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ON THE BRANCHING THEOREM OF THE SYMPLECTIC GROUPS[1]

BY
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1. **Introduction.** In [1], Zhelobenko introduced the concept of a Gauss decomposition Z^tDZ of a topological group and gave characterizations of irreducible representations of the classical groups. In this setting, vectors of representation spaces are polynomial solutions of a system of differential equations and the problem of obtaining branching theorem with respect to a subgroup G_0 is to find all polynomial solutions that are invariant under $Z \cap G_0$ and have dominant weight with respect to $D \cap G_0$.

Branching theorems are obtained for the classical groups in [1] and in the cases $GL(n)\supset GL(n-1)$, $SO(2k)\supset SO(2k-1)$ and $SO(2k+1)\supset SO(2k)$ invariants of $Z \cap G_0$ were explicitly constructed. However, in the proof of the case $Sp(2k)\supset Sp(2k-2)$, the principle of correspondence with $GL(k+1)\supset GL(k-1)$ was employed and the problem of explicit construction of the invariants was left open.

In this paper, an explicit construction of all the invariants of $Z \cap Sp(2k-2)$ that correspond to dominant weights with respect to $D \cap Sp(2k-2)$ is given for the case $Sp(2k)\supset Sp(2k-2)$.

In section 4, the invariants constructed are used to obtain the branching theorem with respect to another subgroup G_1 which is isomorphic to $Sp(2k-2)\times Sp(2)$. This case was studied by J. Lepowsky [2], [3].

2. **Preliminaries.** The symplectic group Sp(n) (where $n=2k$) consists of all complex $n\times n$ matrices that preserve the skew symmetric form

$$[x, y] = x_1y_n+\cdots+x_ky_{k+1}-x_{k+1}y_k-\cdots-x_ny_1.$$

Let Z be the subgroup of upper triangular matrices, Z^t be the set of transpositions of elements in Z and D be the subgroup of diagonal matrices of $Sp(2k)$. The following two theorems, constructed out of Zhelobenko's work, will be used.

THEOREM 1. *Every irreducible representation of $Sp(2k)$ is induced by some character α of D, i.e., if T_g is a finite dimensional irreducible representation of $Sp(2k)$, then T_g is defined in a class of functions on Z by right multiplication, i.e.,*

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for any $f(z)$ in the class, any $g \in G$ and any $z \in Z$,

(2.1) $$T_g f(z) = \alpha(z, g) f(z \cdot g)$$

where (z, g) and $z \cdot g$ are components of zg in D and Z respectively relative to the decomposition $Z^t DZ$. In particular

(2.2) $$T_\delta f(z) = \alpha(\delta) f(\delta^{-1} z \delta), \quad \textit{for all } \delta \textit{ in } D.$$

THEOREM 2. *Let*

$$\begin{pmatrix} \delta_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \delta_k & & & \\ & & & & & \delta_k^{-1} & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & \delta_1^{-1} \end{pmatrix}$$

be the general form of an element in D, then every irreducible representation of $Sp(2k)$ is induced by a character δ of D of the form $\delta_1^{m_1} \cdots \delta_k^{m_k}$ where m_i $(i=1, \ldots, k)$ are integers satisfying $m_1 \geq \cdots \geq m_k \geq 0$. If z_{ij} $(i<j, i, j=1, \ldots, 2k)$ are entries of elements in Z, then the functions of this representation space consist of all polynomials $f(z)$ of all z_{ij} satisfying the differential equations

(2.3)
$$\begin{gathered} \left(\frac{\partial}{\partial z_{12}} + z_{23}\frac{\partial}{\partial z_{13}} + \cdots + z_{2,2k}\frac{\partial}{\partial z_{1,2k}}\right)^{m_1-m_2+1} f(z) = 0, \\ \left(\frac{\partial}{\partial z_{23}} + \cdots + z_{3,2k-1}\frac{\partial}{\partial z_{2,2k-1}}\right)^{m_2-m_3+1} f(z) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left(\frac{\partial}{\partial z_{k,k+1}}\right)^{m_k+1} f(z) = 0. \end{gathered}$$

