_{22}\,a_{31}
\end{aligned} \tag{1.3}
$$

To illustrate one method of determining the algebraic signs of the terms, consider the case $n = 3$ as an example. In the last line of Eq. (1.3), the terms of the expansion are rearranged so that they all have the form $a_{1i}\,a_{2j}\,a_{3k}$. If the permutation $\left(\begin{smallmatrix}1 & 2 & 3\\ i & j & k\end{smallmatrix}\right)$ is even, the term has a plus sign; if the permutation is odd, the sign is minus. For the second term in Eq. (1.3), the permutation $\left(\begin{smallmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{smallmatrix}\right)$ is odd (only one interchange $2 \leftrightarrow 3$ is required), so the term has a minus sign as shown. Using this method to evaluate the determinant of a matrix doesn't apply if the elements are strictly numeric, but it is fine if the elements are symbolic with explicit subscripts. The abstract form is important in the quantum theory of multi-electron atoms where abstract permutations are directly related to the interchange of identical electrons.

1.6.5 The Kronecker Delta

Matrix notation presents an opportunity to introduce a useful new symbol. Consider $\mathbf{TT}^{-1} = \mathbf{E}$. The matrix for the identity has 1 everywhere along the main diagonal and 0 everywhere else. In matrix form the product is therefore written as

$$
\sum_j t_{ij}\, t_{jk}{}^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.
$$

The *Kronecker delta symbol* δ_{ij} is useful to denote the matrix elements of the identity matrix. It is defined as

$$
\delta_{ij} = \delta_{ji} = \begin{cases} 1 & i = j \\ 0 & i \neq j \,. \end{cases}
$$

Using the Kronecker delta the matrix elements are

$$
\sum_j t_{ij}\, t_{jk}{}^{-1} = \delta_{ik}\,.
$$

1.7 Special Matrices

This section defines some special matrices that arise in applications.

1.7.1 The Complex Conjugate of a Matrix

The range of matrix elements can be enlarged to include complex numbers of the form $\alpha + \beta \boldsymbol{i}$ where α and β are real numbers that have a real part α and an imaginary

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part $\beta \boldsymbol{i}$ and where $\boldsymbol{i} = \sqrt{-1}$. The *complex conjugate* is $(\alpha + \beta \boldsymbol{i})^* = \alpha - \beta \boldsymbol{i}$. If u is a complex number, its complex conjugate is denoted by u^*. The complex conjugate of a matrix A is A^*, and its elements are a^*_{ij} as in this example:

$$A = \begin{pmatrix} 1 + 2\boldsymbol{i} & 4 - \boldsymbol{i} \\ 3 + 4\boldsymbol{i} & 6 \end{pmatrix} \qquad A^* = \begin{pmatrix} 1 - 2\boldsymbol{i} & 4 + \boldsymbol{i} \\ 3 - 4\boldsymbol{i} & 6 \end{pmatrix}.$$

1.7.2 The Transpose of a Matrix

Consider a matrix A with elements a_{ij}. The *transpose* $\tilde{A}$ of A is the matrix formed by interchanging the rows and columns of A so that the elements of $\tilde{A}$ are $\tilde{a}_{ij} = a_{ji}$. It follows that elements on the main diagonal are unchanged. Here is an example of a matrix and its transpose:

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 3 & 6 & 0 \\ 2 & 1 & 5 \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 1 \\ 7 & 0 & 5 \end{pmatrix}.$$

The transpose of a matrix product AB is $\tilde{B}\tilde{A}$. Proof is left to the problems.

