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Local Dimensions of Measures of Finite Type II: Measures Without Full Support and With Non-regular Probabilities

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Abstract. Consider a finite sequence of linear contractions $S_j(x) = \rho x + d_j$ and probabilities $p_j > 0$ with $\sum p_j = 1$. We are interested in the self-similar measure $\mu = \sum p_j \mu \circ S_j^{-1}$, of finite type. In this paper we study the multi-fractal analysis of such measures, extending the theory to measures arising from non-regular probabilities and whose support is not necessarily an interval.

Under some mild technical assumptions, we prove that there exists a subset of supp μ of full μ and Hausdorff measure, called the *truly essential class*, for which the set of (upper or lower) local dimensions is a closed interval. Within the truly essential class we show that there exists a point with local dimension exactly equal to the dimension of the support. We give an example where the set of local dimensions is a two element set, with all the elements of the truly essential class giving the same local dimension. We give general criteria for these measures to be absolutely continuous with respect to the associated Hausdorff measure of their support, and we show that the dimension of the support can be computed using only information about the essential class.

To conclude, we present a detailed study of three examples. First, we show that the set of local dimensions of the biased Bernoulli convolution with contraction ratio the inverse of a simple Pisot number always admits an isolated point. We give a precise description of the essential class of a generalized Cantor set of finite type, and show that the k-th convolution of the associated Cantor measure has local dimension at $x \in (0,1)$ tending to 1 as k tends to infinity. Lastly, we show that within a maximal loop class that is not truly essential, the set of upper local dimensions need not be an interval. This is in contrast to the case for finite type measures with regular probabilities and full interval support.

1 Introduction

In this paper we continue the investigations, begun in [10], of the multifractal analysis of equicontractive self-similar measures of finite type. For self-similar measures arising from an IFS that satisfies the open set condition the multifractal analysis is well understood. In particular, the set of attainable local dimensions is a closed interval whose endpoints can be computed with the Legendre transform.

For measures that do not satisfy the open set condition, the multifractal analysis is more complicated and the set of local dimensions need not be an interval. This phenomenon was discovered first for the 3-fold convolution of the classical Cantor measure in [13] and was further explored in [2, 8, 15, 20], for example. In [17], Ngai

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and Wang introduced the notion of finite type, a property stronger than the weak separation condition (WSC), but satisfied by many interesting self-similar measures that fail the open set condition. Examples include Bernoulli convolutions with contraction factor the inverse of a Pisot number and self-similar Cantor-like measures with ratio the inverse of an integer.

Building on earlier work, such as [9,12,14,19], Feng undertook a study of equicontractive, self-similar measures of finite type in [3–5], with his main focus being Bernoulli convolutions. Motivated by this research, in [10] (and [11]) a general theory was developed for the local dimensions of self-similar measures of finite type assuming the associated self-similar set was an interval and the underlying probabilities $\{p_j\}_{j=0}^{m}$ generating the measure μ were regular, meaning $p_0 = p_m = \min p_j$. There it was shown that the set of local dimensions at points in the "essential class" (a set of full Lebesgue measure in the support of μ and often the interior of its support) was a closed interval and that the set of local dimensions at periodic points was dense in this interval. Formulas were given for the local dimensions. These formulae are particularly simple at periodic points.

In this paper we refine the techniques of [10] so that we do not require any assumptions on the probabilities and we relax the requirement that the support of μ (the self-similar set) is an interval. These assumptions were very significant to the approach taken in much of the earlier work, and complications arise when these assumptions do not hold. In Section 3, we introduce the notion of the "truly essential class". Our main theoretical result is that under a mild technical assumption, (that is required only when the support is not an interval) the set of local dimensions at the points in the truly essential class is a closed interval and the set of local dimensions at the periodic points is dense in that interval. We show that the truly essential class is the relative interior of the essential class, and we prove that it has full μ and Hausdorff s-measure, where s is the Hausdorff dimension of the self-similar set.

We prove that there is always a point at which the local dimension of μ coincides with the Hausdorff dimension of $\operatorname{supp}\mu$ and give an example of a measure where this occurs at all the truly essential points (but not at all points of the support). A sufficient condition is given for a finite type measure to be absolutely continuous with respect to the associated Hausdorff measure, and an example is given that satisfies this condition when $s = 1$, even though the self-similar set is not an interval. We also give a formula for calculating the Hausdorff dimension of the support from just the knowledge of the essential class.

The proofs of these facts rely upon formulas that we develop in Section 2 for calculating local dimensions. These formulas are relatively simple for periodic points, although necessarily more complicated than under the assumptions of regular probabilities and full support.

Related results were given by Feng in [5]. There, Feng constructed a (typically, countably infinite) family of closed intervals, I_j, with disjoint interiors, where $\bigcup I_j$ is of full measure and on each of these closed intervals the set of attainable local dimensions of the restricted measure $\mu_j := \mu|_{I_j}$ was a closed interval. From his construction one can see that $\bigcup I_j \cap K$ is our essential class and $\bigcup \operatorname{int}(I_j) \cap K$ is contained in our truly essential class. Note, however, that the local dimension of the restricted measure μ_j

at an end point of I_j is not necessarily the same as the local dimension of μ at this point, even when it is a truly essential point. The techniques of this paper enable us to compute the local dimensions of these boundary points, without assuming regular probabilities or that K is an interval, as required in [10].

In [4], Feng had shown that the set of local dimensions of the uniform Bernoulli convolutions with contraction factor the inverse of a simple Pisot number (meaning the minimal polynomial is $x^n - \sum_{j=0}^{n-1} x^j$) is always a closed interval. As one application of our main result, in Section 4 we prove, in contrast, that biased Bernoulli convolutions with these contraction factors always admit an isolated point in their set of local dimensions.

In Section 5 we present a detailed study of the local dimensions of finite type Cantor-like measures, extending the work done in [2,10,20]. In those papers, it was shown, for example, that if $p_0 < p_j$ for $j \neq 0, m$, then the local dimension at 0 is isolated. Here we give further conditions that ensure there is an isolated point. But we also give examples where the measure has no isolated points, and we give a family of examples that have exactly two distinct local dimensions. We also show that the local dimensions of the rescaled k-fold convolutions of a Cantor-like measure converge to 1 at points in $(0,1)$. Previously, in [1], it was shown that these local dimensions were bounded.

In Section 6, we illustrate, by means of a detailed example, the complications and differences that can arise when studying the local dimensions outside of the truly essential class and in Section 7 investigate the connection between finite type and Pisot contractions.

Feng and Lau [7] studied yet more general IFS that only satisfy the WSC and showed that in this case there is also an open set U such that the set of attainable local dimensions of the restricted measure, $\mu|_U$, is a closed interval. We note that in the examples given in that paper, the set U is much smaller than our truly essential class.

2 Notation and Preliminary Results

We begin by introducing the definition of finite type as well as basic notation and terminology that will be used throughout the paper.

2.1 Finite Type

Consider the iterated function system (IFS) consisting of the contractions $S_j : \mathbb{R} \to \mathbb{R}$, $j = 0, \ldots, m$, defined by

$$S_j(x) = \rho x + d_j,$$

where $0 < \rho < 1$, $0 = d_0 < d_1 < d_2 < \cdots < d_m$ and $m \geq 1$ is an integer. The unique, non-empty, compact set K satisfying

$$K = \bigcup_{j=0}^{m} S_j(K)$$

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is known as the *associated self-similar set*. By rescaling the d_j if needed, we can assume that the convex hull of K is $[0,1]$. We will not assume that $K = [0,1]$ or even that it has non-empty interior.

It was shown in [17, Thm. 1.2] that if $s = \dim_H K$ and H^s denotes the Hausdorff s-measure restricted to K, then $0 < H^s(K) < \infty$. Upon normalizing we can assume $H^s(K) = 1$. Further, we note that $0 < s \le 1$. We remark that in the special case that $K = [0,1]$, then $s = 1$ and H^s is the normalized Lebesgue measure.

Suppose probabilities $p_j > 0$, $j = 0,\dots,m$ satisfy $\sum_{j=0}^m p_j = 1$. Throughout this paper, our interest will be in the self-similar measure μ associated with the family of contractions $\{S_j\}$ given above, which satisfies the identity

$$\mu = \sum_{j=0}^{m} p_j \mu \circ S_j^{-1}. \tag{2.1}$$

These non-atomic probability measures have support K.

We put $\mathcal{A} = \{0,\dots,m\}$. Given an n-tuple $\sigma = (j_1,\dots,j_n) \in \mathcal{A}^n$, we write S_σ for the composition $S_{j_1} \circ \cdots \circ S_{j_n}$ and let $p_\sigma = p_{j_1} \cdots p_{j_n}$.

Definition 2.1 The iterated function system (IFS),

$$\{S_j(x) = \rho x + d_j : j = 0,\dots,m\},$$

is said to be of *finite type* if there is a finite set $F \subseteq \mathbb{R}$ such that for each positive integer n and any two sets of indices $\sigma = (j_1,\dots,j_n)$, $\sigma' = (j_1',\dots,j_n') \in \mathcal{A}^n$, either

$$\rho^{-n}\left|S_\sigma(0) - S_{\sigma'}(0)\right| > c \quad \text{or} \quad \rho^{-n}\left(S_\sigma(0) - S_{\sigma'}(0)\right) \in F,$$

where $c = (1-\rho)^{-1}(\max d_j - \min d_j)$ is the diameter of K.

If the IFS is of finite type and μ is an associated self-similar measure satisfying (2.1), we also say that μ is of *finite type*.

Here we have given the general definition of finite type for an equicontractive IFS in $\mathbb{R}$. This simplifies to $c = 1$ in the case where the convex hull of K is $[0,1]$. It is worth noting here that the definition of finite type is independent of the choice of probabilities.

Finite type is a property that is stronger than the weak separation condition, but weaker than the open set condition [18]. Examples include (uniform or biased) Bernoulli convolutions with contraction factor the reciprocal of a Pisot number and Cantor-like measures associated with Cantor sets with contraction factors reciprocals of integers. See Sections 4 and 5 where these are studied in detail.

2.2 Characteristic Vectors and the Essential Class

The structure of measures of finite type is explained in detail in [3–5] and [10]; we will give a brief overview here.

For each integer n, let $h_1,\dots,h_{s_n}$ be the collection of elements of the set $\{S_\sigma(0), S_\sigma(1) : \sigma \in \mathcal{A}^n\}$, listed in increasing order. Put

$$\mathcal{F}_n = \left\{[h_j, h_{j+1}] : 1 \le j \le s_n - 1 \text{ and } (h_j, h_{j+1}) \cap K \neq \varnothing\right\}.$$

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Elements of $\mathcal{F}_n$ are called *net intervals of level n*. By definition, a net interval contains net subintervals of every lower level. For each $\Delta \in \mathcal{F}_n$, $n \geq 1$, there is a unique element $\widehat{\Delta} \in \mathcal{F}_{n-1}$ that contains Δ, called the *parent* (of *child* Δ). We will define the *left-most child* of parent $\widehat{\Delta} = [a, b] \in \mathcal{F}_{n-1}$ to be the child $\Delta = [a, b'] \in \mathcal{F}_n$. We will similarly define the *right-most child*. It is worth noting that it is possible for a child to be both the left and the right-most child. It is further worth observing that because we are not assuming the self-similar set is the interval $[0, 1]$, it is possible for a parent to have no left-most child or no right-most child.

Given $\Delta = [a, b] \in \mathcal{F}_n$, we denote the *normalized length* of Δ by

$$\ell_n(\Delta) = \rho^{-n}(b - a).$$

By the *neighbour set* of Δ we mean the ordered k-tuple

$$V_n(\Delta) = (a_1, \dots, a_k),$$

where

$$\{a_1, \dots, a_k\} = \left\{ \rho^{-n}(a - S_\sigma(0)) : \sigma \in \mathcal{A}^n, \Delta \subseteq S_\sigma([0, 1]) \right\}.$$

Given $\Delta_1, \dots, \Delta_m$, (listed in order from left to right) all the net intervals of level n which have the same parent and normalized length as Δ, let $r_n(\Delta)$ be the integer r with $\Delta_r = \Delta$. The *characteristic vector of* Δ is the triple

$$\mathcal{C}_n(\Delta) = (\ell_n(\Delta), V_n(\Delta), r_n(\Delta)).$$

Often we suppress $r_n(\Delta)$ giving the *reduced characteristic vector* $(\ell_n(\Delta), V_n(\Delta))$.

If the measure is of finite type, there will be only finitely many distinct characteristic vectors. We denote the set of such vectors by Ω,

$$\Omega = \{\mathcal{C}_n(\Delta) : n \in \mathbb{N}, \Delta \in \mathcal{F}_n\}.$$

By an *admissible path, η, of length $L(\eta) = L$*, we will mean an ordered L-tuple, $\eta = (\gamma_j)_{j=1}^L$, where $\gamma_j \in \Omega$ for all j and the characteristic vector, γ_j, is the parent of γ_{j+1}. Each $\Delta \in \mathcal{F}_n$ can be uniquely identified by an admissible path of length $n + 1$, say $(\mathcal{C}_0(\Delta_0), \dots, \mathcal{C}_n(\Delta_n))$, where $\Delta = \Delta_n$, $\Delta_0 = [0, 1]$, $\Delta_j \in \mathcal{F}_j$, and $\Delta_j = \widehat{\Delta_{j+1}}$ for all j. This is called the *symbolic representation* of Δ; we will frequently identify Δ with its symbolic representation.

Similarly, the *symbolic representation* for $x \in K$ will mean the sequence

$$[x] = \left(\mathcal{C}_0(\Delta_0), \mathcal{C}_1(\Delta_1), \dots \right)$$

of characteristic vectors where $x \in \Delta_n$ for all n and $\Delta_j \in \mathcal{F}_j$ is the parent of Δ_{j+1}. The notation $[x \mid N]$ will mean the admissible path consisting of the first N characteristic vectors of $[x]$. We will often write $\Delta_n(x)$ for the net interval in $\mathcal{F}_n$ containing $x \in K$; its symbolic representation is $[x \mid n]$.

If x is an endpoint of $\Delta_n(x)$ for some n (and then for all larger integers) we call x a *boundary point*. We remark that if x is a boundary point, then there can be two different symbolic representations for x, one approaching x from the left, *i.e.*, by taking right-most descendents at all levels beyond level n, and the other approaching x from the right, by taking left-most descendents. If x is not a boundary point, then the symbolic representation is unique.