Due to the symplectic restriction, not all entries of $z \in Z$ are independent; one may choose as independent variables the entries

$$\begin{gathered} z_{12}, \ldots\ldots\ldots\ldots, z_{1,2k} \\ z_{23} \cdots\cdots\cdots\cdots z_{1,2k-1} \\ \cdots\cdots\cdots \\ z_{k,k+1}. \end{gathered}$$

3. **Branching theorem and construction of invariants.** Consider the subgroup G_0 of $Sp(2k)$ consisting of all matrices that leave x_k and x_{k+1} fixed. This subgroup

is isomorphic to $Sp(2k-2)$. Let the irreducible representation of $Sp(2k)$ induced by the character $\delta_1^{m_1}\cdots\delta_k^{m_k}$ be denoted by $(m_1,\ldots,m_k)$. In [1] (see also [4]), it was proved that the irreducible representations of G_0 appearing in the irreducible representation $(m_1,\ldots,m_k)$ of $Sp(2k)$ are the representations $(q_1,\ldots,q_{k-1})$ corresponding to all possible patterns

$$\begin{pmatrix} m_1 \cdots\cdots\cdots m_k \\ p_1 \cdots\cdots\cdots p_k \\ q_1 \cdots\cdots q_{k-1} \end{pmatrix}, \tag{3.1}$$

where p_i and q_i are integers satisfying

$$m_1 \geq p_1 \geq \cdots \geq m_k \geq p_k \geq 0,$$

$$p_1 \geq q_1 \geq \cdots \geq p_{k-1} \geq q_{k-1} \geq p_k$$

and $(q_1,\ldots,q_{k-1})$ denotes the irreducible representation of G_0 induced by $\delta_1^{q_1}\cdots\delta_{k-1}^{q_{k-1}}$. In what follows, M, P, and Q will denote the rows of (3.1).

PROPOSITION 1. *Every matrix z in the subgroup Z of $Sp(n)$ can be transformed to a matrix whose entries depend only on*

$$\begin{matrix} z_{1k}, & z_{1,k+1} \\ z_{2k}, & z_{2,k+1} \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \\ z_{k-1,k}, & z_{k-1,k+1} \\ & z_{k,k+1} \end{matrix} \tag{3.2}$$

by a right multiplication of an element of $Z\cap G_0$.

Proof. For $k=2$, with the symplectic restriction, one may write

$$z = \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ & 1 & z_{23} & z_{13}-z_{23}z_{12} \\ & & 1 & -z_{12} \\ & & & 1 \end{pmatrix}.$$

Multiply z on the right by

$$z_0 = \begin{pmatrix} 1 & 0 & 0 & -z_{14} \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 0 \end{pmatrix},$$

it is easy to see that entries of zz_0 do not depend on z_{14}. Now suppose that the statement is true for $k-1$. Notice that the truncation X of z formed by elements z_{ij} whose indices take only the values $2, 3, \ldots, n-1$ is symplectic. Write

$$z = \begin{pmatrix} 1 & t & z_{1n} \\ & X & c^t \\ & & 1 \end{pmatrix},$$

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where t is a row vector and c^t is a column vector. Factorize z as

$$\begin{pmatrix} 1 & 0 & 0 \\ & X & 0^t \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t & z_{1n} \\ & I_{n-2} & t^* \\ & & 1 \end{pmatrix},$$

where t^* is the column

$$\begin{pmatrix} z_{1,n-1} \\ \cdot \\ \cdot \\ \cdot \\ z_{1,k+1} \\ -z_{1,k} \\ \cdot \\ \cdot \\ \cdot \\ -z_{12} \end{pmatrix}$$

