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SOLVABILITY OF FACTORIZED FINITE GROUPS

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Abstract. Using classification theorems of simple groups, we give a proof of a conjecture on factorized finite groups which is an extension of a well known theorem due to P. Hall.

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Let G be a finite group and $G = G_1G_2$, where G_1 and G_2 are subgroups of G. There are a number of results in which one can deduce the solvability of G from suitable conditions on G_1 and G_2 (see for instance [**1**]). According to a famous result of P. Hall ([**5**] and [**6**]), a finite group G is solvable if and only if $G = P_1P_2....P_m$ where $P_i \in Syl_{p_i}G$ and $P_iP_j = P_jP_i$ for all $i, j \in \{1, ..., m\}$.

Using classification theorems of simple groups, in this note we present an extension of the cited theorem of Hall. For this, we consider the following definition.

Let $\mathcal{S}$ be the class of all solvable groups. Two subgroups G_1, G_2 of a given group G are *$\mathcal{S}$-connected* whenever for each $x \in G_1$, $y \in G_2$ we have $< x, y > \in \mathcal{S}$.

Considering this definition we prove the following theorem, which proves the conjecture formulated in [**2**].

THEOREM. *Let $G = G_1G_2....G_m$ be a group such that $G_1, ..., G_m$ are solvable subgroups of G. If $G_1, ..., G_m$ are pairwise permutable and pairwise $\mathcal{S}$-connected, then G is solvable.*

2. Preliminary results.

In this section, we collect some of the results that are needed. If G is the product of two solvable subgroups, it is known that G is not necessarily solvable. Particular cases of finite groups factorizable by two subgroups were studied by many authors. Kazarin [**8**] studied the general case and obtained the following result.

2.1. LEMMA (Kazarin [**8**].) *Let $G = G_1G_2$ be a group with G_1 and G_2 solvable subgroups of G. If all composition factors of G are known groups, then the nonabelian simple composition factors of G belong to the following list of groups:*

(a) $\mathbf{PSL}(2, q)$ *with* $q > 3$,
(b) $\mathbf{M}_{11}$,

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(c) $\mathbf{PSL}(3, q)$ *with* $q < 9$,
(d) $\mathbf{PSp}(4, 3)$,
(e) $\mathbf{PSU}(3, 8)$,
(f) $\mathbf{PSL}(4, 2)$.

A consequence of Kazarin's result is the following lemma.

2.2. Lemma (Fisman [**3**].) *Let* $G = G_1G_2....G_m$ *be a group such that* G_iG_j *is a solvable subgroup, for every* $i, j \in \{1, 2, ..., m\}$. *Then* G *is solvable.*

Remark 1. Let $G = \mathbf{PSL}(2, q)$ with $q = p^l$ and p a prime number. The following properties of G are well known.

(a) $|G| = \dfrac{q(q+1)(q-1)}{d}$ where $d = (2, q-1)$.

(b) A Sylow-p-subgroup P of G is elementary abelian of order $q = p^l$ and P is disjoint from its conjugates. Further $|G : \mathbf{N}_G(P)| = q + 1$.

(c) If r is a prime distinct from p or 2, then a Sylow-r-subgroup of G is cyclic.

(d) If p is odd, then a Sylow-2-subgroup of G is dihedral.

(e) G contains cyclic subgroups U of orders $s = \dfrac{q+1}{d}$ and $s = \dfrac{q-1}{d}$. For each $1 \neq u \in U$, we have that $\mathbf{N}_G(< u >)$ is a dihedral group of order $2s$.

For a proof see [**7**, Satz 8.2/8.3/8.4, p.192].

2.3. Lemma *Let* $G = G_1G_2 = G_1N = G_2N$ *be a group, where* G_1 *and* G_2 *are solvable subgroups of* G *and* N *is the unique minimal normal subgroup of* G *and* N *is nonsolvable. Then*

(a) G_1 *acts transitively as a permutation group on the set of normal subgroups of* N *and* $G_1 \cap N = \prod_i^m L_i$ *for* $N = \prod_i^m N_i$ (*with* $N_i \cong N_j$) *for every* $i, j \in \{1, ..., m\}$ *and* $L_i = N_i \cap G_1$.

(b) $|N_1|$ *divides* $|Out(N_1)||N_1 \cap G_1||N_1 \cap G_2|$.

For a proof see [**8**, Lemmas 2.3 and 2.5].

2.4. Lemma *Let* $G = < x >< y >$ *be a group. Then* G *is supersolvable. In particular, the Sylow-p-subgroup of* G, *where* p *is the largest prime divisor of* $|G|$, *is normal in* G.

For a proof see [**7**, Satz 10.1, p.722].

Remark 2. Let G be a solvable group, $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. If $\mathbf{N}_G(P) = P$ and $\mathbf{N}_G(Q) = Q$, then $p = q$.

This is a corollary of Carter's Theorem [**7**, Satz 12.2, p.736].

3. Proof of the theorem. Suppose that the Theorem 1 is false and let G be a counterexample of smallest order with m least possible. By Lemma 2.2, we have that

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$G = G_1G_2$. Clearly the hypothesis is inherited by factor groups. Hence G has a unique minimal normal subgroup N, and N is nonsolvable. Since $G_1N = G_1(G_1N \cap G_2)$ and G_1, $G_1N \cap G_2$ are $\mathcal{S}$-connected solvable subgroups of G_1N, we have that $G_1N = G = G_2N$, by the minimality of G.