It is worth emphasizing that $[x \mid N]$ is defined as the truncation of $[x]$ as opposed to defining it as a sequence $(\mathcal{C}_0(\Delta_0), \mathcal{C}_1(\Delta_1), \dots, \mathcal{C}_N(\Delta_N))$ with $x \in \Delta_i$. To see this

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distinction, recall that it is possible for $\Delta_N = [h_i, h_{i+1}]$ to have no right-most children. Let $x = h_{i+1}$ be the right-most endpoint of Δ_N. Then x is also the left-most endpoint of the adjacent net interval, $\Delta'_N = [h_{i+1}, h_{i+2}]$. As Δ_N has no right-most child, we do not have a net interval of depth $N+1$ with $x \in \Delta_{N+1} \subseteq \Delta_N$. As K has no isolated points and $x \in K$, for all $M \geq N$ we must have net intervals $x \in \Delta_M \subseteq \Delta'_N$. In such a case, the boundary point x has a unique symbolic representation.

A non-empty subset $\Omega' \subseteq \Omega$ is called a *loop class* if whenever $\alpha, \beta \in \Omega'$, then there are characteristic vectors γ_j, $j = 1, \ldots, n$, such that $\alpha = \gamma_1$, $\beta = \gamma_n$ and $(\gamma_1, \ldots, \gamma_n)$ is an admissible path with all $\gamma_j \in \Omega'$. A loop class $\Omega' \subseteq \Omega$ is called an *essential class* if, in addition, whenever $\alpha \in \Omega'$ and $\beta \in \Omega$ is a child of α, $\beta \in \Omega'$. Of course, an essential class is a maximal loop class.

In [5, Lemma 6.4], Feng proved the important fact that there is always precisely one essential class, which we will denote by Ω_0. If $[x] = (\gamma_0, \gamma_1, \gamma_2, \ldots)$ with $\gamma_j \in \Omega_0$ for all large j, we will say that x is an *essential point* (or is *in the essential class*) and similarly speak of a net interval being essential. A path $(\gamma_j)_{j=1}^{L}$ is in the essential class if all $\gamma_j \in \Omega_0$. We similarly speak of a point, net interval, or path as being in a given loop class. The finite type property ensures that every element in the support of μ is contained in a maximal loop class.

We remark that the essential class is dense in the support of μ. This is because the uniqueness of the essential class ensures that every net interval contains a net subinterval in the essential class. In Proposition 3.6 we will show that the essential class has full μ measure and full Hausdorff s-measure in K, where s is the Hausdorff dimension of K.

2.3 Transition Matrices

A very important concept in the multifractal analysis of measures of finite type are the so-called transition matrices. These are defined as follows: Let $\Delta = [a, b]$ be a net interval of level n with parent $\widehat{\Delta} = [c, d]$. Assume $V_n(\Delta) = (a_1, \ldots, a_N)$ and $V_{n-1}(\widehat{\Delta}) = (c_1, \ldots, c_M)$. The *primitive transition matrix*, $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))$, is a $M \times N$ matrix whose jk entry is given by

$$T_{jk} := \Big(T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)) \Big)_{jk} = p_\ell$$

if $\ell \in \mathcal{A}$ and there exists $\sigma \in \mathcal{A}^{n-1}$ with $S_\sigma(0) = c - \rho^{n-1} c_j$ and $S_{\sigma\ell}(0) = a - \rho^n a_k$, and $T_{jk} = 0$ otherwise. We note that in [10] the transition matrices are normalized so that the minimal non-zero entry is 1. That is, we used $p_*^{-1} T$ instead of T, where $p_* = \min p_j$.

We observe that each column of a primitive transition matrix has at least one non-zero entry. The same is true for each row if $\operatorname{supp}\mu = [0, 1]$, but not necessarily otherwise; see Example 3.10.

Given an admissible path $\eta = (\gamma_1, \ldots, \gamma_n)$, we write

$$T(\eta) = T(\gamma_1, \ldots, \gamma_n) = T(\gamma_1, \gamma_2) \cdots T(\gamma_{n-1}, \gamma_n)$$

and refer to such a product as a *transition matrix*. We will say the transition matrix $T(\gamma_1, \ldots, \gamma_n)$ is *essential* if all γ_j are essential characteristic vectors.

By the norm of a matrix T we mean

$$\|T\| = \sum_{jk} |T_{jk}|.$$

A matrix is called *positive* if all its entries are strictly positive. An admissible path η is called *positive* if $T(\eta)$ is a positive matrix. Here is an elementary lemma that shows the usefulness of positivity.

The notation $\mathrm{sp}(T)$ means the spectral radius of the matrix T,

$$\mathrm{sp}(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Lemma 2.2 *Assume A, B, C are transition matrices and B is positive.*

(i) *There are constants $a, b > 0$, depending on the matrices A and B respectively, so that $\|AC\| \geq a\|C\|$ and $\|ABC\| \geq b\|A\|\|C\|$.*

(ii) *If each row of A has a non-zero entry, then there is a constant c, depending on matrix C, such that $\|AC\| \geq c\|A\|$.*

(iii) *There is a constant $C_1 = C_1(B)$ such that if AB is a square matrix, then*

$$\mathrm{sp}(AB) \leq \|AB\| \leq C_1 \,\mathrm{sp}(AB).$$

(iv) *Suppose B is a square matrix. There is a constant $C_2 = C_2(B)$ such that*

$$\mathrm{sp}(B^n) \leq \|B^n\| \leq C_2 \,\mathrm{sp}(B^n) \text{ for all } n.$$

Proof Parts (i) and (ii) follow by simply writing the expressions for $\|AC\|$ and $\|ABC\|$ in terms of the entries of A, B, C, and noting that a transition matrix has non-negative entries and each column has a non-zero entry.

Parts (iii) and (iv) follow as in [10, Lemma 3.15]. ■

2.4 Basic Facts about Local Dimensions of Measures of Finite Type

Definition 2.3 Given a probability measure μ, by the *upper local dimension* of μ at $x \in \operatorname{supp}\mu$ we mean the number

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_{r\to 0^+} \frac{\log \mu([x-r, x+r])}{\log r}.$$

Replacing the lim sup by lim inf gives the *lower local dimension*, denoted $\underline{\dim}_{\mathrm{loc}}\mu(x)$. If the limit exists, we call the number the *local dimension* of μ at x and denote this by $\dim_{\mathrm{loc}} \mu(x)$.

It is easy to see that

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{n\to\infty} \frac{\log \mu([x-\rho^n, x+\rho^n])}{n \log \rho} \text{ for } x \in \operatorname{supp}\mu,$$

and similarly for the upper and lower local dimensions.

Notation Throughout the paper, when we write $F_n \sim G_n$ we mean there are positive constants c_1, c_2 such that

$$c_1 F_n \leq G_n \leq c_2 F_n \text{ for all } n.$$

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To calculate local dimensions, it will be helpful to know $\mu(\Delta)$ for net intervals Δ.

Proposition 2.4 *Let $\Delta_n = [a,b] \in \mathcal{F}_n$, with $V_n(\Delta_n) = (a_1,\dots,a_N)$. Then*

$$\mu(\Delta_n) = \sum_{i=1}^{N} \mu[a_i, a_i + \ell_n(\Delta_n)] \sum_{\substack{\sigma \in \mathcal{A}^n \\ \rho^{-n}(a - S_\sigma(0)) = a_i}} p_\sigma.$$

Furthermore, if $[\Delta_n] = (\gamma_0,\gamma_1,\dots,\gamma_n)$ and

$$P_n(\Delta_n) = \sum_{i=1}^{N} \sum_{\sigma \in \mathcal{A}^n : \rho^{-n}(a - S_\sigma(0)) = a_i} p_\sigma,$$

then

$$\begin{aligned} \mu(\Delta_n) \sim P_n(\Delta_n) &= \| T(\gamma_0,\gamma_1,\dots,\gamma_n) \| \\ &= p_*^n \| T^*(\gamma_0,\gamma_1,\dots,\gamma_n) \| \end{aligned}$$

where $p_* = \min p_j$.

Proof This follows in a similar fashion to Lemma 3.2, Corollary 3.4, and the discussion prior to Corollary 3.10 of [10], noting that

$$\mu\big([a_i, a_i + \ell_n(\Delta)]\big) \geq \mu\big(S_\sigma^{-1}([a,b])\big) > 0. \qquad \blacksquare$$

The analogue of Proposition 2.4 was very useful in [10], as it was the key idea in proving the following formula.

Corollary 2.5 ([10, Cor. 3.10]) *Suppose μ is a self-similar measure satisfying identity (2.1), that has support $[0,1]$, and is of finite type and has probabilities satisfying $p_0 = p_m = \min p_j$. If $x \in \operatorname{supp}\mu$, then*

$$\dim_{\mathrm{loc}} \mu(x) = \frac{\log p_0}{\log \rho} + \lim_{n\to\infty} \frac{\log \| T^*([x \mid n]) \|}{n \log \rho} = \lim_{n\to\infty} \frac{\log \| T([x \mid n]) \|}{n \log \rho}$$

and similarly for the upper and lower local dimensions.

This corollary need not be true, however, if the assumptions of $\operatorname{supp}\mu = [0,1]$ and regular probabilities, *i.e.,* $p_0 = p_m = \min p_j$, are not all satisfied. Instead, we proceed as follows.

Terminology Assume $\{h_j\} = \{S_\sigma(0), S_\sigma(1) : \sigma \in \mathcal{A}^n\}$ with $h_j < h_{j+1}$ and suppose $\Delta_n = [h_i, h_{i+1}]$ is a net interval of level n. Let Δ_n^- be the empty set if $(h_{i-1}, h_i) \cap K$ is empty and otherwise, let $\Delta_n^- = [h_{i-1}, h_i]$. Similarly, define Δ_n^+ to be the net interval immediately to the right of Δ_n (or the empty set), with the understanding that if Δ_n is the left or right-most net interval in $\mathcal{F}_n$, then Δ_n^- (resp. Δ_n^+) is the empty set. We refer to $\Delta_n^-(x), \Delta_n(x), \Delta_n^+(x)$ as *adjacent net intervals* (even if some are the empty set).

If x belongs to the interior of $\Delta_n(x)$, we put

$$M_n(x) = \mu(\Delta_n(x)) + \mu(\Delta_n^+(x)) + \mu(\Delta_n^-(x)).$$

If x is a boundary point of $\Delta_n(x) = [h_i, h_{i+1}]$, we put

$$M_n(x) = \mu(\Delta_n(x)) + \mu(\Delta_n'(x)), \tag{2.2}$$

where $\Delta_n'(x) = \Delta_n^-(x)$ if $x = h_i$ and $\Delta_n'(x) = \Delta_n^+(x)$ if $x = h_{i+1}$. We will refer to $\Delta_n'(x)$ as the *other net interval containing* x, even if it is empty and so formally not a net interval.

Theorem 2.6 *Let μ be a self-similar measure of finite type and let $x \in K$. Then*

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{n\to\infty} \frac{\log M_n(x)}{n \log \rho},$$

provided the limit exists. The lower and upper local dimensions of μ at x can be expressed similarly in terms of lim inf *and* lim sup.

Proof Assume, first, that

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{n\to\infty} \frac{\log \mu[x-\rho^n, x+\rho^n]}{n \log \rho} = D$$

exists.

By the finite type assumption, there are constants $0 < c < C$ such that $c\rho^n < \ell(\Delta_n) < C\rho^n$ for all $\Delta_n \in \mathcal{F}_n$. Pick j and k such that $\rho^j < c$ and $2C < \rho^{-k}$.

If x is a boundary point, then for sufficiently large n, x is an endpoint of $\Delta_n(x)$ and

$$[x-\rho^{n+j}, x+\rho^{n+j}] \subseteq \Delta_n(x) \cup \Delta_n'(x) \subseteq [x-\rho^{n-k}, x+\rho^{n-k}],$$

where the notation is as in (2.2). If x is not a boundary point, then

$$[x-\rho^{n+j}, x+\rho^{n+j}] \subseteq \Delta_n^-(x) \cup \Delta_n(x) \cup \Delta_n^+(x) \subseteq [x-\rho^{n-k}, x+\rho^{n-k}].$$

In either case,

$$\mu[x-\rho^{n+j}, x+\rho^{n+j}] \le M_n(x) \le \mu[x-\rho^{n-k}, x+\rho^{n-k}].$$

This in turn implies that

$$\begin{aligned}\Big(\frac{n+j}{n}\Big)\Big(\frac{\log \mu[x-\rho^{n+j}, x+\rho^{n+j}]}{(n+j)\log \rho}\Big) &\ge \frac{\log M_n(x)}{n\log\rho} \\ &\ge \Big(\frac{n-k}{n}\Big)\Big(\frac{\log \mu[x-\rho^{n-k}, x+\rho^{n-k}]}{(n-k)\log \rho}\Big).\end{aligned}$$

The limit of the left-hand side and the right-hand side both go to D, hence the limit of the middle expression exists and is equal to D.

It follows similarly that if $\lim_{n\to\infty} \log M_n(x)/n\log\rho$ exists, then also

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{n\to\infty} \frac{\log \mu[x-\rho^n, x+\rho^n]}{n\log\rho} = \lim_{n\to\infty} \frac{\log M_n(x)}{n\log\rho}.$$

The arguments for the lower and upper local dimensions are similar. ∎

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2.5 Periodic Points

Recall that in [10], $x \in K$ is called a *periodic point* if x has symbolic representation

$$[x] = (\gamma_0, \dots, \gamma_J, \theta^-, \theta^-, \dots),$$

where θ is an admissible cycle (a non-trivial path with the same first and last letter) and θ^- is the path with the last letter of θ deleted. We refer to θ as a *period* of x. Boundary points are necessarily periodic and there are only countably many periodic points. Note that a periodic point is essential if and only if it has a period that is a path in the essential class.

If there is a choice of θ for which $T(\theta)$ is a positive matrix, we call x a *positive, periodic point.*

Of course, a period for a periodic point x is not unique. For example, if $\theta = (\theta_1, \dots, \theta_L, \theta_1)$ is a period, then so is (θ^-, θ) and so is $(\theta_2, \dots, \theta_L, \theta_1, \theta_2)$. However, these different choices for the period give the same symbolic representation for x. But if x is a boundary point, then x may have two different symbolic representations, one for which $[x \mid N] = \Delta_N(x)$ and the other having $[x \mid N] = \Delta'_N(x)$, and these two representations arise from (fundamentally) different periods.

We note that the transition matrices associated with the two periods associated with the same symbolic representation for x will have the same normalized (for their length) spectral radius. This need not be the case for periods associated with different symbolic representations.

Here is the analogue of [10, Proposition 4.14] when there is no assumption of regularity.