Multiplying z on the right by

$$z_1 = \begin{pmatrix} 1, -z_{12}, -z_{13}, \ldots, -z_{1,k-1}, 0, 0, -z_{1,k+2}, \ldots, & -z_{1n} \\ I_{n-2} & t_1^* \\ & 1 \end{pmatrix},$$

one obtains

$$zz_1 = \begin{pmatrix} 1 & 0 & 0 \\ & X & 0^t \\ & & 1 \end{pmatrix}\begin{pmatrix} 1, 0, 0, \ldots, z_{1k}, z_{1,k+1}, 0, \ldots, & 0 \\ & \cdot \\ & \cdot \\ & \cdot \\ I_{n-2} & -z_{1,k+1} \\ & -z_{1,k} \\ & \cdot \\ & \cdot \\ & \cdot \\ & 1 \end{pmatrix}$$

It is clear that the second factor of zz_1 commutes with every matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ & Y & 0^t \\ & & 1 \end{pmatrix}$$

in $Z \cap G_0$. By induction assumption, the proof is thus completed.

COROLLARY 1. *When an irreducible representation of* $Sp(2k)$ *is restricted to* G_0, *the invariants of* $Z \cap G_0$ *can only depend on* (3.2).

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From corollary 1, polynomials corresponding to patterns (3.1) must depend only on (3.2). For clarity, write (3.2) as

$$\begin{array}{ll} a_1, & b_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{k-1}, & b_{k-1} \\ & b_k. \end{array}$$

Thus when restricted to these polynomials, (2.3) becomes

$$\begin{aligned} &\left(a_2\frac{\partial}{\partial a_1}+b_2\frac{\partial}{\partial b_1}\right)^{m_1-m_2+1} f(a,b)=0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\left(\frac{\partial}{\partial b_k}\right)^{m_k+1} f(a,b)=0. \end{aligned} \tag{3.3}$$

To construct these polynomials, for each fixed pattern (3.1), define the following functions:

$$f_i(M,P,Q)=\begin{cases} a_i^{p_i-q_i}b_i^{m_i-p_i}, & (q_i\ge m_{i+1},\ q_{i-1}\ge m_i) \\ a_i^{p_i-q_i}b_i^{q_{i-1}-1-p_i}, & (q_i\ge m_{i+1},\ q_{i-1}< m_i) \\ a_i^{p_i-m_{i+1}}b_i^{q_{i-1}-1-p_i}(b_{i+1}a_i-a_{i+1}b_i)^{m_{i+1}-q_i}, & (q_i< m_{i+1},\ q_{i-1}< m_i) \\ a_i^{p_i-m_{i+1}}b_i^{m_i-p_i}(b_{i+1}a_i-a_{i+1}b_i)^{m_{i+1}-q_i}, & (q_i< m_{i+1},\ q_{i-1}\ge m_i). \end{cases} \tag{3.4}$$

Where $a_k=a_{k+1}=b_{k+1}=1$, $q_0=m_1$ and $q_k=m_{k+1}=0$. Now consider the function

$$F(M,P,Q)\equiv\prod_{i=1}^{k} f_i(M,P,Q) \tag{3.5}$$

It will be proved that (3.5) corresponds (3.1).

THEOREM 3. *The functions* (3.5) *constructed for each pattern* (3.1) *satisfy* (3.3) *and have weights* $(q_1,q_2,\ldots,q_{k-1})=Q$ *with respect to* $D\cap G_0$. *Furthermore, they are linearly independent.*

Proof. To show that these functions satisfy (3.3), consider the first differential equation

$$\left(a_2\frac{\partial}{\partial a_1}+b_2\frac{\partial}{\partial b_1}\right)^{m_1-m_2+1} f(a,b)=0.$$

For $F(M,P,Q)$ to satisfy this differential equation, it is sufficient that f_1 of $F(M,P,Q)$ satisfies it. One considers the following cases.

(i) If $q_1\ge m_2$, then

$$(p_1-q_1)+(m_1-p_1)=m_1-q_1<m_1-m_2+1,$$

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hence each term of the expansion of the differential operator annihilates f_1.

(ii) For $q_1<m_2$, notice that

$$\left(a_2\frac{\partial}{\partial a_1}+b_2\frac{\partial}{\partial b_1}\right)(b_2a_1-a_2b_1)^k = 0, \qquad (k = 0, 1, 2, \ldots)$$

and

$$(p_1-m_2)+(m_1-p_1) < m_1-m_2+1,$$

hence again each term of the expansion of the differential operator annihilates f_1. In a similar way, one can prove that $F(M, P, Q)$ satisfies the rest of (3.3).