By Lemma 2.1, we have that the composition factors of G belong to the following list:

(a) $\mathbf{PSL}(2, q)$ with $q > 3$,
(b) $\mathbf{M}_{11}$,
(c) $\mathbf{PSL}(3, q)$ with $q < 9$,
(d) $\mathbf{PSp}(4, 3)$,
(e) $\mathbf{PSU}(3, 8)$,
(f) $\mathbf{PSL}(4, 2)$.

From now on we denote by L a nonabelian simple composition factor of G. By the above arguments $L \leq G$. Put $H = L \cap G_1$ and $K = L \cap G_2$.

(**I**) *Assume that* $L \cong \mathbf{PSL}(2, q)$ *with* $q = p^a$ *an odd number.*
Let $r \in \pi(L) - \{2, p\}$ and R be an r-subgroup of L. Since $(\frac{q+1}{2}, \frac{q-1}{2}) = 1$ we have that $|\mathbf{N}_L(R)| = 2s$ with $s = \frac{q+1}{2}$ or $s = \frac{q-1}{2}$, by Remark 1 (e). In particular p is not a divisor of $|\mathbf{N}_L(R)|$.

Let x be a p-element of H and y an r-element of H or K. Since $M =< x, y >$ is solvable there exist $P_1 \in Syl_p(M)$ and $R_1 \in Syl_r(M)$ such that $P_1R_1 = R_1P_1$. Since p is not a divisor of $|\mathbf{N}_L(R_1)|$, by Remark 2 we have that r divides $|\mathbf{N}_M(P_1)|$. Hence, if $P \in Syl_p(L)$, then r divides $|\mathbf{N}_L(P)|$ by Remark 1 (a). It follows that every odd prime number in $\pi(L)$ divides $|\mathbf{N}_L(P)| = \frac{q(q-1)}{2}$, by Lemma 2.3 (b). Again, since $(\frac{q+1}{2}, \frac{q-1}{2}) = 1$ we have $q + 1 = 2^s$. We shall obtain a contradiction proving that 2 divides $|\mathbf{N}_L(P)|$.

Let w be a 2-element of H or K and $M =< x, w >$. Let $S_1 \in Syl_2(M)$ and $P_1 \in Syl_p(M)$ be such that $P_1S_1 = S_1P_1$. Since 2 does not divide $|\mathbf{N}_L(P)|$ it follows that 2 does not divide $|\mathbf{N}_M(P_1)|$ by Remark 1 (a). Hence p divides $|\mathbf{N}_M(S_1)|$. Let $P_2 \in Syl_p(\mathbf{N}_M(S_1))$. If there exist a subgroup Z of S_1 of order two normalized by some subgroup P_3 of P_2, then $P_3 \trianglelefteq ZP_3$ and we obtain a contradiction. Therefore, since S_1 is cyclic or dihedral, we have that S_1 is of order 4 and P_2 acts faithfully on S_1. Hence $P_2 \leq \mathbf{GL}(2, 2)$ and $p = 3$. It follows that $q = 3$, a contradiction.

(**II**) *Assume that* $L \cong \mathbf{PSL}(2, 2^n)$.
Let p be the largest prime and $r \neq p$ an odd prime both dividing $\omega|L|$, x be a p-element of H and y an r-element of H or K. Since $M =< x, y >$ is solvable, there are $P_1 =< x_1 >\in Syl_p(M)$ and $R_1 =< y_1 >\in Syl_r(G_2)$ such that $P_1R_1 = R_1P_1$. Hence, by Lemma 2.4 it follows that $P_1 \trianglelefteq R_1P_1$. We deduce that every odd prime in $\pi(L)$ divides ω (by Remark 1(e)) and $2^n = 2$, a contradiction.

(**III**) *Assume that L is isomorphic to some group of the following list:* $\{\mathbf{PSL}(3, q)$ *with* $q < 9$ *(here* $q \neq 2$ *since* $\mathbf{PSL}(3, 2) \cong \mathbf{PSL}(2, 7))$*,* $\mathbf{M}_{11}$*,* $\mathbf{PSp}(4, 3)$*,* $\mathbf{PSL}(4, 2)$ *or* $\mathbf{PSU}(3, 8)\}$.

By Lemma 2.3 (b) we have that $|L|$ divides $|Out(L)||H||K|$. Therefore, for every $\{p, q\} \subseteq \pi(L)$ there is a solvable $\{p, q\}$-subgroup S of L. Since

PSL(3, 3) does not have a {2, 13}-subgroup,
PSL(3, 4) does not have a {5, 7}-subgroup,
PSL(3, 5) does not have a {5, 31}-subgroup,
PSL(3, 7) does not have a {7, 19}-subgroup,
PSL(3, 8) does not have a {7, 73}-subgroup,
$\mathbf{M}_{11}$ does not have a {3, 11}-subgroup,
PSp(4, 3) does not have a {3, 5}-subgroup,
PSU(3, 8) does not have a {7, 19}-subgroup,
PSL(4, 2) does not have a {5, 7}-subgroup,

we have a contradiction.

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