Proposition 2.7 *If x is a periodic point with period θ, then the local dimension exists and is given by*

$$\dim_{\mathrm{loc}} \mu(x) = \frac{\log \operatorname{sp}(T(\theta))}{L(\theta^-) \log \rho},$$

where if x is a boundary point of a net interval with two different symbolic representations given by periods θ and ϕ, then θ is chosen to satisfy

$$\frac{\log \operatorname{sp}(T(\theta))}{L(\theta^-)} \geq \frac{\log \operatorname{sp}(T(\phi))}{L(\phi^-)}.$$

Proof First, suppose x is a boundary periodic point with two different symbolic representations given by periods θ, ϕ. There is no loss of generality in assuming the two periods have the same lengths $L = L(\theta^-)$ and pre-period path of length J. Assume $\operatorname{sp}(T(\theta)) \geq \operatorname{sp}(T(\phi))$. Given large n, let $m = [(n-J)/L]$, so x has symbolic representations

$$(\gamma_0, \gamma_1, \dots, \gamma_{J-1}, \underbrace{\theta^-, \dots, \theta^-}_{m}, \theta_1, \dots, \theta_t),$$
$$(\gamma_0, \gamma'_1, \dots, \gamma'_{J-1}, \underbrace{\phi^-, \dots, \phi^-}_{m}, \phi_1, \dots, \phi_t)$$

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for suitable $t \le L$. From Proposition 2.4,

$$\mu(\Delta_n(x)) \sim \Big\| T(\gamma_0,\dots,\gamma_{J-1},\underbrace{\theta^-,\dots,\theta^-}_{m},\theta_1,\dots,\theta_t)\Big\|,$$

$$\mu(\Delta'_n(x)) \sim \Big\| T(\gamma_0,\dots,\gamma'_{J-1},\underbrace{\phi^-,\dots,\phi^-}_{m},\phi_1,\dots,\phi_t)\Big\|.$$

Lemma 2.2 implies that there are positive constants c_j, independent of n, such that

$$\begin{aligned}\|(T(\theta))^{m+1}\| &\le \Big\| T(\underbrace{\theta^-,\dots,\theta^-}_{m},\theta_1,\dots,\theta_t)\Big\| \, \| T(\theta_t,\dots,\theta_L,\theta_1)\| \\ &\le c_1 \Big\| T(\gamma_0,\dots,\gamma_J,\underbrace{\theta^-,\dots,\theta^-}_{m},\theta_1,\dots,\theta_t)\Big\| \le c_2 \| (T(\theta))^m\|,\end{aligned}$$

and consequently,

$$c_3 \|(T(\theta))^{m+1}\| \le \mu(\Delta_n(x)) \le c_4 \|(T(\theta))^m\|.$$

Similarly,

$$c'_3 \|(T(\phi))^{m+1}\| \le \mu(\Delta'_n(x)) \le c'_4 \|(T(\phi))^m\|,$$

If $\mathrm{sp}(T(\theta)) > \mathrm{sp}(T(\phi))$, then for large enough m, $\|(T(\theta))^m\| > \|(T(\phi))^m\|$, and hence

$$c_3 \|(T(\theta))^{m+1}\| \le \mu(\Delta_n(x)) + \mu(\Delta'_n(x)) = M_n(x) \le 2c_4 \|(T(\theta))^m\|.$$

Since $\|(T(\theta))^m\|^{1/m} \to \mathrm{sp}(T(\theta))$, Theorem 2.6 gives

$$\dim_{\mathrm{loc}} \mu(x) = \lim_n \frac{\log M_n(x)}{n \log \rho} = \lim_m \frac{\log \|T(\theta)^m\|}{mL \log \rho} = \frac{\log \mathrm{sp}(T(\theta))}{L \log \rho}.$$

If, instead, $\mathrm{sp}(T(\theta)) = \mathrm{sp}(T(\phi))$, then for each n,

$$\begin{aligned} C_1 \max\Big(\|(T(\theta))^{m+1}\|, \|(T(\phi))^{m+1}\|\Big) &\le M_n(x) \\ &\le C_2 \max\Big(\|(T(\theta))^{m}\|, \|(T(\phi))^{m}\|\Big).\end{aligned}$$

As both

$$\frac{\log \|(T(\theta))^m\|}{mL \log \rho}, \frac{\log \|T(\phi))^m\|}{mL \log \rho} \longrightarrow_{m\to\infty} \frac{\log \mathrm{sp}(T(\theta))}{L \log \rho},$$

the result again follows.

If x is a boundary periodic point with only one symbolic representation, then $\Delta'_n(x)$ is empty for large n and the arguments are similar, but easier.

Now, assume x is periodic, but not a boundary point, say

$$[x] = (\gamma_0,\dots,\gamma_J,\theta^-,\theta^-,\dots),$$

where $\theta = (\theta_1,\dots,\theta_L)$. Let $n = J+1+mL+t$ for $1 \le t \le L$. Then

$$(\gamma_0,\dots,\gamma_J,\underbrace{\theta^-,\dots,\theta^-}_{m})$$

is a common ancestor to all of $\Delta_n(x)$, $\Delta_n^{\pm}(x)$, at most L levels back. Thus, if $\Delta_n'(x)$ denotes any of $\Delta_n(x)$ or $\Delta_n^{\pm}(x)$, an application of Lemma 2.2 implies that

$$\mu(\Delta_n'(x)) \le C_L \Big\| T(\gamma_0, \dots, \gamma_J, \underbrace{\theta^-, \dots, \theta^-}_{m}) \Big\| \le C_L' \| T(\theta)^m \|,$$

where C_L, C_L' are constants depending only on L. But, also,

$$\big\| (T(\theta))^{m+1} \big\| \le c_L \big\| (T(\theta))^m T(\theta_1, \dots, \theta_t) \big\| \le c_L' \mu(\Delta_n(x)) \le c_L'' \big\| T((\theta))^m \big\|,$$

and consequently,

$$\mu(\Delta_n^{\pm}(x)) \le C_L' \big\| (T(\theta))^m \big\| \le C_L'' \big\| (T(\theta))^{m+1} \big\| \le C_L''' \mu(\Delta_n(x)).$$

Therefore, $M_n(x)$ is comparable to $\mu(\Delta_n(x))$ and hence to $\|T((\theta))^m\|$ (with constants of comparability independent of n). The argument is completed as before. ■

3 Local Dimensions at Truly Essential Points

In this section we will obtain our main theoretical results on the structure of local dimensions, analogues of those found in [10, Section 5]. Because local dimensions may depend on adjacent net intervals, $\Delta_n^-(x)$ and $\Delta_n^+(x)$, rather than only on $\Delta_n(x)$, we introduce a subset of the essential class that we call the truly essential class. We will see that this subset has full μ and Hausdorff s-measure for $s = \dim_H K$. Our main results state that under a weak technical assumption the local dimensions at periodic points are dense in the set of (upper and lower) local dimensions at truly essential points and that the set of local dimensions at truly essential points is a closed interval. Furthermore, we prove that there is always a truly essential point at which the local dimension agrees with the Hausdorff dimension of the self-similar set, and we give criteria for when the measure μ is absolutely continuous with respect to the Hausdorff measure.

3.1 Truly Essential Points

Definition 3.1 Suppose K is the self-similar set associated with an IFS of finite type.

(i) We will say that $x \in K$ is a *boundary essential point* if x is a boundary point of $\Delta_n(x) \in \mathcal{F}_n$ for some n, and both $\Delta_n(x)$ and the other n-th level net interval containing x, $\Delta_n'(x)$, are essential (where if $\Delta_n'(x)$ is empty we understand it to be essential).

(ii) We will say that $x \in K$ is an *interior essential point* if x is not a boundary point and there exists an essential net interval with x in its interior.

(iii) We call x a *truly essential point* if it is either an interior essential point or a boundary essential point.

Obviously, truly essential points are essential and if x is in the interior of some essential net interval, then it is truly essential. In particular, any essential point that is not truly essential must be a boundary point. Hence, there can be only countably many of these and they are periodic.

Any point in the relative interior of the essential class (with respect to the space K) is either contained in the interior of some essential interval, or is a boundary essential point. Hence the relative interior of the essential class is equal to the set of truly essential points. If the essential class is a (relatively) open set, then the essential class coincides with the truly essential class. This is the situation, for example, with the Bernoulli convolutions and Cantor-like measures discussed in Sections 4 and 5. Another IFS where the set of essential points is equal to the set of truly essential points is given in Example 3.10.

However, as the example below demonstrates, these two sets need not be equal.

Example 3.2 Consider the maps $S_i(x) = x/4 + d_i/8$ with $d_i = i$ for $i = 0, \dots, 3$, $d_4 = 5$, and $d_5 = 6$. The reduced transition diagram has 4 reduced characteristic vectors (RCV). The reduced characteristic vectors are

- RCV 1: $(1, (0))$,
- RCV 2: $(1/2, (0))$,
- RCV 3: $(1/2, (0, 1/2))$,
- RCV 4: $(1/2, (1/2))$.

The transition maps are

- RCV 1 $\to [2a, 3a, 3b, 3c, 4a, 2b, 3d, 4b]$,
- RCV 2 $\to [2, 3a, 3b, 3c]$,
- RCV 3 $\to [3a, 3b, 3c, 3d]$,
- RCV 4 $\to [4a, 2, 3, 4b]$.

By this we mean, for example, that the reduced characteristic vector 1 has 8 children. Listed in order from left to right, they are the reduced characteristic vectors $2, 3, 3, 3, 4, 2, 3, 4$ etc. By $3a$ we mean the first occurrence of the child of type 3, $3b$ the second, etc. If there is only one child of that type, we do not need to distinguish them, but different children of the same type can have different transition matrices, so must be identified. It is easy to see from the transition maps that the essential class is $\{3a, 3b, 3c, 3d\}$. See Figure 1 for the transition diagram.

Now consider the boundary periodic point x having symbolic representation $(1, 4a, 4a, 4a, \dots)$, with $4a$ being the left-most child of 4. This also has symbolic representation $(1, 3c, 3d, 3d, \dots)$, $3d$ being the right-most child of 3. One of these symbolic representations is in the essential class, whereas the other is not. As such, this point is an essential point, but it is not a truly essential point.

The significance of an interior essential point x is that $\Delta_n^-(x), \Delta_n(x)$ and $\Delta_n^+(x)$ have a common essential ancestor for some n. Conversely, if $\Delta_n^-(x), \Delta_n(x)$ and $\Delta_n^+(x)$ have a common essential ancestor for some n, then x belongs to the relative interior of the essential class and thus is truly essential.

A periodic point x that is an interior essential point admits a period θ with the property that if $[x] = (\gamma, \theta^-, \theta^-, \dots)$ and θ_1 is the first letter of θ, then the net interval (with symbolic representation) (γ, θ) is in the interior of the net interval (γ, θ_1). We will call such a period θ *truly essential*. Equivalently, θ is truly essential if and only if θ is a path that does not consist solely of right-most descendents or solely of left-most descendents.

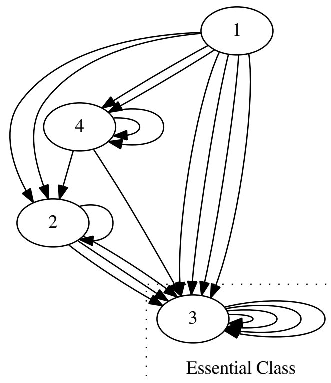

Figure 1: Transition diagram for Example 3.2

It was shown [10, Proposition 4.5] that under the assumption that the self-similar set was an interval, the essential class had full Lebesgue measure. In fact, this is true for the truly essential class, with Lebesgue measure replaced by either the self-similar measure μ or the (normalized) Hausdorff s-measure, where $s = \dim_H K$, as the next Proposition shows. To prove this, we first need some preliminary lemmas.

Lemma 3.3 *There exists an integer J such that for each net interval $\Delta \in \mathcal{F}_n$ there exists a $\sigma \in \mathcal{A}^{J+n}$ with $S_\sigma([0,1]) \subseteq \Delta$.*

Proof Consider a net interval $\Delta \in \mathcal{F}_n$. As there is some $x \in K$ in the interior of Δ, there is an index t and $\sigma \in \mathcal{A}^{n+t}$ such that $S_\sigma([0,1]) \subseteq \Delta$. Since $S_\sigma(0)$ is not isolated in K, there must be a level $n+T$ net interval $\Delta_0 \subseteq \Delta$, with left end $S_\sigma(0)$. Choose the index T minimal with this property.

Assume Δ has symbolic representation $(\gamma_0, \gamma_1, \ldots, \gamma_n)$ and Δ_0 has representation $(\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots, \gamma_{n+T})$. Let $\Delta' = (\gamma_0, \chi_1, \ldots, \chi_{m-1}, \gamma_n) \in \mathcal{F}_m$ be any other net interval with symbolic representation ending with the same characteristic vector γ_n. It will also have a descendent, Δ_0', with representation ending with the path $(\gamma_n, \ldots, \gamma_{n+T})$. Since 0 is in the neighbour set of Δ_0, the same is true for Δ_0' and thus its left endpoint is an image of 0 under S_τ for some $\tau \in \mathcal{A}^{m+T}$. As the pairs (Δ, Δ') and (Δ_0, Δ_0') have the same finite type structure (up to normalization), it follows that $S_\tau([0,1]) \subseteq \Delta'$. Hence, Δ' has the same minimal index T, in other words, T depends only upon the final characteristic vector associated with Δ.

As there are only finitely many characteristic vectors, we can take J to be the maximum of these indices T taken over all the characteristic vectors. ∎

Lemma 3.4 *There exists a positive constant c such that for all $\Delta \in \mathcal{F}_n$ and all n we have $c\rho^{sn} \le H^s(\Delta \cap K) \le \rho^{sn}$.*

Proof Fix an n-th level net interval $\Delta \in \mathcal{F}_n$. By construction, there exists some $\sigma \in \mathcal{A}^n$ such that $\Delta \subseteq S_\sigma([0,1])$. Then $\Delta \cap K \subseteq S_\sigma(K)$ and hence $H^s(\Delta \cap K) \leq \rho^{ns} H^s(K) = \rho^{ns}$.

Choose J as in the previous lemma. Then there exists some $\tau \in \mathcal{A}^{n+J}$ such that $S_\tau([0,1]) \subseteq \Delta$. Hence, $S_\tau(K) \subseteq \Delta \cap K$ and therefore $\rho^{(J+n)s} H^s(K) \leq H^s(\Delta \cap K)$. Taking $c = \rho^{Js} > 0$, we are done. ∎

Corollary 3.5 *We have* $\dim_H(\Delta \cap K) = s$ *for all net intervals* Δ.

Proof This is immediate, since $0 < H^s(\Delta \cap K) < \infty$. ∎

Proposition 3.6 *Suppose μ is a self-similar measure of finite type, with support K of Hausdorff dimension s. The set of points in K that are not truly essential is a subset of a closed set having zero μ and H^s-measure.*

Proof As we already observed, every net interval contains a descendent net subinterval that is essential. This essential net interval contains some $x \in K$ in its interior and hence contains a further subinterval that is in its interior. For the purposes of this proof, we will call this an interior essential net interval. The finite type property ensures we can always find an interior essential net subinterval within a bounded number of generations, say at most J.

We claim that there exists some $\lambda > 0$ such that the proportion of the measure of this net subinterval to the measure of the original interval is $\geq \lambda$. This is because all J'th level descendent net subintervals have comparable measure to the original net interval. For measure H^s, this property is shown in Lemma 3.4, and for the measure μ it follows from the definition.