To show that $F(M, P, Q)$ has weight $Q=(q_1, \ldots, q_{k-1})$ with respect to $D \cap G_0$, use (2.2). For

$$\delta_0 = \begin{pmatrix} \delta_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \delta_{k-1} & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & \delta_{k-1}^{-1} & & \\ & & & & & & \ddots & \\ & & & & & & & \delta_1^{-1} \end{pmatrix}$$

in $D \cap G_0$, one has

$$T_{\delta_0}f(z) = \delta_1^{m}\cdots\delta_{k-1}^{m_{k-1}}f(\delta_0^{-1}z\delta_0), \tag{3.6}$$

where $f(\delta_0^{-1}z\delta_0)$ can be written as

$$\delta_1^{r_1}\cdots\delta_{k-1}^{r_{k-1}}f(z)$$

for some negative integers $r_1, \ldots, r_{k-1}$ and the weights of $f(z)$ with respect to $D \cap G_0$ is $(m_1+r_1, \ldots, m_{k-1}+r_{k-1})$. Multiplying out $\delta_0^{-1}z\delta_0$, one finds that for $i\leq k-1$, a_i and b_i are changed to $\delta_i^{-1}a_i$ and $\delta_i^{-1}b_i$ respectively. To obtain the weight of $F(M, P, Q)$ with respect to $D \cap G_0$, one first considers δ_1 and the following cases.

(i) If $q_1\geq m_2$, then by (3.6) and substitution of $\delta_1^{-1}a_1$ and $\delta_1^{-1}b_1$ for a_1 and b_1 in (3.4) one obtains

$$r_1 = -(p_1-q_1)-(m_1-p_1) = -m_1+q_1,$$

thus the power of δ_1 is q_1.

(ii) If $q_1<m_2$, then r_1 is

$$-(p_1-m_2)-(m_1-p_1)-(m_2-q_1) = -m_1+q_1,$$

again, the power of δ_1 is q_1.

The powers of $\delta_2, \ldots, \delta_{k-1}$ may be obtained by using (3.6) and (3.4). It turns out that they are $q_2, \ldots, q_{k-1}$ respectively.

Finally, to show linear independence of these functions, it is sufficient to consider functions having the same weight with respect to $D \cap G_0$. Hence it suffices to consider functions $\{F_1, \ldots, F_i\} = \mathscr{F}_Q$ corresponding to patterns with the same Q. Suppose F_i in $\mathscr{F}_Q$ is a linear combination of $S \subseteq \mathscr{F}_Q$. One again examines the following different cases.

(i) If $q_1 \geq m_2$, then the powers of a_1 and b_1 in the functions F_j of $\mathscr{F}_Q$ are $p_1^{(j)} - q_1$ and $m_1 - p_1^{(j)}$. Since all functions of $\mathscr{F}_Q$ are polynomials, F_i can be a linear combination of S only when every pattern corresponding to functions in S has the same $p_1 = p_1^{(i)}$.

(ii) If $q_1 < m_2$, then the highest power of a_1 appearing in F_i is $p_1^{(i)}$.

In a similar way, one can examine all cases in (3.4) and conclude that functions in S must correspond to F_i. But then elements of S must be a scalar times F_i, hence $\mathscr{F}_Q$ is a linear independent set.

