ALTITUDES OF A GENERAL n-SIMPLEX *

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Abstract

The purpose of this paper is to prove that the altitudes of an n-simplex (a simplex in an n-space) S form an *associated set* of $n+1$ lines (see Baker, [4] for $n = 4$) such that any $(n-2)$-space meeting n of them meets the $(n+1)$th too. As an immediate consequence 2 quadrics are associated with S, one touching its primes at the respective feet of its altitudes and the other touching $n(n+1)$ primes, n parallel to each of its altitudes and 2 through each of its $(n-2)$-spaces. Certain special cases are also mentioned.

1. Introduction

1.1. The associated character of the altitudes of an n-simplex S was anticipated much earlier as confirmed by Professor H. S. M. Coxeter in a private letter dated 19.3.1959 wherein he says: 'I am sure the altitudes of a simplex are $n+1$ associated lines. In hyperbolic or elliptic space, they would join corresponding vertices of two absolute polar simplexes, and the Euclidean case would follow by a limiting process.'

Again the $(n-2)$-spaces normal to the plane faces of S at their respective orthocentres were observed (Mandan [28]) to meet its altitudes, each parallel to $\binom{n}{3}$ of them, as a further indication.

For $n = 4$, it is already an established fact (Mandan [16]). For $n = 3$, the altitudes of a tetrahedron form a hyperbolic group (Court [8]) or 4 generators of one system of a quadric satisfying the desired conditions of an associated set. For $n = 2$, the altitudes of a triangle are well known to concur and thus satisfy in a sense the necessary condition to form an associated set.

1.2. In what follows we shall make use of the following known ideas and propositions proved previously for an n-space.

(a) A line and hyperplane or a prime are perpendicular or normal to each other, if their traces in the prime at infinity, p say, are pole and polar for the $(n-2)$-sphere at infinity or the *absolute polarity* (Mandan [13], [22], [25]), (p) say.

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(b) The joins of the corresponding vertices of a pair of polar reciprocal simplexes S, S' for a quadric Q form in general an associated set of $n+1$ lines such that ∞^{n-3} $(n-2)$-spaces can be drawn through each point of each line to meet them and therefore ∞^{n-2} $(n-2)$-spaces exist in all meeting them (Beatty [6]; Coxeter and Todd [12]; see also Baker [5] for the dual proposition). In analogy with *Gergonne's theorem* (Court [10]) in a plane, we may name it too after Gergonne in all spaces when Q is inscribed to S (cf. Baker [3], p. 53, Ex. 14). Several special cases of degeneration also arise in accordance with certain special relationship which may exist between several elements of S or S' in regard to Q (Mandan [17]).

(c) If through the vertices i of an n-simplex S $n+1$ lines a_i be drawn such that ∞^{n-3} $(n-2)$-spaces pass through every i meeting them, a_i form an associated set (Mandan [29]).

2. Proof of the proposition

The $(n-1)$-simplex (j') formed of the n traces j' of the n altitudes a_j of an n-simplex S through its n vertices j, in p, is seen to be the polar reciprocal w.r.t. (p) of the one (T_{ij}) formed of the n traces T_{ij} in p of its n edges ij through its $(n+1)$th vertex i (§ 1.2a). Therefore the n joins $j'T_{ij}$ form in general an associated set of n lines in p such that ∞^{n-3} $(n-3)$-spaces (t') can be drawn to meet them (§ 1.2b). The ∞^{n-3} $(n-2)$-spaces (t) joining (t') to i then meet the said n joins such that the primes determined by (t) and a join $j'T_{ij}$ contain the altitude $a_{j-jj'}$ of S which therefore meets (t). Consequently all altitudes of S meet (t) and through each vertex of S ∞^{n-3} $(n-2)$-spaces like (t) can be drawn to meet them. Hence (§ 1.2c) we have

THEOREM 1. *The altitudes of a simplex form in general an associated set as defined above.*

3. Associated quadrics

3.1. An immediate consequence of the preceding proposition and the third Brianchon's theorem (Mandan [29]) is the following

THEOREM 2. *If, through the n $(n-2)$-spaces in a prime* (i) *of an n-simplex, n hyperplanes be drawn parallel to its corresponding altitude; or if, through the common $(n-2)$-space of a pair of its primes (i), (j), the pair of hyperplanes be drawn perpendicular to (i), (j); then the $n(n+1)$ such hyperplanes touch a quadric.*

COROLLARY 1. *If, through the 3 edges in a face of a tetrahedron, 3 planes be drawn parallel to its corresponding altitude; or if, through the common*

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edge of a pair of its faces (i), (j) *the pair of planes be drawn perpendicular to* (i), (j); *then the 12 such planes touch a quadric* (cf. Baker [3], p. 54, Ex. 15; Court [9]).