We now exhibit a Cantor-like construction. We begin with $[0,1]$. Consider the first level at which there is a net interval that is interior essential. Remove the interiors of all the net intervals of this level that are interior essential. The resulting closed subset of $[0,1]$ is a finite union of closed intervals, say C_1, whose measures, either H^s or μ, total at most $1-\lambda$. We repeat the process of removing the interiors of the interior essential net intervals at the next level at which there are interior essential, net intervals in each of the intervals of C_1. The resulting closed subset now has measure at most $(1-\lambda)^2$.

After repeating this procedure k times, one can see that the non-interior essential points are contained in a finite union of closed intervals, denoted C_k, whose total measure is at most $(1-\lambda)^k$. It follows that the non-interior essential points are contained in the closed set $\bigcap_{k=1}^{\infty} C_k$, and this set has both μ and H^s-measure 0. ∎

Remark 3.7 Observe that we have actually proved that the complement of the interior of the essential class (in K) has μ and H^s-measure zero.

Another consequence of Lemma 3.4 is to obtain a new formula for the Hausdorff dimension of self-similar set of finite type. In [17] a formula was given which required knowing the complete transition graph. In fact, it suffices to know the transition graph of the essential characteristic vectors. For the purpose of this proof we introduce the following notation. Let $\gamma_1, \dots, \gamma_r$ be a complete list of the reduced characteristic

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vectors. Define a $r \times r$ matrix I by $(I)_{jk}$ = the number of children of γ_j that are of type γ_k. We call I the *incidence matrix of the essential class*.

Proposition 3.8 *Let K be a self-similar set of finite type and let I be the incidence matrix of the essential class. Then*

$$s = \dim_H K = \frac{\log(\mathrm{sp}(I))}{|\log \rho|}.$$

Example 3.9 Consider the example $S_j(x) = x/3 + b_j$ with $b_j \in \{0, 2/87, 2/3\}$. This IFS has 2280 reduced characteristic vectors, hence to compute the dimension using the full set of reduced characteristic vectors would require finding the eigenvalues of a 2280×2280 matrix. But there are only 2 essential vectors, and the incidence matrix of the essential class is equal to $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$.Using the proposition above one can easily deduce that the dimension of the self-similar set is 1, although the set is not the full interval $[0,1]$.

Proof of Proposition 3.8. Choose Δ_0 an essential net interval of level n_0 with the property that all essential characteristic vectors are descendents of Δ_0 at level $N + n_0$. It can be seen from the proof of [5, Lemma 6.4] that such a net interval exists. For $n \geq N$, let

$$E_n = \{\Delta \in \mathcal{F}_{n+n_0} : \Delta \subseteq \Delta_0\}.$$

From Lemma 3.4 we have (for $|E_n|$ denoting the cardinality of E_n),

$$c\rho^{sn}|E_n| \leq \sum_{\Delta \in E_n} H^s(\Delta \cap K) \leq C\rho^{sn}|E_n|$$

for positive constants c, C. Since the sets $\Delta \in E_n$ have disjoint interiors,

$$0 < H^s(\Delta_0 \cap K) = \sum_{\Delta \in E_n} H^s(\Delta \cap K) < \infty,$$

thus there are positive constants A, B such that

$$A \leq \rho^{sn}|E_n| \leq B \text{ for all } n.$$

Consequently,

$$\frac{\frac{1}{n}\log|E_n|}{|\log\rho|} + \frac{\log A}{n\log\rho} \geq s \geq \frac{\frac{1}{n}\log|E_n|}{|\log\rho|} + \frac{\log B}{n\log\rho}.$$

Without loss of generality we can assume Δ_0 has symbolic representation with last letter γ_1. Then $|E_n|$ is the sum of the entries of row 1 of I^n, so

$$|E_n| = \|[1, 0, \ldots, 0]I^n\| = \|[1, 0, \ldots, 0]I^N I^{n-N}\|.$$

But $[1, 0, \ldots, 0]I^N$ is a vector with all non-zero entries, since Δ_0 has all the essential characteristic vectors as descendents at level $N + n_0$. Hence, $|E_n| \sim \|I^{n-N}\|$ and since

$$\frac{\frac{1}{n}\log\|I^{n-N}\|}{|\log\rho|} \to \frac{\log(\mathrm{sp}(I))}{|\log\rho|},$$

we deduce that this is the value of s. ∎

3.2 Positive Row Property

Throughout the remainder of this section, we will assume, without loss of generality, that $\Delta \subsetneqq \widehat{\Delta}$ whenever the net interval Δ is a child of $\widehat{\Delta}$. To see that this assumption is without loss of generality, we note that as μ is of finite type, there will be an integer N such that all net intervals will have at least two descendents N levels deeper. Consider the new IFS with contractions $S_{i_1} \circ \cdots \circ S_{i_N}$ and probabilities $p_{i_1} \ldots p_{i_N}$. This IFS gives rise to the same self-similar measure μ. Moreover, the set of net intervals of level kN of the original construction are precisely the level k net intervals in the new construction. This new construction has the desired property.

In the theorems of this subsection we will also assume that the self-similar measure of finite type has the property that each essential primitive transition matrix has a non-zero entry in each row. This is the weak technical condition referred to in the introduction and we call it the *positive row property*. The property holds automatically when the self-similar set $K = [0,1]$ (see [10, Sec. 3.2]), such as for (even non-regular) Bernoulli convolutions and Cantor-like measures. This stronger assumption is not necessary, though, as we see in Example 7.1.

The positive row property can fail to hold when $K \neq [0,1]$ and can even fail when there is a positive essential transition matrix, as the example below demonstrates.

Example 3.10 Consider the self-similar measure associated with the IFS

$$\left\{ S_j(x) = x/3 + d_j : d_j = 0, 4/9, 5/9, 2/3 \right\}$$

and uniform probabilities. This measure is of finite type. Its support is a proper subset of $[0,1]$, since $(1/3, 4/9) \cap K$ is empty. However, if $I = [2/3, 1]$, then $I \subseteq \bigcup_{j=0}^{3} S_j(I)$, and this implies that $[2/3, 1] \subseteq K$. Thus, K has positive Lebesgue measure. The reduced characteristic vectors are

- RCV 1: $(1, (0))$,
- RCV 2: $(1/3, (0))$,
- RCV 3: $(1/3, (0, 1/3))$,
- RCV 4: $(1/3, (0, 1/3, 2/3))$,
- RCV 5: $(1/3, (1/3, 2/3))$,
- RCV 6: $(1/3, (2/3))$.

The transition maps are

- RCV 1 $\to [1, X, 2, 3, 4, 5, 6]$,
- RCV 2 $\to [1]$,
- RCV 3 $\to [2, 3, 4]$,
- RCV 4 $\to [4, 4, 4]$,
- RCV 5 $\to [4, 4, 4]$,
- RCV 6 $\to [4, 5, 6]$.

The 'X' denotes that between the child of type 1 and the child of type 2, in the parent 1, there is an interval $[h_j, h_{j+1}]$ that is not a net interval, as $(h_j, h_{j+1}) \cap K = \varnothing$.

The transition diagram is shown in Figure 2. There are three (non-reduced) essential characteristic vectors denoted $4a, 4b, 4c$ and one reduced characteristic vector 4. The primitive transition matrices for the essential class are given below. For $x = a, b, c$,

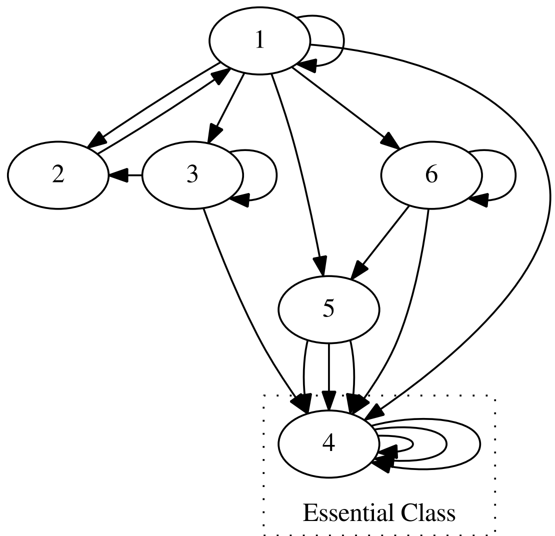

Figure 2: Transition diagram for Example 3.10

the matrix $T(4,4x)$ is any of $T(4a,4x)$, $T(4b,4x)$, or $T(4c,4x)$, as these three matrices coincide. Note that $T(4,4a)$ has a row of zeroes, while the essential transition matrix $(T(4c,4b)T(4b,4b)T(4b,4c))^2$ is positive. Hence, this example does not satisfy the positive row property, although there is a positive essential transition matrix

$$T(4,4a) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad T(4,4b) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \quad T(4,4c) = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 \end{bmatrix}$$

If x is an essential point, but not a truly essential point, then x cannot be in the interior of an essential interval. This means x will be on the boundary of both an essential net interval, $\Delta_n(x)$, and a non-essential net interval, $\Delta'_n(x)$, for all n sufficiently large. It is easy to see from the transition maps that there are no such points. Hence, the truly essential set coincides with the essential set.

3.3 Main Results

We begin by establishing the existence of special paths that we call truly essential.

Lemma 3.11 *Suppose μ is a self-similar measure of finite type satisfying the positive row property. Given any two essential characteristic vectors, γ_1, γ_2, there is a positive essential path $\eta = (\eta_1, \dots, \eta_k)$, that does not consist of solely left-most descendents, or solely right-most descendents, and having $\eta_1 = \gamma_1$ and $\eta_k = \gamma_2$.*

Proof In [10, Proposition 4.12] it was shown that there is an admissible essential path η_0 that begins and ends at γ_1 and is positive. Since any net interval contains an element of K in its interior, there must be an essential path, η', beginning with γ_1 and ending at, say χ, which does not consist of solely left-most or solely right-most descendents. Now take any essential path η'' from χ to γ_2. Put $\eta = \eta_0\eta'\eta''$. This is a positive path, since the product (in either order) of any positive matrix by a matrix with a non-zero entry in each row and column is again positive. ∎

We will call a path $\eta = (\eta_1, \dots, \eta_s)$, as described in the lemma above, a *truly essential positive path*. If $\Delta = (\gamma_0, \dots, \gamma_N, \eta_1, \dots, \eta_s)$, then the two adjacent intervals of Δ are both descendents of the essential interval $(\gamma_0, \dots, \gamma_N, \eta_1)$. Consequently any x whose symbolic representation begins $(\gamma_0, \dots, \gamma_N, \eta)$ is truly essential. Further, if x is a periodic point with period $\theta\eta\phi$ for some θ and ϕ, then x is an interior essential point.

Notation For the remainder of this section, F will denote a fixed, finite set of truly essential, positive paths with the property that given any two essential characteristic vectors, there is a path in F joining them in either order.

Theorem 3.12 *Suppose μ is a self-similar measure of finite type satisfying the positive row property. Then the set of local dimensions of μ at interior essential, positive, periodic points is dense in the set of all lower local dimensions of μ at truly essential points. A similar statement holds for the (upper) local dimensions.*

Proof We will first assume that $\dim_{\mathrm{loc}} \mu(x)$ exists. The arguments for upper and lower local dimensions are similar.

Step 1: To begin, we will show that if x is a boundary essential (necessarily periodic) point, then its local dimension can be approximated by that of a truly essential, positive, periodic point.

First, suppose x has two different symbolic representations, say,

$$(\gamma_0, \gamma_1, \dots, \gamma_J, \theta^-, \theta^-, \dots) \quad \text{and} \quad (\gamma_0, \gamma_1', \dots, \gamma_{J'}', \phi^-, \phi^-, \dots).$$

There is no loss of generality in assuming that the two periods have the same lengths $L = L(\theta^-)$ and pre-period path of length J. Without loss of generality assume that $\mathrm{sp}(T(\theta)) \geq \mathrm{sp}(T(\phi))$, so Proposition 2.7 gives

$$\dim_{\mathrm{loc}} \mu(x) = \frac{\log \mathrm{sp}(T(\theta))}{L(\theta^-) \log \rho}.$$

Let $\eta \in F$ be a truly essential, positive path chosen so that $\theta^-\eta\theta$ is well defined. Consider the periodic point

$$[y_n] = (\gamma_0, \dots, \gamma_J, \psi_n^-, \psi_n^-, \dots),$$

where

$$\psi_n = (\underbrace{\theta^-, \dots, \theta^-}_{n} \eta, \theta_1),$$

θ_1 being the first letter of θ. As noted above, this construction produces an interior essential point and therefore

$$\dim_{\text{loc}} \mu(y_n) = \frac{\log \operatorname{sp}(T(\psi_n))}{L(\psi_n^-) \log \rho}.$$

As η is a positive path, Lemma 2.2 implies that

$$\operatorname{sp}(T(\psi_n)) \leq \|T(\psi_n)\| \leq c_1 \|(T(\theta))^n\| \leq c_2 \operatorname{sp}(T(\theta^n)),$$
$$\operatorname{sp}(T(\psi_n)) \geq c_3 \|T(\psi_n)\| \geq c_4 \operatorname{sp}(T(\theta))^n,$$

where the constants are positive and independent of n. It follows that

$$\dim_{\text{loc}} \mu(y_n) = \frac{\log C_n \operatorname{sp}(T(\theta))^n}{(nL(\theta^-) + L(\eta)) \log \rho},$$

where the constants, C_n, are bounded above and bounded below from zero. Hence,

$$\begin{aligned}
\dim_{\text{loc}} \mu(y_n) &= \frac{\log C_n}{(nL(\theta^-) + L(\eta)) \log \rho} + \frac{\log \operatorname{sp}(T(\theta))}{(L(\theta^-) + \frac{1}{n} L(\eta)) \log \rho} \\
&= \frac{\log C_n}{(nL(\theta^-) + L(\eta)) \log \rho} + \dim_{\text{loc}} \mu(x) \frac{L(\theta^-)}{L(\theta^-) + \frac{1}{n} L(\eta)} \\
&\to \dim_{\text{loc}} \mu(x) \text{ as } n \to \infty.
\end{aligned}$$

The case where x has a unique representation is similar. This completes Step 1.

Step 2: Now, suppose x is an interior essential point with $\Delta_N(x)$ and its two adjacent N-th level intervals having common essential ancestor at level J. If the symbolic representation for x begins with the path $(\gamma_0, \ldots, \gamma_J)$, then for any $n > N$, all three of $\Delta_n(x), \Delta_n^+(x), \Delta_n^-(x)$ have symbolic representation also beginning with $(\gamma_0, \ldots, \gamma_J)$.