4. **An application of the invariants.** Let G_1 be the subgroup of $Sp(n)$ generated by G_0 and all elements of $Sp(n)$ leaving $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2k}$ invariant. Then $G_1 \simeq Sp(2k-2) \times Sp(2)$ and an irreducible representation of G_1 is characterized by integers $(q_1, \ldots, q_{k-1}; q)$ satisfying $q_1 \geq q_2 \geq \cdots \geq q_{k-1} \geq 0$ and $q \geq 0$. Let $m(q_1, \ldots, q_{k-1}; q)$ denote the multiplicity of the representation $(q_1, \ldots, q_{k-1}; q)$ in the representation $(m_1, \ldots, m_k)$ of $Sp(2k)$. To find $m(q_1, \ldots, q_{k-1}; q)$, it suffices [1, p. 12, Corollary] to look for independent functions in $(m_1, \ldots, m_k)$ that satisfy

$$T_{z_0} f(z) = f(zz_0) = f(z_0), \qquad \forall z_0 \in Z(G_1) \tag{4.1}$$

and

$$T_\delta f(z) = \delta_1^{q_1} \cdots \delta_{k-1}^{q_{k-1}} \delta_k^q f(z), \qquad \forall \delta \in D(G_1) \tag{4.2}$$

Since $Z \cap G_1 \supset Z \cap G_0$ and $D \cap G_1 \supset D \cap G_0$, these functions are constructable from the functions $\prod_1^k f_i(M, P, Q)$.

Let $Q = (q_1, \ldots, q_{k-1})$ be fixed, $\mathscr{F}_Q$ be the collection of all functions $\prod_1^k f_i(M, P, Q)$ with this Q and $V(\mathscr{F}_Q)$ be the space spanned by $\mathscr{F}_Q$; $m(q_1, \ldots, q_{k-1}; q)$ is then equal to the number of independent functions in $V(\mathscr{F}_Q)$ that satisfy (4.1) and (4.2).

Every z_0 in $Z(G_1)$ can be written as $z_1 z_2$ where $z_1 \in Z(G_0)$ and z_2 is of the form

$$\begin{pmatrix} I_{k-1} & & & \\ & 1 & c & \\ & 0 & 1 & \\ & & & I_{k-1} \end{pmatrix}$$

where all other entries are zero. If $f(z)$ is in the space spanned by the functions

$\prod_1^k f_i(M, P, Q)$, then

$$T_{z_1z_2}f(z) = T_{z_2}f(zz_1) = T_{z_2}f(z) = f(zz_2).$$

Under right multiplication by z_2, the variables $a_1, \ldots, a_{k-1}, b_1, \ldots, b_k$ are changed as

$$(4.3)\qquad \begin{array}{ll} a_1 \to a_1, & b_1 \to b_1 + a_1c \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{k-1} \to a_{k-1}, & b_{k-1} \to b_{k-1} + a_{k-1}c \\ & b_k \to b_k + c. \end{array}$$

Hence (4.1) is equivalent to invariance under the transformation (4.3).

Lemma 1. *A polynomial function* $f(a_1, \ldots, a_{k-1}; b_1, \ldots, b_k)$ *is invariant under the transformation* (4.3) *iff* $f(a_1, \ldots, a_{k-1}; b_1, \ldots, b_k)$ *is of the form*

$$(4.4)\qquad \sum_{s,t} r_{t_1,\ldots,t_{k-1}}^{s_1,\ldots,s_{k-1}} a_1^{s_1} \cdots a_{k-1}^{s_{k-1}} (b_1 - a_1 b_k)^{t_1} \cdots (b_{k-1} - a_{k-1} b_k)^{t_{k-1}}.$$

Proof. Under (4.3), $b_i - a_i b_k$ is transformed to $b_i + a_i c - a_i(b_k + c) = b_i - a_i b_k$. Therefore, (4.4) is invariant.

Conversely, suppose a polynomial

$$f(a_1, \ldots, a_{k-1}; b_1, \ldots, b_k) = \sum_{s,t} r_{t_1,\ldots,t_k}^{s_1,\ldots,s_{k-1}} a_1^{s_1} \cdots a_{k-1}^{s_{k-1}} b_1^{t_1} \cdots b_k^{t_k}$$

is invariant under the transformation (4.3). By setting $c = -b_k$, it follows that all $r_{t_1,\ldots,t_k}^{s_1,\ldots,s_{k-1}}$ for which $t_k \neq 0$ are zero and $f(a_1, a_{k-1}; b_1, \ldots, b_k)$ is of the form (4.4).