COROLLARY 2. *If, through the pair of vertices of a side of a triangle, the pair of lines be drawn parallel to its corresponding altitude; or if through the common vertex of a pair of its sides* (i), (j) *the pair of lines be drawn perpendicular to* (i), (j); *then the 6 such lines touch a conic* (cf. Baker [2], p. 25, Ex. 2; Court [9]).

3.2. In analogy with the *orthic triangle* of a triangle and the *orthic tetrahedron* of a tetrahedron (Court [7]), we may define the *orthic simplex* of a simplex as one formed of the feet of its altitudes. As a limit of the second Brianchon's theorem (Mandan [29]) or from the converse of the Gergonne's theorem (§ 1.2b) we may deduce

THEOREM 3. *There exists a quadric* Q *inscribed to a simplex* S *and circumscribed to its orthic simplex* S' *such that* S, S' *are polar reciprocals of each other for* Q.

DEFINITION. Q may be called the *orthic quadric* of the simplex S.

4. Isodynamic simplex

4.1. It may happen that the n joins $j'T_{ij}$ of § 2 concur at a point P (Mandan [17]) and then the join iP is obviously the common transversal of the altitudes of the simplex S, in which case they are said to form a *semi-associated set* (Mandan [29]).

4.2. If the tangential simplex of a simplex formed of the tangent primes of its circumhypersphere at its vertices be its *anticevian* for a point L (Mandan [21]) or perspective to it from L, it is said to be *isodynamic* with L as its *Lemoine point*; and the join of L to its circumcentre, called its *Brocard diameter*, meets its altitudes (Mandan [23]).

4.3. Again a pair of tetrahedra in any r-space ($r > 3$) are said to be *projective*, if the 4 joins of their vertices in a certain one-to-one correspondence are met by a line such that their 4 *arguesian points* common to the 4 pairs of their corresponding planes are collinear in their *arguesian line* (Mandan [18]).

Thus follows

THEOREM 4. *The altitudes of a semi-isodynamic* n*-simplex* S *form a semi-associated set such that each of its tetrahedra is projective to the corresponding one of its orthic* n*-simplex* S' *from its Brocard diameter giving rise to* $\binom{n+1}{3}$ *arguesian points lying by fours on its* $\binom{n+1}{4}$ *arguesian lines,*

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$n-2$ through each point and lying by fives in its $\binom{n+1}{5}$ 'arguesian planes', $n-3$ through each line and lying by sixes in its $\binom{n+1}{6}$ 'arguesian solids', $\cdots$ and so on.

5. Special cases

5.1. If the altitudes of a simplex S happen to be doubly semi-associated, they concur at its orthocentre H making S *orthocentric or orthogonal* (Mandan [15], [28]) and the *orthic axes* of its triangles and the *orthic planes* of its tetrahedra all lie in its *orthic prime* h (Mandan [24]). H, h are then pole and polar for S as well as for its orthic quadric Q (§ 3.2). Hence Q is the *polar quadric* of h for S (Mandan [31]). H may be then said to be its *Gergonne point* w.r.t. Q in analogy with such a point associated with a triangle T w.r.t. a conic inscribed to T (Court [11]; Mandan [21]), and S be called *isogonic* w.r.t. Q with H as its *Fermat point* (Mandan [23], [26]).

Thus we have the following

THEOREM 5. *An orthogonal simplex S is isogonic w.r.t. its orthic quadric Q with its Gergonne point w.r.t. Q (= its Fermat point) at its orthocentre H. Q is the polar quadric of its orthic prime h w.r.t. S. The orthic simplex S' of S forms its 'cevian simplex' for H, being inscribed to S and perspective to S from H. The vertices of S or S' and H form a 'self-conjugate' set of points for Q such that the join of any two points contains the pole of the hyperplane determined by the rest of them* (cf. Baker [3]; Mandan [17]).

5.2. An edge ij of an n-simplex S is said to be *conjugate* to its opposite $(n-2)$-space (ij) for a quadric Q, if the polar line of (ij) for Q meets ij such that the 2 joins of its vertices i, j to the corresponding ones of its polar reciprocal n-simplex S' for Q meet at F_2 (say); S may be said to be *bi-isogonic* w.r.t. Q with F_2 as its *bi-Fermat point* when Q is inscribed to S and therefore circumscribed to S' (Mandan [17], [30]).