Without loss of generality assume that

$$\max\big\{\mu(\Delta_n(x)), \mu(\Delta_n^+(x)), \mu(\Delta_n^-(x))\big\} = \mu(\Delta_n^+(x))$$

along a subsequence not renamed. (The other cases are similar.) Of course, then we have

$$\mu(\Delta_n^+(x)) \leq M_n(x) \leq 3\mu(\Delta_n^+(x)).$$

Suppose

$$\Delta_n^+(x) = (\gamma_0, \ldots, \gamma_J, \chi_{J+1}^{(n)}, \ldots, \chi_n^{(n)})$$

and let η be a path in F joining $\chi_n^{(n)}$ to γ_J. We remark here that the $\chi_{J+1}^{(n)}, \ldots, \chi_n^{(n)}$ will depend on n, as Δ_n^+ may not be a descendent of Δ_{n-1}^+. Put

$$\theta_n = (\gamma_J, \chi_{J+1}^{(n)}, \ldots, \chi_{n-1}^{(n)}, \eta)$$

and denote by y_n the interior essential, positive, periodic point with symbolic representation

$$[y_n] = (\gamma_0, \ldots, \gamma_{J-1}, \theta_n^-, \theta_n^-, \ldots).$$

Of course,

$$\dim_{\text{loc}} \mu(y_n) = \frac{\log \operatorname{sp}(T(\theta_n))}{L(\theta_n) \log \rho}.$$

Lemma 2.2 implies that there is a constant $c_1 > 0$, independent of n, such that

$$c_1\|T(\theta_n)\| \le \mathrm{sp}(T(\theta_n)) \le \|T(\theta_n)\|.$$

As $L(\theta_n) = n - J + L(\eta)$, it follows by similar reasoning that

$$\dim_{\mathrm{loc}} \mu(y_n) = \frac{C_n}{n\log\rho} + \frac{\log\|T(\theta_n)\|}{n\log\rho} \longrightarrow \lim_n \frac{\log\|T(\theta_n)\|}{n\log\rho}. \tag{3.1}$$

Yet another application of Lemma 2.2 shows that

$$\begin{aligned} M_n(x) &\le 3\mu(\Delta_n^+(x)) \le c_1\|T(\gamma_0,\dots,\gamma_J,\chi_{J+1}^{(n)},\dots,\chi_n^{(n)})\| \\ &\le c_2\|T(\gamma_J,\chi_{J+1}^{(n)},\dots,\chi_n^{(n)},\eta)\| \le c_3\|T(\theta_n)\|, \end{aligned}$$

where the constants are independent of n, and similarly,

$$M_n(x) \ge \mu(\Delta_n^+(x)) \ge c\|T(\theta_n)\|.$$

Thus, $M_n(x) \sim \|T(\theta_n)\|$ and hence it follows from (3.1) that

$$\dim_{\mathrm{loc}} \mu(x) = \lim_n \frac{\log\|M_n(x)\|}{n\log\rho} = \lim_n \frac{\log\|T(\theta_n)\|}{n\log\rho} = \lim_n \dim_{\mathrm{loc}} \mu(y_n) \qquad \blacksquare$$

Theorem 3.13 *Suppose μ is a self-similar measure of finite type satisfying the positive row property. Assume that (x_n) are interior essential, positive, periodic points. There is an interior essential point x such that*

$$\overline{\dim}_{\mathrm{loc}}\mu(x) = \limsup_n \dim_{\mathrm{loc}} \mu(x_n),$$
$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_n \dim_{\mathrm{loc}} \mu(x_n).$$

Proof This is similar to the proof of [10, Theorem 5.5] with some technical complications that we highlight here. To begin, suppose x_n has truly essential, positive period θ_n where, without loss of generality,

$$\Big|\frac{\log \mathrm{sp}(T(\theta_{2n}))}{L(\theta_{2n}^-)} - \limsup_k \frac{\log \mathrm{sp}(T(\theta_k))}{L(\theta_k^-)}\Big| < \frac{1}{n},$$
$$\Big|\frac{\log \mathrm{sp}(T(\theta_{2n+1}))}{L(\theta_{2n+1}^-)} - \liminf_k \frac{\log \mathrm{sp}(T(\theta_k))}{L(\theta_k^-)}\Big| < \frac{1}{n},$$

all even labelled paths θ_{2n}^- have the same first letter and the same last letter, and similarly for the odd labelled paths.

Choose truly essential, positive paths η^e and η^o from the finite set F so that η^o joins the last letter of an odd path to the first letter of even path and η^e does the opposite.

Let $L_n = 2L(\theta_{n+1}^-)+L(\eta^o)+L(\eta^e)$. Choose C_n such that for all $\ell \le L_n$ and $j > \ell$ we have $\mu(\Delta_j) \ge C_n\mu(\Delta_{j-\ell})$ when $\Delta_{j-\ell} \in \mathcal{F}_{j-\ell}$ is the ancestor of $\Delta_j \in \mathcal{F}_j$. Now choose $k_n \ge 2$ sufficiently large so that in addition to the requirements of k_n in the proof of [10, Thm. 5.5], we also have

$$\frac{3+\log C_n}{k_n} \longrightarrow 0.$$

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Suppose $x \in K$ has symbolic representation

$$[x] = \big(\gamma_0, \dots, \gamma_J, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \eta^o, \underbrace{\theta_2^-, \dots, \theta_2^-}_{k_2}, \eta^e, \dots\big).$$

We remark that as η^e and η^o are truly essential paths, the point x is interior essential.

Suppose

$$[x \mid j] = \big(\gamma_0, \dots, \gamma_J, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \eta^o, \dots, \underbrace{\theta_n^-, \dots, \theta_n^-}_{k_n}, \eta', (\theta_{n+1}^-, \theta_{n+1}^-)\big),$$

where η' is either η^o or η^e, as appropriate, and the notation $(\theta_{n+1}^-, \theta_{n+1}^-)$ means any subpath of the path $\theta_{n+1}^-, \theta_{n+1}^-$. As η' is truly essential, $\Delta_j(x)$ and its two adjacent intervals have common ancestor

$$[x \mid j-\ell] = \big(\gamma_0, \dots, \gamma_J, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \eta^o, \dots, \underbrace{\theta_n^-, \dots, \theta_n^-}_{k_n}\big)$$

for some $\ell \le 2L(\theta_{n+1}^-) + L(\eta')$. If, instead,

$$[x \mid j] = \big(\gamma_0, \dots, \gamma_J, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \eta^o, \dots, \underbrace{\theta_n^-, \dots, \theta_n^-}_{k_n}, \eta', \underbrace{\theta_{n+1}^-, \dots, \theta_{n+1}^-}_{p_n}, (\theta_{n+1}^-, \eta'')\big)$$

with $2 \le p_n \le k_{n+1}$ (where we may include a subset of $\eta'' = \eta^e$ or η^o if $p_n = k_{n+1} - 1$), then $\Delta_j(x)$ and its two adjacent intervals have common ancestor

$$[x \mid j-\ell] = \big(\gamma_0, \dots, \gamma_J, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \eta^o, \dots, \eta', \underbrace{\theta_{n+1}^-, \dots, \theta_{n+1}^-}_{p_n - 1}\big),$$

where $\ell \le 2L(\theta_{n+1}^-) + L(\eta'')$.

In either case, for all such j, there is some $\ell \le L_n$ such that $\Delta_{j-\ell}(x)$ is a common ancestor of $\Delta_j(x)$ and its two adjacent intervals. Since

$$\begin{aligned} C_n \mu\big(\Delta_{j-\ell}(x)\big) &\le \mu\big(\Delta_j(x)\big) \le M_j(x) \\ &\le \mu\big(\Delta_j^+(x) \cup \Delta_j(x) \cup \Delta_j^-(x)\big) \\ &\le 3\mu\big(\Delta_{j-\ell}(x)\big) \text{ for all } l \le L_n, \end{aligned}$$

it will be sufficient to study the behaviour of the subsequences

$$\Big\| T(\gamma_0, \dots, \gamma_J, \underbrace{\theta_1^-, \dots, \theta_1^-}_{k_1}, \eta^o, \dots, \eta', \underbrace{\theta_{n+1}^-, \dots, \theta_{n+1}^-}_{p_n}) \Big\|$$

for $p_n \le k_{n+1}$, and this we do in the same manner as in [10]. ∎

It was shown in [10, Thm. 5.7] that the set of local dimensions at essential points was a closed interval. Here we prove the same conclusion for the set of local dimensions at truly essential points.

Theorem 3.14 *Suppose μ is a self-similar measure of finite type satisfying the positive row property. Let y, z be interior essential, positive, periodic points. Then the set of local dimensions of μ at truly essential points contains the closed interval with endpoints* $\dim_{\mathrm{loc}} \mu(y)$ *and* $\dim_{\mathrm{loc}} \mu(z)$.

Proof Let y and z have truly essential, positive periods ϕ and θ, respectively, with $T(\phi) = A$ and $T(\theta) = B$. Let η_1, η_2 be truly essential, positive paths joining the last letter of ϕ to the first letter of θ and vice versa. Given $0 < t < 1$, choose subsequences $m_k, n_k \to \infty$ such that

$$\frac{L(\theta^-)m_k}{L(\theta^-)m_k + L(\phi^-)n_k} \longrightarrow t.$$

Put

$$T(\psi_k) = B^{m_k} T(\eta_1) A^{n_k} T(\eta_2)$$

and consider a truly essential, positive, periodic point x_k with period ψ_k. Using Lemma 2.2 we deduce that

$$\mathrm{sp}\big(B^{m_k} T(\eta_1) A^{n_k} T(\eta_2)\big) \sim \mathrm{sp}(B)^{m_k}\,\mathrm{sp}(A)^{n_k}.$$

Coupled with Proposition 2.7, this implies

$$\begin{aligned}\lim_k \dim_{\mathrm{loc}} \mu(x_k) &= \lim_k \frac{\log \mathrm{sp}(B)^{m_k} + \log \mathrm{sp}(A)^{n_k}}{(L(\theta^-)m_k + L(\phi^-)n_k)\log\rho}\\ &= t\dim_{\mathrm{loc}}\mu(z) + (1-t)\dim_{\mathrm{loc}}\mu(y).\end{aligned}$$

Now appeal to the previous theorem to complete the proof. ∎

The three theorems combine to yield the following important corollary.

Corollary 3.15 *Let μ be a self-similar measure of finite type satisfying the positive row property. Let $I = \inf\{\dim_{\mathrm{loc}}\mu(x) : x$ interior essential, positive, periodic$\}$ and $S = \sup\{\dim_{\mathrm{loc}}\mu(x) : x$ interior essential, positive, periodic$\}$. Then*

$$\{\dim_{\mathrm{loc}}\mu(x) : x \text{ interior essential}\} = \{\dim_{\mathrm{loc}}\mu(x) : x \text{ truly essential}\} = [I,S].$$

A similar statement holds for the lower and upper local dimensions.

It is worth commenting here that this need not be the case for the set of upper local dimensions of a maximal loop class (outside of the truly essential class). An example is given in Section 6.

Remark 3.16 In Example 3.2, the local dimension of the boundary point x with symbolic representations $(1,4a,4a,4a,\dots)$ and $(1,3c,3d,3d,\dots)$ is

$$\frac{|\log(\max \mathrm{sp}(T(4a,4a)),\mathrm{sp}(T(3d,3d)))|}{\log 4}.$$

Regardless of the choice of probabilities, this local dimension is always contained within the interval that is the set of local dimensions of truly essential points. It would be interesting to know if there were any examples of self-similar measures of finite type and essential points x where $\dim_{\mathrm{loc}}\mu(x)$ is not contained in the set of local dimensions of the truly essential points.

3.4 Local Dimension and the Dimension of the Support

In this section we show that, assuming the positive row property, the essential class must contain a point x such that $\dim_{\mathrm{loc}} \mu(x) = \dim_H(K)$.

Lemma 3.17 *Let μ be a self-similar measure of finite type, with $s = \dim_H K$. Let E denote the set of truly essential points and put*

$$G_1 = \{x \in E \mid \overline{\dim}_{\mathrm{loc}}\mu(x) > s\},$$
$$G_2 = \{x \in E \mid \underline{\dim}_{\mathrm{loc}}\mu(x) < s\}.$$

Then $\mu(G_1) = 0 = H^s(G_2)$.

Proof We recall that there are only countably many boundary essential points, and every non-atomic measure assigns mass zero to a countable set. Hence the statement will be true if and only if it is true for E= the set of interior essential points.

Let $x \in G_1$, say with $\overline{\dim}_{\mathrm{loc}}\mu(x) = s(1+\varepsilon)$ for some $\varepsilon > 0$. Then there will exist infinitely many n such that

$$\frac{\log \mu(\Delta_n(x))}{n \log \rho} > s\Big(1 + \frac{\varepsilon}{2}\Big).$$

By Lemma 3.4, we have

$$\lim_n \frac{\log H^s(\Delta_n(x))}{n \log \rho} = s.$$

This implies that there are infinitely many n such that

$$\frac{\log \mu(\Delta_n(x))}{\log \rho} > \Big(1 + \frac{\varepsilon}{3}\Big) \frac{\log H^s(\Delta_n(x))}{\log \rho},$$

and therefore

$$\mu\big(\Delta_n(x)\big) \le H^s\big(\Delta_n(x)\big) H^s\big(\Delta_n(x)\big)^{\varepsilon/3}.$$

Since $H^s(\Delta_n) \to 0$ as $n \to \infty$, it follows that for all $0 < a < 1$ there exists an n such that

$$\mu\big(\Delta_n(x)\big) \le aH^s\big(\Delta_n(x)\big).$$

In a similar way, if $x \in G_2$ and $b > 1$, then there exists an n such that

$$\mu\big(\Delta_n(x)\big) \ge bH^s\big(\Delta_n(x)\big).$$

Define

$$E^a_{1,n} = \bigcup\big\{\Delta \in \mathcal{F}_n : \mu(\Delta) \le aH^s(\Delta)\big\}, \qquad E^a_1 = \bigcup_n E^a_{1,n},$$
$$E^b_{2,n} = \bigcup\big\{\Delta \in \mathcal{F}_n : \mu(\Delta) \ge bH^s(\Delta)\big\}, \qquad E^b_2 = \bigcup_n E^b_{2,n}.$$

The comments above show that $G_1 \subseteq E^a_1$ for all $0 < a < 1$ and $G_2 \subseteq E^b_2$ for all $b > 1$. Put

$$F^a_{1,1} = E^a_{1,1}, F^a_{1,n} = E^a_{1,n} \setminus \bigcup_{k=1}^{n-1} F^a_{1,k},$$

and similarly define $F^b_{2,n}$. Then E^a_1 is the disjoint union of the sets $F^a_{1,n}$ and similarly for E^b_2. Further, we observe that each set $F^a_{1,n}$ is a union of intervals, Δ, with disjoint interiors and the property that $\mu(\Delta) \le aH^s(\Delta)$. Hence σ-additivity and the continuity of H^s imply

$$\mu(G_1) \le \mu(E_1^a) \le \sum_n \mu(F_{1,n}^a) \le a \sum_n H^s(F_{1,n}^a)$$
$$= aH^s(E_1^a) \le aH^s(K) \le a.$$

As $0 < a < 1$ is arbitrary, we have that $\mu(G_1) = 0$.