1.7.3 The Adjoint of a Matrix

The *adjoint* of a matrix A is its transpose with complex conjugates of its elements. The adjoint is symbolized $A^\dagger$. The result is the same whether the transpose or the complex conjugate is done first:

$$A = \begin{pmatrix} 1 + 2\boldsymbol{i} & 4 - \boldsymbol{i} \\ 3 + 4\boldsymbol{i} & 6 \end{pmatrix} \qquad A^\dagger = \begin{pmatrix} 1 - 2\boldsymbol{i} & 3 - 4\boldsymbol{i} \\ 4 + \boldsymbol{i} & 6 \end{pmatrix}.$$

1.7.4 Hermitian Matrices

If a matrix A is equal to its adjoint, so that $A = A^\dagger$, the matrix is said to be *Hermitian*. Here are two examples of Hermitian matrices, one real, the other complex. All Hermitian matrices are square:

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 6 & 0 \\ 2 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 - \boldsymbol{i} & 2 \\ 3 + \boldsymbol{i} & 0 & -4\boldsymbol{i} \\ 2 & 4\boldsymbol{i} & 3 \end{pmatrix}.$$

The left-hand matrix is a *real symmetric* matrix; all real symmetric matrices are Hermitian. Note also that for every Hermitian matrix, the diagonal elements must be real even if the matrix is complex.

Hermitian matrices are fundamental in quantum mechanics. They guarantee that physical quantities calculated according to quantum mechanics are always real numbers as they must be.

1.7.5 Unitary and Orthogonal Matrices

If the adjoint of a matrix A is equal to its inverse so that $A^\dagger = A^{-1}$, the matrix is said to be *unitary*. It follows that the product of a unitary matrix with its adjoint is the identity. If A is real $A = A^*$, then its transpose equals its inverse $\tilde{A} = A^{-1}$, and A is said to be *real orthogonal* or simply *orthogonal*. All unitary and orthogonal matrices are square.

The identity matrix E is an orthogonal matrix because $E^\dagger = E$ and the product $EE = E$ shows that E is its own inverse, $E^{-1} = E$.

Here is the product of a matrix with its adjoint. It is equal to the identity, showing that the matrix is orthogonal.

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Orthogonal matrices have the property that if their rows or columns are considered to be the components of n-dimensional vectors with respect to a set of unit basis vectors $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \ldots, \hat{\boldsymbol{e}}_n$, the row and column vectors are both unit orthogonal vectors (called an *orthonormal set*). Take, for instance, the first row of the left-hand matrix in the example. Its scalar product with itself is $(\frac{1}{2}\hat{\boldsymbol{e}}_1 - \frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_2 + 0) \cdot (\frac{1}{2}\hat{\boldsymbol{e}}_1 - \frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_2 + 0) = (\frac{1}{4} + \frac{3}{4}) = 1$. The scalar product of its first and second column vectors is $(\frac{1}{2}\hat{\boldsymbol{e}}_1 + \frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_2 + 0) \cdot (-\frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_1 + \frac{1}{2}\hat{\boldsymbol{e}}_2 + 0) = (-\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}) = 0$.

1.8 A Brief History of Group Theory

The major part of group theory was developed by mathematicians in the nineteenth and early twentieth centuries. Space does not permit citing all the famous names, but a few are especially worthy of note. The French mathematician and political activist Évariste Galois (1811–32) applied the name "group" to this branch of mathematics and is considered to be the founder of group theory. Sadly, he died at the young age of 20 after losing a duel. Ferdinand Georg Frobenius (1849–1917) in Germany introduced the fundamental concepts of characters and representations.

Amalie ("Emmy") Noether (1882–1935), "the greatest woman mathematician who ever lived," should also be mentioned. German-born, she taught in Germany until forced from her post in 1933. She spent the brief remaining years of her life at Bryn Mawr College in Pennsylvania and at the Institute for Advanced Study near Princeton University. Her greatest achievement was her "wonderful theorem" proving that the symmetries of a physical system lead to conservation laws obeyed by the system, another indication of the importance of group theory in physics.

The advent of quantum mechanics in the 1920s transformed group theory from an abstract mathematical discipline to a powerful tool for understanding nature, particularly phenomena in atomic physics, solid state physics, nuclear physics, and strange particle physics. In the early days of quantum mechanics some physicists did not appreciate the power of group theory in physics and called it a "pest," perhaps because their universities had not offered physics courses in group theory.