For any $\delta \in D$, multiplying out $\delta^{-1} z \delta$, the variables $a_1, \ldots, a_{k-1}, b_1, \ldots, b_k$ are changed as

$$(4.5)\qquad \begin{array}{ll} a_1 \to a_1 \delta_1^{-1} \delta_k, & b_1 \to b_1 \delta_1^{-1} \delta_k^{-1}, \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{k-1} \to a_{k-1} \delta_1^{-1} \delta_k, & b_{k-1} \to b_{k-1} \delta_{k-1}^{-1} \delta_k^{-1}, \\ & b_k \to b_k \delta_k^{-2}. \end{array}$$

If $f(a_1, \ldots, a_{k-1}; b_1, \ldots, b_k)$ satisfies (4.1) and (4.2), then from (4.5), the equations

$$(4.6)\qquad \left.\begin{array}{ll} s_i + t_i = m_i - q_i, & (i = 1, \ldots, k-1) \\ (t_1 + \cdots + t_{k-1}) - (s_1 + \cdots + s_{k-1}) = m_k - q & \end{array}\right\}$$

must be satisfied for each summand $a_1^{s_1} \cdots a_{k-1}^{s_{k-1}} (b_1 - a_1 b_k)^{t_1} \cdots (b_{k-1} - a_{k-1} b_k)^{t_{k-1}}$.

Using (4.5), the weight of the function $\prod_1^k f_i(M, P, Q)$ is found to be

$$(4.7)\qquad \delta_1^{q_1} \cdots \delta_{k-1}^{q_{k-1}} \delta_k^{2(p_1 + \cdots + p_k) - (m_1 + \cdots + m_k) - (q_1 + \cdots + q_{k-1})}$$

Thus if $f(a_1, \ldots, a_{k-1}; b_1, \ldots, b_k) = \sum_{F_i \in \mathscr{F}_Q} r_i F_i$, then P of each F_i must satisfy

$$2(p_1^{(i)}+\cdots+p_k^{(i)})-(m_1+\cdots+m_k)-(q_1+\cdots+q_{k-1}) = q \tag{4.8}$$

Suppose that $Q=(q_1, \ldots, q_{k-1})$ satisfies

$$q_1 \geq m_2, \ldots, q_{k-1} \geq m_k. \tag{4.9}$$

In this case, the functions in $\mathscr{F}_Q$ are monomials. Let $\mathscr{B}$ be the subset of $\mathscr{F}_Q$ that satisfy (4.8), then

$$\mathscr{B} = \left\{ a_1^{p_1-q_1} b_1^{m_1-p_1} \cdots a_{k-1}^{p_{k-1}-q_{k-1}} b_{k-1}^{m_{k-1}-p-k-1} b_k^{m_k-p_k} \,\middle|\, 2\sum_1^k p_i - \sum_1^k m_i - \sum_1^{k-1} q_i = q \right\}.$$

LEMMA 2. *If the function* (4.4) *satisfies* (4.2) *and belongs to the space* $V(\mathscr{B})$ *spanned by* $\mathscr{B}$, *then every summand*

$$a_1^{s_1} \cdots a_{k-1}^{s_{k-1}} (b_1 - a_1 b_k)^{t_1} \cdots (b_{k-1} - a_{k-1} b_k)^{t_{k-1}} \tag{4.10}$$

of it also belongs to $V(\mathscr{B})$.

Proof. If (4.4) satisfies (4.2), then every summand of it also satisfies (4.2). Since each summand is invariant under (4.3), it belongs to $V(\mathscr{F}_Q)$. $V(\mathscr{B})$ is obviously the subspace of $V(\mathscr{F}_Q)$ that satisfies (4.2), thus each summand of (4.4) belongs to $V(\mathscr{B})$.

Thus assuming (4.9) is satisfied to find $m(q_1, \ldots, q_{k-1}; q)$, it suffices to find the number of independent polynomials of the form (4.10) that are in $V(\mathscr{B})$. Notice that the power of b_k of any element in $\mathscr{B}$ does not exceed m_k.