Similarly ,

$$H^s(G_2) \le H^s(E_2^b) \le \frac{1}{b}\mu(E_2^b) \le \frac{1}{b}\mu(K) \le \frac{1}{b},$$

and as $b > 1$ is arbitrary, we have that $H^s(G_2) = 0$. ■

Theorem 3.18 *Let μ be a self-similar measure of finite type satisfying the positive row property. Then there exists a truly essential element x with $\dim_{\mathrm{loc}} \mu(x) = \dim_H K$.*

Proof In fact, we will show a stronger result, that there exists an interior essential point x such that $\dim_{\mathrm{loc}} \mu(x) = \dim_H K$.

Let E be the set of interior essential points. According to Corollary 3.15, the set of local dimensions at the interior essential points is an interval. So it suffices to show that the supremum of this interval is at least $\dim_H K$, and the infimum is at most $\dim_H K$.

Assume, for a contradiction, that the infimum is strictly greater than $\dim_H K$. This implies for all $x \in E$,

$$\dim_H K < \underline{\dim}_{\mathrm{loc}}\mu(x) \le \overline{\dim}_{\mathrm{loc}}\mu(x),$$

and hence $E \subseteq G_1$. This fact, combined with Proposition 3.6 and Lemma 3.17, gives $1 = \mu(E) \le \mu(G_1) = 0$, a contradiction.

Similarly, if the supremum of the local dimensions of E was strictly less than $\dim_H K$, then $E \subseteq G_2$, and hence $1 = H^s(E) \le H^s(G_2) = 0$, a contradiction. ■

It would be interesting to know if the set of such points has full μ measure. Notice that Lemma 3.17 implies that this is true if μ is absolutely continuous with respect to H^s. Our next result gives conditions under which this latter statement is true.

Proposition 3.19 *Suppose μ is a self-similar measure of finite type, with $\dim_H K = s$. Assume that the norm of any product of n essential, primitive transition matrices is bounded above by $C\rho^{sn(1-\varepsilon_n)}$, where $\sup_n n\varepsilon_n < \infty$ and $C > 0$ is a constant. Then μ is absolutely continuous with respect to H^s.*

Proof By [16, p. 35], $\mu \ll H^s$ if and only if $D(x) < \infty$ for μ almost all x, where

$$D(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{H^s(B(x,r))}.$$

Appealing to Proposition 3.6, we see that it suffices to prove $D(x) < \infty$ for all interior essential points x. Standard arguments show it will be sufficient to prove

$$\liminf_{n\to\infty} \frac{\mu(\Delta_n(x) \cup \Delta_n^+(x) \cup \Delta_n^-(x))}{H^s(\Delta_n(x) \cup \Delta_n^+(x) \cup \Delta_n^-(x))} < \infty.$$

For x an interior essential point, choose J such that for all n sufficiently large, $\Delta_{J+n}(x), \Delta^+_{J+n}(x)$ and $\Delta^-_{J+n}(x)$ have a common essential ancestor at level J. Thus, for $\Delta'_{n+J}(x)$ denoting any of $\Delta_{n+J}(x)$ or its two adjacent net intervals

$$\mu(\Delta'_{n+J}(x)) \sim \big\| T(\gamma_0, \dots, \gamma_J, \gamma'_{J+1}, \dots, \gamma'_{n+J}) \big\| \\ \sim \big\| T(\gamma_J, \gamma'_{J+1} \dots, \gamma'_{n+J}) \big\| \le C\rho^{sn(1-\varepsilon_n)}$$

for a constant C not dependent on n. Here the last inequality comes from the hypothesis of the proposition. Since Lemma 3.4 implies $H^s(\Delta_n) \sim \rho^{ns}$ for any n-th level net interval,

$$\liminf_{n\to\infty} \frac{\mu(\Delta_n(x) \cup \Delta^+_n(x) \cup \Delta^-_n(x))}{H^s(\Delta_n(x) \cup \Delta^+_n(x) \cup \Delta^-_n(x))} \le \liminf_{n\to\infty} C \frac{\rho^{sn(1-\varepsilon_n)}}{\rho^{ns}} < \infty,$$

as $\sup n\varepsilon_n < \infty$. ∎

Remark 3.20 We note that this proposition did not require the assumption of the positive row property. Moreover, similar arguments show that $H^s|_{\operatorname{supp}\mu}$ is absolutely continuous with respect to μ if the norm of any product of n essential, primitive transition matrices is bounded below by $C\rho^{sn(1+\varepsilon_n)}$, where $\sup_n n\varepsilon_n < \infty$ and $C > 0$ is a constant.

In the next example, the self-similar measure is mutually absolutely continuous to Lebesgue measure restricted to $\operatorname{supp}\mu$, and the local dimension is identical at all the truly essential points.

Example 3.21 Consider the example $S_j(x) = x/4 + b_j/12$ where $b_j \in \{0,1,2,7,8,9\}$, and associate with these the probabilities $p_0 = p_1 = p_4 = p_5 = 1/8$, $p_2 = p_3 = 1/4$. This measure does not have full interval support, although the support is still of dimension one. To see this, we observe that $K = [0, 5/12] \cup [7/12, 1]$. There is one reduced characteristic vector within the essential class. The four transition matrices from this vector to itself are:

$$\begin{bmatrix} \frac18 & 0 & 0 \\ 0 & 0 & \frac14 \\ \frac18 & \frac14 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac18 & \frac18 & 0 \\ 0 & 0 & 0 \\ \frac18 & \frac18 & \frac14 \end{bmatrix}, \quad \begin{bmatrix} \frac14 & \frac18 & \frac18 \\ 0 & 0 & 0 \\ 0 & \frac18 & \frac18 \end{bmatrix}, \quad \begin{bmatrix} 0 & \frac14 & \frac18 \\ \frac14 & 0 & 0 \\ 0 & 0 & \frac18 \end{bmatrix}.$$

We notice that all column sums of all of these matrices are exactly the same at $1/4$. Hence the norm of any n-fold product of these matrices is comparable to 4^{-n}. This gives that the local dimension at all truly essential points is 1, and the measure μ is mutually absolutely continuous with respect to Lebesgue measure on its support. It is worth observing that this is true, despite this example not satisfying the positive row property. Note that the points outside the essential class do not necessarily have local dimension 1. For instance, $\dim_{\mathrm{loc}} \mu(0) = \log 8/\log 4 = 3/2$.

Another illustration of this is seen in Example 5.11, where this phenomena occurs for a Cantor-like measure when H^s is the normalized Lebesgue measure.

4 Biased Bernoulli Convolutions with Simple Pisot Contractions

In this section we will assume that μ is a Bernoulli convolution generated by the IFS

$$\{S_0(x) = \rho x, S_1(x) = \rho x + (1-\rho)\}$$

and probabilities $p, 1-p$, where ρ is the inverse of a simple Pisot number (one whose minimal polynomial is of the form $x^k - x^{k-1} - \cdots - x - 1$) and $0 < p < 1$. The self-similar set is $[0,1]$, hence the positive row property holds for all these Bernoulli convolutions.

Feng in [4] showed that if $p = 1/2$, then μ has no isolated point in its multifractal spectrum. In contrast, we will show here that if $p \neq 1/2$, there is always an isolated point, either $\dim_{\mathrm{loc}} \mu(0)$ or $\dim_{\mathrm{loc}} \mu(1)$, depending on whether p is less than or greater than $1/2$.

In [4, Sect. 5], Feng determined the characteristic vectors, transition graph, and primitive transition matrices for the case $p = 1/2$. Using this information, it is not difficult to determine the primitive transition matrices for the general case. In what follows, we use Feng's notation to label the characteristic vectors as a, b, d, c_i, $\overline{c_1}$, e_j, f_j, g, where $i = 1, \ldots, k$ and $j = 1, \ldots, k-1$. Here, all but a, b, d are in the essential class.

Lemma 4.1 *The primitive transition matrices for the vectors in the essential class are given by:*

$$T(c_{j-1}, c_j) = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \textit{ for } 2 \le j \le k, \qquad T(c_k, g) = \begin{bmatrix} p \\ 1-p \end{bmatrix},$$
$$T(c_k, c_1) = \begin{bmatrix} p & 0 \\ 1-p & p \end{bmatrix}, \qquad T(c_k, \overline{c_1}) = \begin{bmatrix} 1-p & p \\ 0 & 1-p \end{bmatrix},$$
$$T(g, f_1) = T(f_j, f_{j+1}) = [p], \qquad T(f_j, c_1) = [\,1-p \;\; p\,] \textit{ for } j \le k-2,$$
$$T(g, e_1) = T(f_j, e_1) = [1-p], \qquad T(e_j, f_1) = [p] \textit{ for } j \le k-1,$$
$$T(g, c_1) = T(e_j, c_1) = [\,1-p \;\; p\,], \qquad T(e_j, e_{j+1}) = [1-p] \textit{ for } j \le k-2.$$

Proof We leave this as an exercise for the reader, as it follows in a straightforward manner from the information gathered in [4]. The main points to observe are that if the i-th neighbour of a parent coincides with the j-th neighbour of a child, then $T_{ij} = p$, while if they differ by (common) normalized distance $1-\rho$, then $T_{ij} = 1-p$. We also remind the reader that for simple Pisot numbers, ρ^{-1}, with minimal polynomial of degree k, $1-\rho = \rho - \rho^{k+1}$.

We illustrate this with $T(c_{j-1}, c_j)$. From [4] it can be seen that the (normalized) neighbours of c_j are 0 and $1 - \rho^{k-j+1}$, and c_j is the only child of the parent c_{j-1}. If we renormalize so they can be compared, we see that the two 0 neighbours coincide and the non-0 neighbours differ by $1-\rho$. Thus, T is diagonal with the entries being p and $1-p$, respectively. ∎

Notation Given a matrix T, denote by $\|T\|_{\min}$ the pseudo-norm

$$\|T\|_{\min} = \min_j \sum_i |T_{ij}|,$$

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where the sum is over all the rows of the matrix. That is, $\|T\|_{\min}$ is the minimal column sum of T. Obviously, $\|T\| \geq \|T\|_{\min}$. A useful property is that $\|T_1T_2\|_{\min} \geq \|T_1\|_{\min}\|T_2\|_{\min}$.

Lemma 4.2 *There exists an integer N such that if $x \in (0,1)$, then*

$$[x] = (\gamma_1, \dots, \gamma_M, \eta_1, \eta_2, \dots),$$

where $\gamma_1, \dots, \gamma_M$ are characteristic vectors, η_j are essential paths of length at most N whose first letter, denoted $\eta_{j,1}$, equals c_1, $\overline{c_1}$ or f_1, and

$$\left\| T(\eta_j, \eta_{j+1}, \eta_{j+2,1}) \right\|_{\min} \geq \min\left(p^{L-1}(1-p), (1-p)^{L-1}p \right),$$

where $L = L(\eta_j, \eta_{j+1})$.

Proof One can see from the transition maps given in [4, Sec 5.1] that the symbolic representation for any $x \in (0,1)$ begins either as $[x] = (a, c_1, \dots)$ or $[x] = (a, *, *, \dots, *, y, \dots)$, where $*$ denotes (all) $b's$ or $d's$ and y is either c_1, e_1 or f_1. In the case when $y = e_1$ the path must continue as $(e_1, \dots, e_{j-1}, z)$ where $z = c_1$ or f_1 and $j \leq k-2$, or as $(e_1, \dots, e_{k-1}, f_1)$. Whichever is the case, one can see that each essential x must eventually admit either a (first) c_1 or f_1. This will be the first letter of η_1. Now define η_j to begin with the j-th occurrence of either c_1 (or $\overline{c_1}$ in Feng's notation) or f_1. We need to check that with this construction the η_j are paths of bounded length (independent of x) and have the required property on the pseudo-norm of the transition matrices.

First, suppose a path η_j begins with c_1. Then it must continue as $(c_1, \dots, c_k)$. If c_k is followed by c_1 (or $\overline{c_1}$), then we stop and take $(c_1, \dots, c_k)$ as η_j having length k. Otherwise, c_k is followed by g, and if that is followed by c_1 or f_1, then $\eta_j = (c_1, \dots, c_k, g)$ has length $k+1$. The only other possibility is that g is followed by e_1, but in that case, as we saw above, the path will continue as $(e_2, \dots, , e_j)$ with $j \leq k-1$, before continuing with either c_1 or f_1 (necessarily with f_1 if $j = k-1$). Such a path η_j has length at most $k+1+k-1 = 2k$.

To summarize, the paths η_j that begin with c_1, together with the first letter of η_{j+1}, are of the form $(c_1, \dots, c_k)$ with next letter either c_1 or $\overline{c_1}$, $(c_1, \dots, c_k, g)$ with next letter either c_1 or f_1, or $(c_1, \dots, c_k, g, e_1, \dots, e_j)$ with $j \leq k-1$ and next letter either c_1 or f_1 (necessarily f_1 if $j = k-1$).

The arguments are similar for the paths that begin with f_1, with these paths having length at most $2k-2$.

Now we verify the claimed pseudo-norm property. For this we apply the previous lemma to analyze the product of the appropriate transition matrices. Of course, any primitive transition matrix has pseudo-norm at least $\min(p, 1-p)$.

For paths η_J that begin with c_1, we will see that even

$$\left\| T(\eta_J, \eta_{J+1,1}) \right\|_{\min} \geq \min\left(p^{L-1}(1-p), (1-p)^{L-1}p \right) \text{ for } L = L(\eta_J),$$

and this will certainly imply the claim. To prove this, we consider the different paths individually.

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Case 1: $(\eta_J, \eta_{J+1,1}) = (c_1, \dots, c_k, y)$ with $y_1 = \eta_{J+1,1} = c_1$ or $\overline{c_1}$: If $y = c_1$, then

$$T(\eta_J, \eta_{J+1,1}) = T(c_1, \dots, c_k) T(c_k, c_1) = \begin{bmatrix} p^k & 0 \\ (1-p)^k & (1-p)^{k-1} p \end{bmatrix}$$

and hence has pseudo-norm with the required lower bound. The argument when the first letter of $\eta_{J+1} = \overline{c_1}$ is similar.