Mathematical physicist Eugene Wigner (1902–95, Nobel laureate in physics 1963) was one of the first to realize the value of group theory, which he used to explain the energy levels of electrons in atoms. Born in Budapest, he later became an American citizen. He was at Princeton University for much of his career and he shared the 1963 Nobel Prize for applying group theory to nuclear structure.

1.9 Brief Bios

Arthur Cayley (1821–95), a British mathematician, was the first to establish criteria for defining a group. In his earlier days as a lawyer he wrote many papers in mathematics as a hobby but later left his law practice, eventually becoming a professor of mathematics at Cambridge University.

The German mathematician Leopold Kronecker (1823–91) obtained his doctorate in mathematics at the age of 22, but then spent the next eight years managing valuable properties inherited from a wealthy uncle. By age 30, Kronecker had enough money for a comfortable life and returned to mathematics.

Hermitian matrices are named after the French mathematician Charles Hermite (1822–1901). Hermite spent his time studying great mathematicians like Lagrange and Gauss. He neglected humdrum course material and as a consequence did poorly on the standard examinations. Luckily, professors recognized his abilities, and Hermite went on to become one of the greatest mathematicians.

Summary of Chapter 1

Chapter 1 introduces fundamental concepts of group theory.

a) Group theory deals with abstract operations. Combinations of operations ("products") can be displayed in a product table.

b) The meaning of symmetry has been termed "change without change" because applying a symmetry operation leaves an object the same as at the start.

c) To be classed as a group, a set of operations must satisfy certain axioms. The product of any two members of the set must also be a member of the set. There must be an identity member in the set. Each member of the set must have an inverse in the set.

d) Examples of groups are presented, especially the triangle rotation group, the **32** group, and the permutation group $\mathbf{S}_3$ of order 6.

e) Groups are described as discrete if they have a finite number of group members, or as continuous with an uncountable number of members that depend on a continuous parameter, for example, a rotation angle.

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f) Isomorphism and homomorphism are possible relations between different groups. Two groups are isomorphic if there is a unique one-to-one correspondence between the members of one group and the members of the other, so that each obeys the same product table. A homomorphism between two groups also obeys the same product table, but a member of one group corresponds to more than one member of the other.

g) A group has a faithful representation by matrices where there is an isomorphism between the group elements and the matrices so that the matrices obey the group's product table.

h) Matrix algebra is the principal mathematical tool in applications of group theory. Examples are presented of matrix notation, matrix addition $A + B$, matrix multiplication AB, and determinant of a matrix $|A|$. Several special matrices are defined: complex conjugate A^*, transpose $\tilde{A}$, adjoint $A^\dagger = \tilde{A}^*$, Hermitian $A = A^\dagger$, unitary $A^{-1} = A^\dagger$, orthogonal $A^{-1} = \tilde{A}$.

i) The Kronecker delta $\delta_{ij} = \delta_{ji} = 1$ for $i = j$ or 0 for $i \neq j$.

Problems and Exercises

1.1 Evaluate $\mathbf{T}e^x$ for the transformation $\mathbf{T}x = -x$.

1.2 Evaluate $\mathbf{T}\cos x$ for the transformation $\mathbf{T}x = -x$.

1.3 Express the inverse $(\mathbf{AB})^{-1}$ of the product $\mathbf{AB}$ in terms of $\mathbf{A}$ and $\mathbf{B}$.

1.4 Show that the positive and negative real integers (including 0) form a group under the operation of addition.

1.5 Show that the real integers $1, 2, \ldots$ do not form a group under the operation of multiplication.

1.6 Show that the group members $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ for the 3-fold rotation of an equilateral triangle described in Section 1.2.1 can be written $\{\mathbf{E}, \mathbf{B}, \mathbf{B}^2\}$.

1.7 Write the product table for a group of order 2.

1.8 Write a product table for the 4-fold rotations of a square about its geometric center. Is this a cyclic group?

$\mathbf{E}$: $rotate\, by$ 0°

$\mathbf{A}$: $rotate\, by$ 90°

$\mathbf{B}$: $rotate\, by$ 180°

$\mathbf{C}$: $rotate\, by$ 270°

1.9 Prove that a given group member occurs only once in a given column of the product table.

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1.10 Cayley proved that every discrete group of order n can be found in the product table for S_n. Illustrate this result for S_3 (Table 1.5 or Table 1.6). What does your result say about the number of distinct groups of order 3?