Again the pair of altitudes of S from its vertices i, j meet at its *bi-orthocentre* making it *bi-orthocentric* (Mandan [14], [16], [28]) with ij as its *special edge*, if and only if ij is perpendicular to (ij) or the trace $T(ij)$ in p of (ij) lies in the polar $(n-2)$-space of T_{ij} for (p) (§ 2) in which case ij may be said to be *conjugate* to (ij) for (p). Thus follows

THEOREM 6. *An n-simplex S is bi-orthocentric, if and only if its special edge is conjugate to its opposite $(n-2)$-space for its orthic quadric Q or for the absolute polarity (p), so that S becomes bi-isogonic w.r.t. Q with bi-Fermat point at its bi-orthocentre.*

5.3. If r altitudes of an n-simplex S from its r vertices concur at its *r-orthocentre* H_r, S is said to be *r-orthocentric* and denoted as S_r with its

$(r-1)$-space s_{r-1} of its said r vertices called *special* such that its *special r-altitude* perpendicular to s_{r-1} and its opposite $(n-r)$-space passes through H_r. If q other altitudes of S_r also concur at H_q, it is denoted as $S_{q \cdot r}$; it is *semi-orthocentric* with H_q, H_r as the pair of its *semi-orthocentres* when $q = n-r+1$, and *uni-orthocentric of order one* with H_q, H_r as its *unisemi-orthocentres* when $q = n-r$, in which case its $(n+1)$th altitude concurs with its special r- and $(n-r)$-altitudes at its *uni-orthocentre*. If $q < n-r$ and the rest of the $n-q-r+1$ altitudes of $S_{q \cdot r}$ also concur at $H_{n-q-r+1}$, it is said to become *demi-orthocentric of order one* with H_q, $H_{n-q-r+1}$, H_r as its 3 *demi-orthocentres* such that its special q-, r-, $(n-q-r+1)$-altitudes concur at its *di-orthocentre*. Similarly we may define a uni-orthocentric simplex $S_{q \cdot r \cdots (n-q-r-\cdots)}$ and demi-orthocentric $S_{q \cdot r \cdots (n-q-r-\cdots+1)}$ of any higher order and these are said to be *proper*, if they possess a uni-orthocentre and a di-orthocentre respectively (Mandan [14], [15], [19]).

In the same style we may develope *semi-*, *uni-*, *demi-isogonic simplexes* of various types and orders w.r.t. a quadric Q inscribed to them respectively (cf. Mandan [30]), as we have defined a bi-isogonic one (§ 5.2). Thus follows

THEOREM 7. *A semi-, uni- or demi-orthocentric simplex becomes respectively semi-, uni- or demi-isogonic w.r.t. its orthic quadric with semi-, uni- and unisemi-; or, di- and demi-Fermat points at its semi-, uni- and unisemi-, or, di- and demi-orthocentres.*

5.4. (a) From the propositions of incidence alone Baker [1], p. 39, Ex. 7) has established that when $n = 2r$, a definite $(r-1)$-space can be drawn to meet $r+1$ lines of general position, each in one point, and when $n = 2r-1$, a definite $(r-1)$-space through a point to meet r lines.

(b) A q-space $\bar{q}$ and an r-space $\bar{r}$ $(q \geqq r)$ in an n-space are said to be *conjugate* for a quadric Q, if the polar of $\bar{q}$ for Q meets $\bar{r}$ in a point, and for an $(n-2)$-quadric Q' if their traces in the hyperplane of Q' are *conjugate* (cf. § 5.2) for Q' (cf. Mandan [17]).

Now we may prove the following

THEOREM 8. *If an $(r-1)$-space x of a $(2r-1)$-simplex be conjugate to its opposite $(r-1)$-space y, or an opposite r-space z, for its orthic quadric, x is conjugate to y for the absolute polarity too. Its r altitudes from its vertices in x lie in a hyperplane and so do its other r altitudes. Its vertex A (say) common to z, x lies in both the hyperplanes, so that the definite $(r-2)$-space meeting the first r altitudes passes through A.*

THEOREM 9. *If an r-space x of a $(2r)$-simplex be conjugate to its opposite $(r-1)$-space y, or an opposite r-space z, for its orthic quadric, x is conjugate to y for the absolute polarity too. Its $r+1$ altitudes from its vertices in x in the former case lie in a hyperplane and its other r altitudes in a $(2r-2)$-space,* *and in the later case the definite $(r-1)$-space meeting its $r+1$ altitudes from its vertices in z or x contains its vertex common to z, x.*

5.5. Two sets of $r+2$ points P_i, P'_i, each spanning an $(r+1)$-space which has no solid common with that spanned by the other, are said to be *projective* (cf. § 4.3) from an $(r-1)$-space which meets the $r+2$ joins $P_iP'_i$ such that their $r+2$ *arguesian points* common to their corresponding r-spaces are collinear in their *arguesian line* (Mandan [20], [27]). We may then prove (cf. Mandan, [17])

THEOREM 10. *If r consecutive edges of an $(r+2)$gon formed of $r+2$ vertices of an n-simplex be conjugate to their respectively opposite $(r-1)$-spaces for its orthic quadric Q, its $r+2$ altitudes from these vertices are met by an $(n-r-1)$-space through the polar $(n-r-2)$-space for Q of their $(r+1)$-space. Hence if $n = 2r$, their $(r+1)$-simplex is projective to the corresponding one of its orthic n-simplex from the $(r-1)$-space meeting its said $r+2$ altitudes such that the $r+2$ points common to the $r+2$ pairs of the corresponding r-spaces of the two $(r+1)$-simplexes are collinear.*

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