Case 2: $(\eta_J, \eta_{J+1,1}) = (c_1, \dots, c_k, g, y)$ with $y = c_1$ or f_1: If $y = c_1$, then an easy calculation shows

$$T(\eta_J, \eta_{J+1,1}) = T(c_1, \dots, c_k) T(c_k, g) T(g, c_1) = \begin{bmatrix} p^k(1-p) & p^{k+1} \\ (1-p)^{k+1} & p(1-p)^k \end{bmatrix}.$$

If $y = f_1$, then

$$T(\eta_J, \eta_{J+1,1}) = \begin{bmatrix} p^{k+1} \\ (1-p)^k p \end{bmatrix}.$$

The cases $(\eta_J, \eta_{J+1,1}) = (c_1, \dots, c_k, g, e_1, \dots, e_j, y)$ for $y = c_1$ or f_1, or

$$(\eta_J, \eta_{J+1,1}) = (f_1, \dots, f_j, e_1, \dots, e_i, f_1)$$

are similar.

The only cases in which we must consider two consecutive paths, $\eta_J \eta_{J+1}$, are when $\eta_J = (f_1, \dots, f_j)$ and either the next letter is c_1 or the path continues as $(f_1, \dots, f_j, e_1, \dots, e_i)$ with $i, j \geq 1$ and the next letter is c_1. But in that case, the next path, η_{J+1}, is one of the paths beginning with c_1 discussed above, and we already know that then

$$\left\| T(\eta_{J+1}, \eta_{J+2,1}) \right\|_{\min} \geq \min(p^{L-1}(1-p), (1-p)^{L-1} p) \text{ for } L = L(\eta_{J+1}).$$

Combining this bound with the fact that $\|T(\eta_J, \eta_{J+1,1})\|_{\min} \geq \min(p^L, (1-p)^L)$ for $L = L(\eta_J)$ completes the proof. ∎

Theorem 4.3 *Suppose μ is a Bernoulli convolution with contraction factor ρ the inverse of a simple Pisot number and with probabilities $p \neq 1-p$. Then there is an isolated point in the set of local dimensions of μ at either* 0 *or* 1*, depending on which of p or $1-p$ is smaller.*

Proof Without loss of generality assume $p < 1/2$. Standard arguments show that $\dim_{\text{loc}} \mu(0) = \log p / \log \rho$.

Consider any $x \in (0,1)$. As the set of local dimensions of boundary essential points is contained in the set of local dimensions of interior essential points, we can assume without loss of generality that x is an interior essential point. Write $[x] = (\gamma_1, \dots, \gamma_M, \eta_1, \eta_2, \dots)$ with the notation as in the previous lemma. We have the formula

$$\dim_{\text{loc}} \mu(x) = \lim_J \frac{\log \left\| T(\eta_1, \eta_2, \dots, \eta_{2J}, \eta_{2J+1,1}) \right\|}{\sum_{i=1}^{2J} L(\eta_i) \log \rho}$$

should the local dimension of μ at x exist.

Set $L_i = L(\eta_{2i-1}, \eta_{2i})$. Then

$$\|T(\eta_1, \eta_2, \dots, \eta_{2J}, \eta_{2J+1,1})\| \geq \prod_{i=1}^{J} \|T(\eta_{2i-1}, \eta_{2i}, \eta_{2i+1,1})\|_{\min}$$
$$\geq p^{\sum_i (L_i - 1)}(1-p)^J.$$

Hence,

$$\log \|T(\eta_1, \eta_2, \dots, \eta_{2J}, \eta_{2J+1,1})\| = \Big(\sum_i L_i - J\Big)\log p + J\log(1-p),$$

so that

$$\frac{\log \|T(\eta_1, \eta_2, \dots, \eta_{2J}, \eta_{2J+1,1})\|}{\sum_{i=1}^{J} L_i} \geq \log p + \frac{J(\log(1-p) - \log p)}{\sum_{i=1}^{J} L_i}.$$

But $L_i \leq 2N$ (where N is as in the lemma), hence for any J,

$$\frac{\log \|T(\eta_1, \eta_2, \dots, \eta_{2J}, \eta_{2J+1,1})\|}{\sum_{i=1}^{2J} L(\eta_i)\log \rho} \leq \frac{\log p}{\log \rho} + \frac{\log(1-p) - \log p}{2N \log \rho} < \frac{\log p}{\log \rho} = \dim_{\mathrm{loc}} \mu(0),$$

and therefore $\dim_{\mathrm{loc}} \mu(x)$ is bounded away from $\dim_{\mathrm{loc}} \mu(0)$. ∎

5 Cantor-like Measures of Finite Type

The focus of this section will be the Cantor-like self-similar sets and measures generated by the IFS

$$\Big\{S_j(x) = \frac{1}{d}x + \frac{j}{md}(d-1) : j = 0, \dots, m\Big\} \tag{5.1}$$

for integers $d \geq 2$ and probabilities $p_j > 0$, $j = 0, \dots, m$. The self-similar set is the m-fold sum of the Cantor set with contraction factor $1/d$, rescaled to $[0,1]$, and is the full interval when $m \geq d-1$. We will assume this to be the case, for otherwise the IFS satisfies the open set condition and is well understood. This class of measures includes, for example, the m-fold convolution of the uniform Cantor measure associated with the Cantor set generated by $S_0(x) = \frac{1}{d}x$, $S_1(x) = \frac{1}{d}x + \frac{d-1}{d}$. As $K = [0,1]$ when $m \geq d-1$, we see that all of these examples satisfy the positive row property.

These measures were studied by different methods in [2, 20] where it was shown, for example, that $\{\dim_{\mathrm{loc}} \mu(x) : x \in (0,1)\}$ was a closed interval. In [10, Sect. 7] it was shown that the essential class for any of these Cantor-like measures is $(0,1)$, hence all $x \in (0,1)$ are truly essential. Consequently, the fact that $\{\dim_{\mathrm{loc}} \mu(x) : x \in (0,1)\}$ is a closed interval can also be deduced from Corollary 3.15.

In this section we will establish more refined information about the local dimensions of these measures. In particular, we give a new proof of the fact that $\dim_{\mathrm{loc}} \mu(0)$ (or $\dim_{\mathrm{loc}} \mu(1)$) is an isolated point if p_0 (resp., p_m) is the minimal probability, as was shown by other methods in [2, 20]. We give an example to show that there need not be an isolated point if this is not the case, as well as examples of Cantor-like measures whose set of local dimensions consists of (precisely) two points.

For this detailed analysis it is helpful to completely determine the finite type structure of these measures. There are two cases to consider, $m \equiv 0 \bmod (d-1)$ and $m \not\equiv 0 \bmod (d-1)$.

Proposition 5.1 *Assume μ is the self-similar Cantor-like measure of finite type generated by the IFS (5.1), with $m = k(d-1)$ for integer k.*

(i) *The essential class has one reduced characteristic vector, E, with normalized length $1/k$ and neighbour set $(j/k : j = 0,\dots,k-1)$. The reduced characteristic vector E has d children, identical to itself, labelled as $E^{(i)}$, $i = 1,\dots,d$.*

(ii) *There are $m-k+2$ net intervals at level one with reduced characteristic vector E. These are the intervals $[\frac{k-1}{kd},\frac{k}{kd}],\dots,[1-\frac{k}{kd},1-\frac{k-1}{kd}]$.*

(iii) *The primitive transition matrix $T(E,E^{(i)})$ is given by the following formula: For $x,y = 0,\dots,k-1$,*

$$(T(E,E^{(i)}))_{x,y} = \begin{cases} p_{dx-y+i-1} & \text{if } 0 \le dx-y+i-1 \le m,\\ 0 & \text{otherwise.}\end{cases}$$

Example 5.2 Consider the IFS as in (5.1) with $d = 4$ and $m = 9$, $k = 3$. The essential class consists of the one reduced characteristic vector $(1/3,(0,1/3,2/3))$. There are four transition matrices from E to E. They are

$$\begin{bmatrix} p_0 & 0 & 0\\ p_4 & p_3 & p_2\\ p_8 & p_7 & p_6\end{bmatrix},\quad \begin{bmatrix} p_1 & p_0 & 0\\ p_5 & p_4 & p_3\\ p_9 & p_8 & p_7\end{bmatrix},\quad \begin{bmatrix} p_2 & p_1 & p_0\\ p_6 & p_5 & p_4\\ 0 & p_9 & p_8\end{bmatrix},\quad \begin{bmatrix} p_3 & p_2 & p_1\\ p_7 & p_6 & p_5\\ 0 & 0 & p_9\end{bmatrix}.$$

Proof of Proposition 5.1. As noted in the proof of [10, Proposition 7.1],

$$\{S_\sigma(0) : \sigma \in \mathcal{A}^n\} = \Big\{\frac{(d-1)j}{md^n} : 0 \le j \le (d^n-1)k\Big\},$$
$$\{S_\sigma(1) : \sigma \in \mathcal{A}^n\} = \Big\{\frac{(d-1)(j+k)}{md^n} : 0 \le j \le (d^n-1)k\Big\}.$$

First, consider the level n net intervals that lie in $[1/d^n, 1-1/d^n]$. These have the form

$$\Delta^{(j)} = \Big[\frac{(d-1)j}{d^n m},\frac{(d-1)(j+1)}{d^n m}\Big] = \Big[\frac{j}{d^n k},\frac{j+1}{d^n k}\Big]$$

for $j = k,\dots,k(d^n-1)-1$. They have normalized length $(d-1)/m = 1/k$ and normalized neighbours as claimed in Proposition 5.1(i). These net intervals have d children,

$$\Big[\frac{(d-1)(dj+i)}{d^{n+1}m},\frac{(d-1)(dj+i+1)}{d^{n+1}m}\Big] \text{ for } i = 0,\dots,d-1,$$

all of the same type again.

At level 1, the net intervals have the form $[\frac{j-1}{dk},\frac{j}{dk}]$. If $j < k-1$, then there are only j neighbours, because $j-1-i < 0$ if $i \ge j$. If $j > m$, there are $< k$ neighbours, because $(j-1)/dk$ is not an iterate of 0. All other net intervals are type E. This proves (ii).

Now consider the x neighbour of $E^{(j)}$ at level n, for $0 \le x \le k-1$, namely

$$S_{\sigma_x}(0) = \frac{(d-1)(j-x)}{d^n m},$$

and the y neighbour of its i-th child, $E^{(i)}$, for $0 \le y \le k-1$,

$$S_{\sigma_y}(0) = \frac{(d-1)(dj+i-1-y)}{d^{n+1}m}.$$

For any $0 \le w \le m$, it follows that

$$S_{\sigma_x w}(0) = \frac{(d-1)(d(j-x)+w)}{d^{n+1}m}.$$

Hence, whenever $0 \le dx - y + i - 1 = w \le m$, we have $S_{\sigma_x w}(0) = S_{\sigma_y}(0)$, and this proves (iii). ∎

Example 5.3 Suppose $m = k(d-1)$ is even. Then $1/2 = S_\sigma(0)$ for some $\sigma \in \mathcal{A}$ and therefore $1/2$ is a left endpoint of a net interval of level one, and hence is a boundary essential point. Thereafter, $1/2$ is the left endpoint of the left-most child of the parent net interval, and thus $1/2$ has symbolic representation $(E^{(m/2)}, E^{(1)}, E^{(1)}, \dots)$. Similarly, $1/2$ is also the right-most endpoint of the right-most child of the net interval immediately to the left of this net interval. Consequently, $1/2$ also has symbolic representation $(E^{(m/2)-1}, E^{(d)}, E^{(d)}, \dots)$.

When $k = 2$ ($m = 2(d-1)$), for example, then $T(E^{(1)}, E^{(1)}) = \begin{bmatrix} p_0 & 0 \\ p_d & p_{d-1} \end{bmatrix}$ and $T(E^{(d)}, E^{(d)}) = \begin{bmatrix} p_{d-1} & p_{d-2} \\ 0 & p_m \end{bmatrix}$, so that we have

$$\dim_{\mathrm{loc}} \mu(1/2) = |\log(\max(p_0, p_{d-1}, p_m))| / \log d.$$

Proposition 5.4 *Assume μ is the self-similar Cantor-like measure of finite type generated by the IFS (5.1), with $m = k(d-1) + r$, $1 \le r \le d-2$.*

(i) *The essential class consists of two reduced characteristic vectors, E with normalized length r/m and neighbour set $(j(d-1)/m : j = 0, \dots, k)$, and F with normalized length $(d-1-r)/m$ and neighbour set $((r + j(d-1))/m : j = 0, \dots, k-1)$.*

(ii) *At level one the essential net intervals are alternately E and F, beginning with the interval $[\frac{1}{d} - \frac{r}{md}, \frac{1}{d}]$ of type E and ending with $[1 - \frac{1}{d}, 1 - (\frac{1}{d} - \frac{r}{md})]$ also of type E. There are $m - k + 1$ net intervals with characteristic vector E and $m - k$ with characteristic vector F.*

(iii) *Type E has $2r+1$ children labelled (from left to right) $E^{(1)}, F^{(2)}, \dots, E^{(2r+1)}$.*
Type F has $2(d-r)-1$ children labelled $F^{(1)}, E^{(2)}, \dots, F^{2(d-r)-1}$.

(iv) *The non-zero entries of the primitive transition matrices are as follows:*

- *For $i = 0, \dots, r$ and $0 \le x, y \le k$, $(T(E, E^{(2i+1)}))_{xy} = p_{dx-y+i}$ if $0 \le dx - y + i \le m$.*
- *For $i = 1, \dots, r$ and $0 \le x \le k, 0 \le y \le k-1$, $(T(E, F^{(2i)}))_{xy} = p_{dx-y+i-1}$ if $0 \le dx - y + i - 1 \le m$.*
- *For $i = 0, \dots, d-r-1$ and $0 \le x, y \le k-1$, $(T(F, F^{(2i+1)}))_{xy} = p_{dx+r-y+i}$ if $0 \le dx + r - y + i \le m$.*
- *For $i = 1, \dots, d-r-1$ and $0 \le x \le k-1, 0 \le y \le k$, $(T(F, E^{(2i)}))_{xy} = p_{dx+r-y+i}$ if $0 \le dx + r - y + i \le m$.*

Proof The proof is similar to the previous case, but with two characteristic vectors arising because the iterates of 0 and 1 do not coincide. Indeed,

$$\{S_\sigma(1) : \sigma \in \mathcal{A}^n\} = \Big\{ \frac{(d-1)(j+k)+r}{md^n} : 0 \le j \le (d^n-1)k \Big\}.$$

The net intervals whose left endpoint is an iterate of 0 give one characteristic vector and those whose left endpoint is an iterate of 1 is the second. We leave the details for the reader. ∎

Example 5.5 Suppose $k = 1$, $m = d-1+r$, where $1 \le r \le d-2$ and m is even. There are an odd number of net intervals at level one, and by symmetry $1/2$ lies at the centre of the middle interval. This is a net interval of type F, since $2(m-1)+1 \equiv 3 \bmod 4$. At all other levels there are an odd number of net intervals, so again $1/2$ lies at the centre of the middle one and again this is a type F, namely $F^{(2i+1)}$ where $i = (d-r-1)/2$, since $2(d-r)-1 \equiv 1 \bmod 4$. As $T(F^{(2i+1)}, F^{(2i+1)}) = [p_{m/2}]$, we have $\dim_{\text{loc}} \mu(1/2) = \left|\log p_{m/2}\right| / \log d$.