LEMMA 3. *The polynomial* (4.10) *is in* $V(\mathscr{B})$ *iff* $a_1^{s_1+t_1} \cdots a_{k-1}^{s_{k-1}+t_{k-1}} b_k^{t_1+\cdots+t_{k-1}}$ *is in* $\mathscr{B}$.

Proof. If $a_1^{s_1+t_1} \cdots a_{k-1}^{s_{k-1}+t_{k-1}} b_k^{t_1+\cdots+t_{k-1}} \in \mathscr{B}$, then

$$\left.\begin{array}{c} s_i + t_i = m_i - q_i, \qquad (i = 1, \ldots, k-1) \\ (s_1+\cdots+s_{k-1})-(t_1+\cdots+t_{k-1}) = q - m_m, \\ t_1+\cdots+t_{k-1} \leq m_k. \end{array}\right\} \tag{4.11}$$

A general term of the expansion of (4.10) is

$$a_1^{s_1+(t_1-j_1)} \cdots a_{k-1}^{s_{k-1}+(t_{k-1}-j_{k-1})} b_1^{j_1} \cdots b_{k-1}^{j_{k-1}} b_k^{(t_1+\cdots+t_{k-1})-(j_1+\cdots+j_{k-1})}$$

By (13), $[s_i+(t_i-j_i)]+j_i = m_i - q_i$ $(i=1, \ldots, k-1)$, $\sum_1^{k-1} [s_i+(t_i-j_i)] - \sum_1^{k-1} j_i - 2[\sum_1^{k-1} t_i - \sum_1^{k-1} j_i] = q$ and $(t_1+\cdots+t_{k-1})-(j_1+\cdots+j_{k-1}) \leq t_1+\cdots+t_{k-1} \leq m_k$. Thus every general term is in $\mathscr{B}$. The converse is obvious.

It is now clear that when (4.9) is satisfied $m(q_1, \ldots, q_{k-1}; q)$ is equal to the number of non-negative integer solutions $(s_1, \ldots, s_{k-1}; t_1, \ldots, t_{k-1})$ to (4.11). The general case is included in the following:

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THEOREM 4. *$m(q_1, \ldots, q_{k-1}; q)$ is equal to the number of non-negative integer solutions $(s_1, \ldots, s_{k-1}; t_1, \ldots, t_{k-1})$ of*

$$\left.\begin{aligned} s_i+t_i &= m_i''-q_i, \qquad (i = 1, \ldots, k-1) \\ (s_1+\cdots+s_{k-1})-(t_1+\cdots+t_{k-1}) &= q-m_k'' \\ t_1+\cdots+t_{k-1} &\le m_k'' \end{aligned}\right\} \tag{4.12}$$

where

$$m_1'' = m_1-(m_1'+m_2'),$$
$$m_2'' = m_2-(m_2'+m_3'), \ldots, m_{k-1}'' = m_{k-1}-(m_{k-1}'+m_k'),$$
$$m_k'' = m_k-m_k'$$

and

$$m_1' = 0,$$
$$m_2' = \max(0, m_2-q_1), \ldots, m_k' = \max(0, m_k-q_{k-1}).$$

Proof. The case when (4.9) is satisfied is treated previously. For the general case, consider a fixed Q and the subset $\mathscr{S}$ of $\mathscr{F}_Q$ consisting of functions that satisfy (10). By definition, the polynomial

$$(a_1b_2-b_1a_2)^{m_2'} \cdots (a_{k-1}b_k-b_{k-1})^{m_k'} \tag{4.13}$$

is a common factor for all functions in $\mathscr{S}$. (4.13) is invariant under (4.3); under (4.5), it is changed to

$$\delta_1^{-m_2'}\delta_2^{-(m_2'+m_3')} \cdots \delta_{k-1}^{-(m_{k-1}'+m_k')}\delta_k^{-m_k'}(a_1b_2-b_1a_2)^{m_2'} \cdots (a_{k-1}b_k-b_{k-1})^{m_k'}.$$