1.11 There are only two distinct groups of order 4. One of them is the group for rotations of a square, Problem 8. Here is the product table of the other.

	E	**K**	**L**	**M**
E	**E**	**K**	**L**	**M**
K	**K**	**E**	**M**	**L**
L	**L**	**M**	**E**	**K**
M	**M**	**L**	**K**	**E**

Show that $\{\mathbf{E}, \mathbf{K}, \mathbf{L}, \mathbf{M}\}$ indeed form a group. Is it an Abelian group? Is it a cyclic group? Explain.

1.12 Find three subgroups of order 2 in the product table for the permutation group of order 6, Table 1.6.

1.13 Class each of the following permutations as even or odd.

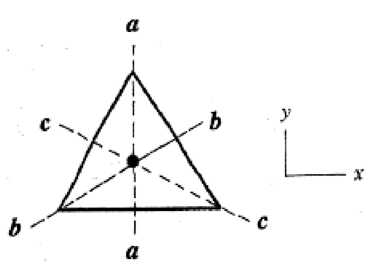
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

(a) (b) (c)

1.14 In the permutation (5 3 6 1 4 2) what number does 6 map to?

1.15 In the permutation (3 2 6 4 1 5) what number does 5 map to?

1.16 Find a matrix representation for the "flip" of an equilateral triangle about its aa axis.



1.17 Consider the group $\mathbf{\Gamma} = \{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$ for the rotations of an equilateral triangle, Table 1.2. Are the following matrices a homomorphic representation of $\mathbf{\Gamma}$?

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$$\begin{matrix} (1) & (-1) & (-1) \\ \mathbf{E} & \mathbf{A} & \mathbf{B} \end{matrix}$$

1.18 A *diagonal* matrix is a matrix that has zero elements except on its diagonal. Show that the product of two diagonal matrices is a diagonal matrix.

1.19 Find matrix representations for the 4-fold rotations of a square as described in Problem 8.

1.20 Which of the following matrices has an inverse?

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 2 \\ 3 & 0 & -1 \\ 2 & -2 & 1 \end{pmatrix}$$

$$(a) \qquad\qquad (b) \qquad\qquad (c)$$

1.21 Consider the matrices A and B.

$$A = \begin{pmatrix} 3 & 2 & -3 \\ -2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 3 & 1 \\ -2 & 0 & 5 \end{pmatrix}$$

Find $A + B$, $A - B$, the product AB, and the product BA.

1.22 Consider the matrices A and B.

$$A = \begin{pmatrix} -1 & 5 & 4 \\ -3 & 3 & 4 \\ 2 & 0 & -3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & -2 & -1 \\ 5 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$$

Find $A + B$, $A - B$, the product AB, and the product BA.

1.23 Consider these two matrices and show that the determinant of their product is equal to the product of their determinants. This is a general result true for the product of any two square matrices.

$$\begin{pmatrix} 2 & -1 & 2 \\ 2 & 0 & 3 \\ 3 & -1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -2 \\ 0 & 3 & -1 \\ 2 & 1 & -2 \end{pmatrix}$$

1.24 For $n \times n$ matrices A and B, show that $\widetilde{AB} = \tilde{B}\tilde{A}$.

1.25 For the matrix A, find A^*, $\tilde{A}$, and $A^\dagger$.

$$A = \begin{pmatrix} 1 & 5 & 2 \\ 3 & 0 & -1 \\ 4 & -2 & 1 \end{pmatrix}$$

1.26 For the matrix A, find A^*, $\tilde{A}$, and $A^\dagger$.

$$A = \begin{pmatrix} 1 & 3 + \boldsymbol{i} & 2\boldsymbol{i} \\ 3 & 0 & -1 + 2\boldsymbol{i} \\ 4 - 3\boldsymbol{i} & -2 & 1 - \boldsymbol{i} \end{pmatrix}$$