Write $\mathscr{S}=(a_1b_2-b_1a_2)^{m_2'} \cdots (a_{k-1}b_k-b_{k-1})^{m_k'}\mathscr{B}'$, where

$$\mathscr{B}' = \{a_1^{p_1-(m_2'+q_1)}b_1^{(m_1-m_1')-p_1} \cdots a_{k-1}^{p_{k-1}-(m_k'+q_{k-1})}b_{k-1}^{(m_{k-1}-m_{k-1}')-p_{k-1}}$$
$$\times\, b_k^{(m_k-m_k')-p_k} \mid 2\sum_1^k p_i-\sum_1^k m_i-\sum_1^{k-1} q_i=q\}.$$

Replacing the set $\mathscr{B}$ in Lemmas 2 and 3 by $\mathscr{B}'$, the result follows immediately.

Branching theorems are usually stated by means of patterns similar to (3.1). The following theorem gives this description for the case studied in Theorem 4.

THEOREM 5 *The irreducible representations of G_1 appearing in $(m_1, \ldots, m_k)$ of $Sp(2k)$ can be put in one-to-one correspondence with all patterns of integers*

$$\left(\begin{array}{c|c} m_1 \cdots\cdots\cdots\cdots m_k & \\ p_1 \cdots\cdots p_{k-1} & \\ q_1 \cdots\cdots q_{k-1} & q \end{array}\right) \tag{4.14}$$

where $m_1 \ge p_1 \ge m_2 \cdots \ge p_{k-1} \ge m_k$, $p_1 \ge q_1 \ge p_2 \cdots \ge p_{k-1} \ge q_{k-1} \ge 0$,

$$q = m_k''+\sum_1^{k-1}(p_i-q_i-m_{i+1}')-\sum_1^{k-1}(m_i''-p_i+m_{i+1}'), \tag{4.15}$$

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and $\sum_1^{k-1}(m_i''-p_i+m_{i+1}')\leq m_k''$ *Furthermore each pattern corresponds to* $(q_1, \ldots, q_{k-1}; q)$ *of* G_1.

Proof. In the particular case when (4.9) holds, recall that $m(q_1, \ldots, q_{k-1}; q)$ is the number of monomials $a_1^{s_1+t_1}\cdots a_{k-1}^{s_{k-1}+t_{k-1}}b_k^{t_1+\cdots+t_{k-1}}$ (where $s_1, \ldots, s_{k-1}, t_1, \ldots, t_{k-1}$ satisfy (4.11)) that belong to $\mathscr{B}$. Hence every $(q_1, \ldots, q_{k-1}; q)$ is associated with partitions of the integers m_i-q_i $(i=1, \ldots, k-1)$ into s_i and t_i such that $\sum_1^{k-1}s_i-\sum_1^{k-1}t_i=q-m_k$ and $t_1+\cdots+t_{k-1}\leq m_k$. Let $t_i=m_i-p_i$, then $s_i=p_i-q_i$, $q=m_k+\sum_1^{k-1}(p_i-q_i)-\sum_1^{k-1}(m_i-p_i)$ and $(q_1, \ldots, q_{k-1}; q)$ can then be associated with the pattern

$$\left(\begin{matrix} m_1 \cdots\cdots\cdots\cdots m_k \\ p_1 \cdots\cdots p_{k-1} \\ q_1 \cdots\cdots q_{k-1} \end{matrix}\;\middle|\; q = m_k+\sum_1^{k-1}(p_i-q_i)-\sum_1^{k-1}(m_i-p_i)\right). \tag{4.16}$$

It is now clear that when (4.9) holds, all $(q_1, \ldots, q_{k-1}; q)$ that are contained in $(m_1, \ldots, m_k)$ can be put in one-to-one correspondence with patterns (4.16) where $m_k-\sum_1^{k-1}(m_i-p_i)\geq 0$.

The general case can be proved analogously by considering the set $\mathscr{B}'$ as defined in the proof of Theorem 1 and letting $s_i=p_i-q_i-m_{i+1}'$, $t_i=m_i''-p_i+m_{i+1}'$ in (4.12).

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