In the proof of the next result we will use the pseudo norm $\|T\|_{\min}$, defined in the previous section, and also the norm

$$\|T\|_{\max} = \max_j \sum_i |T_{ij}|$$

where the sum is over all the rows of the matrix. That is, $\|T\|_{\max}$ is the maximal column sum of T. For matrices with non-negative values it is easy to see that

$$\|T_1T_2\|_{\min} \ge \|T_1\|_{\min}\|T_2\|_{\min}, \quad \|T_1T_2\|_{\max} \le \|T_1\|_{\max}\|T_2\|_{\max},$$
$$\|T\|_{\min} \le \|T\| \le C\|T\|_{\max},$$

where C is the number of columns of T.

Proposition 5.6 *Let $P_i = \sum_{i \equiv j \bmod d} p_j$, $P_{\max} = \max(P_i)$, and $P_{\min} = \min(P_i)$. For any $x \in (0,1)$, we have*

$$\frac{|\log P_{\max}|}{\log d} \le \underline{\dim}_{\text{loc}}\mu(x) \le \overline{\dim}_{\text{loc}}\mu(x) \le \frac{|\log P_{\min}|}{\log d}.$$

Proof From the formulas given in Propositions 5.1 and 5.4, one can see that the column sums of an essential, primitive transition matrix T are of the form P_i. Hence, if T is a product of m essential, primitive transition matrices, then

$$P_{\min}^m \le \|T\| \le CP_{\max}^m,$$

where C is a bound for the number of columns of a primitive transition matrix.

Since any $x \in (0,1)$ is truly essential and the set of local dimensions of boundary essential points is contained in the set of local dimensions of interior essential points (Cor. 3.15), we can assume without loss of generality that x is an interior essential point. Hence, there exists a k so that $\Delta_k(x)$ is an essential net interval and a common ancestor for $\Delta_n^-(x)$, $\Delta_n(x)$ and $\Delta_n^+(x)$ for all $n \ge k$. Consequently,

$\mu(\Delta_n^-(x)), \mu(\Delta_n(x))$ and $\mu(\Delta_n^+(x))$ can all be approximated by the norms of products of $n-k$ primitive transition matrices within the essential class. From this the result follows. ∎

Corollary 5.7 (i) *If $p_0 < P_{\min}$, then $\dim_{\text{loc}} \mu(0)$ is an isolated point.*
(ii) *If $m \geq d$ and $p_0 < p_j$ for $j \neq 0, m$, then $\dim_{\text{loc}} \mu(0)$ is an isolated point.*

Similar statements can be made for p_m and $\dim_{\text{loc}} \mu(1)$.

Proof We have that (i) is immediate, since $\dim_{\text{loc}} \mu(0) = |\log p_0|/\log d$.

For (ii), one can easily check from these formulas that p_0 (and p_m) are never the only non-zero entries in a column when $m \geq d$. Hence the hypothesis of (i) is satisfied. ∎

Remark 5.8 We remark that it is possible for (i) to be satisfied without p_0 being minimal. For instance, if $m \geq 2d$, then every column admits at least two non-zero entries, and hence it would suffice to have $p_0 < 2p_j$ for all j in order for $\dim_{\text{loc}} \mu(0)$ to be an isolated point.

On the other hand, it is also possible for such a measure to have no isolated points. Here is an example.

Example 5.9 Consider the IFS $\{S_j(x) = x/3 + j/6 : j = 0, \dots, 4\}$ and probabilities $p_0 = p_4 = 1/3$, $p_1 = p_2 = p_3 = 1/9$. The essential class is composed of one reduced characteristic vector, with three transition matrices from this vector to itself. The transition matrices are

$$\begin{bmatrix} 1/3 & 0 \\ 1/9 & 1/9 \end{bmatrix}, \quad \begin{bmatrix} 1/9 & 1/3 \\ 1/3 & 1/9 \end{bmatrix}, \quad \begin{bmatrix} 1/9 & 1/9 \\ 0 & 1/3 \end{bmatrix}.$$

One can check that the second matrix has 4/9 as an eigenvalue. Further, all matrices have maximal column sum equal to 4/9. This gives an exact lower bound for the set of local dimensions. One can compute that the local dimension of the essential class, $(0,1)$, contains the interval $I = [\frac{\log(9/4)}{\log 3}, 1.24] \approx [0.738, 1.24]$ and is contained in $[\frac{\log(9/4)}{\log 3}, 2.00]$. We can establish the upper bounds by explicitly finding a point of local dimension 1.24 in the first case, and by using the $\|\cdot\|_{\max}$ norm for the second case. The local dimension of the self-similar measure at the two end points of the support is 1 and $1 \in I$.

Corollary 5.10 *If $P_{\max} = P_{\min}$, then $\{\dim_{\text{loc}} \mu(x) : x \in (0,1)\} = \{1\}$.*

Proof This follows from the observation that

$$d \cdot P_{\min} \leq \sum P_i \leq d \cdot P_{\max} \quad \text{and} \quad \sum P_i = 1. \qquad ∎$$

Here is a family of examples of this phenomena, generalizing [10, Ex. 6.1].

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Example 5.11 Suppose μ is the self-similar measure associated with the IFS (5.1) with $m+1 \equiv 0 \bmod d$ and $p_j = 1/(m+1)$ for all $j = 0, \dots, m \geq d$. Then

$$\dim_{\mathrm{loc}} \mu(x) = 1 \text{ for all } x \in (0,1),$$
$$\dim_{\mathrm{loc}} \mu(0) = \dim_{\mathrm{loc}} \mu(1) = \frac{\log(m+1)}{\log d} > 1,$$

so the set of local dimensions is a doubleton.

Proof The assumption that $m+1 \equiv 0 \bmod d$ ensures that each column of each essential primitive transition matrix T has exactly k non-zero entries, where $m+1 = kd$. Consequently, $P_i = k/(m+1) = 1/d$ for each i and the result follows from the previous corollary. ∎

Remark 5.12 These measures are also an example of the phenomena addressed in Proposition 3.19. The proof above shows there exists a constant C such that $d^{-n} \leq \|T\| \leq Cd^{-n}$ for all n-fold products of primitive transition matrices. As $\dim_H \operatorname{supp} \mu = 1$, the proposition implies μ restricted to the truly essential class is absolutely continuous with respect to Lebesgue measure on $[0,1]$.

Corollary 5.13 *Suppose $\{\mu_n\}$ is a sequence of Cantor-like measures, all with contraction factor $1/d$. Let $P_{\max}^{(n)}$ and $P_{\min}^{(n)}$ be the maximal and minimal column sums associated with μ_n. If $P_{\max}^{(n)} - P_{\min}^{(n)} \to 0$, then the set of local dimensions at any $x \in (0,1)$ tends to 1.*

Proof Similar reasoning to the proof of the previous corollary shows that

$$P_{\min}^{(n)}, P_{\max}^{(n)} \longrightarrow \frac{1}{d}.$$ ∎

Example 5.14 Let μ be the self-similar measure associated with the IFS (5.1) and let μ^k be the k-fold convolution of μ, normalized to $[0,1]$. Then

$$\dim_{\mathrm{loc}} \mu^k(x) \longrightarrow 1 \text{ for all } x \in (0,1) \quad \text{and} \quad \dim_{\mathrm{loc}} \mu^k(x) \longrightarrow \infty \text{ for } x = 0,1.$$

To see this, let $Q(x) = p_0 + p_1 x + \cdots + p_n x^n$. The measure μ^k is also a Cantor-like measure with contraction factor $1/d$. With the contractions ordered in the natural way, the probability of the j^{th} term, denoted $p_j^{(k)}$, is equal to the coefficient of x^j in $Q(x)^k$ and

$$P_i^{(k)} = \sum_{j \equiv i \bmod d} p_j^{(k)} = \frac{1}{d} \sum_{j=1}^{d} Q(\zeta_d^j)^k \zeta_d^{-ji},$$

where ζ_d is a primitive d-th root of unity. It is easy to see that $Q(1) = 1$ and $|Q(\zeta_d^j)| < 1$ for $j \neq d$. Hence we see that $P_i^{(k)} \to 1/d$ as $k \to \infty$ for all i. This in turn implies that $P_{\min}^{(k)} - P_{\max}^{(k)} \to 0$ and hence $\dim_{\mathrm{loc}} \mu^k(x) \to 1$ for each $x \in (0,1)$.

In contrast, $\dim_{\mathrm{loc}} \mu^k(0) = \lim |\log p_0^k| / \log d \to \infty$ and similarly for $\dim_{\mathrm{loc}} \mu^k(1)$.

Example 5.15 Suppose ν is the uniform Cantor measure associated with the IFS $\{S_0(x) = x/d, S_1(x) = x/d + (d-1)/d\}$. Then ν^m is the measure of finite type generated by the IFS (5.1) and probabilities $p_j = 2^{-m}\binom{m}{j}$. Information was given about the minimum and maximum local dimensions (other than at $0, m$) in [2, Thm. 6.1] for $m \le 2d - 1$.

We can extend the maximum local dimension result to $m < 3(d-1)$ when $m - d$ is odd, as follows. First, note that the column sums of essential primitive transition matrices have the form

$$P_j = 2^{-m} \sum_{k=-\infty}^{\infty} \binom{m}{j + kd}$$

and reasoning as in [2, Lem.6.2] shows that these are minimized at $j = [\frac{m-d}{2}]$. We can assume that $m = 2(d-1) + r$ for $1 \le r \le d - 2$. Consider the periodic element x_0 with period $\theta = (F^{(2i+1)}, F^{(2i+1)})$ for $i = (d - r - 1)/2 = (m - d + 1)/2 - r$. Then

$$T(F^{(2i+1)}, F^{(2i+1)}) = \begin{bmatrix} p_{\frac{m-d+1}{2}} & p_{\frac{m-d-1}{2}} \\ p_{\frac{m+d+1}{2}} & p_{\frac{m+d-1}{2}} \end{bmatrix}.$$

The two column sums are equal and minimal among all column sums of essential primitive transition matrices. Thus, $\|T\| \sim \|T\|_{\min}$ and further, this is a lower bound on the norm of any essential primitive transition matrix. Hence $\dim_{\mathrm{loc}} \mu(x_0)$ is maximal over all $x \in (0,1)$.

Since the column sums are maximized when $j = [m/2]$, we deduce from Examples 5.3 and 5.5 that $\dim_{\mathrm{loc}} \nu^m(1/2) = |\log p_{m/2}|/\log d$ is minimal when $m < 2(d-1)$ is even, as was also seen in [2].

6 Maximal Loop Classes Outside the Essential Class

In [10], it is shown that if μ is a self-similar measure of finite type, with full support and regular probabilities, then the set of upper (or lower) local dimensions at points in any positive maximal loop class is an interval. In this section we show that this is not true for finite type measures satisfying only the positive row property. The example we use is a self-similar measure that would be Cantor-like, in the sense of the previous section, if we had allowed some probabilities to be zero.

The measure μ will arise from the maps $S_i(x) = x/4 + d_i/12$ with $d_i = i$ for $i = 0, 1, \ldots, 5$, $d_6 = 8$, and $d_7 = 9$, and probabilities $p_0 = 1/2$, $p_i = 1/14$ for $i = 1, \ldots, 7$. The reduced transition diagram has 7 reduced characteristic vectors. The reduced characteristic vectors are

- RCV 1: $(1, (0))$,
- RCV 2: $(1/3, (0))$,
- RCV 3: $(1/3, (0, 1/3))$,
- RCV 4: $(1/3, (0, 1/3, 2/3))$,
- RCV 5: $(1/3, (1/3, 2/3))$,
- RCV 6: $(1/3, (2/3))$,
- RCV 7: $(2/3, (0, 1/3))$.

The maps are

- RCV 1 $\to [2, 3, 4, 4, 4, 4, 5, 6, 2, 7, 6]$,
- RCV 2 $\to [2, 3, 4, 4]$,
- RCV 3 $\to [4, 4, 4, 4]$,
- RCV 4 $\to [4, 4, 4, 4]$,

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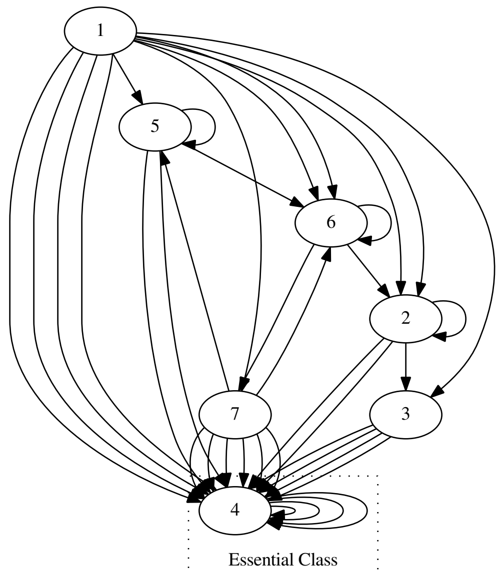

Figure 3: Transition diagram for example in Section 6

- RCV 5 → $[4, 4, 5, 6]$,
- RCV 6 → $[2, 7, 6]$,
- RCV 7 → $[4, 4, 4, 4, 4, 4, 5, 6]$.

We refer the reader to Figure 3 for the transition diagram.

As the probabilities are not regular, the reduced transition diagram does not contain all of the necessary information to compute the local dimension at a point, since to calculate $\dim_{\mathrm{loc}} \mu(x)$, we need to know about $\Delta_n^-(x)$, $\Delta_n^+(x)$, in addition to $\Delta_n(x)$. To keep track of this information, we introduce the triple transition diagram. Each triple consists of a net interval and its adjacent net intervals. If there is no adjacent net interval, then we represent this with an X. The triple transition diagram also displays the transitions from each triple to their triple children and denotes which transitions are right or left-most descendents. See Figure 4 for the triple transition diagram.

We define, in the obvious way, the triple loop classes, triple maximal loop classes and the triple essential class. In this example, the set of points that are in the triple essential class, $[4, 4, 4]$, is the same as the set of truly essential points. To see this, observe that if x is an interior essential point, then there exists an integer n such that x is in the interior of the net interval $\Delta_n(x)$ whose reduced characteristic vector is of type 4. As x is not equal to the end point of $\Delta_n(x)$, there will exist some k such that $\Delta_{n+k}(x)$ and its two neighbours will all have reduced characteristic vector of type 4. Hence, $[\Delta_{n+k}^-(x), \Delta_{n+k}(x), \Delta_{n+k}^+(x)] = [4, 4, 4]$. If, instead, x is a boundary essential point, then there exists an n such that two adjacent $\Delta_n(x)$ and $\Delta_n'(x)$ are the reduced characteristic vector of type 4. In this case, regardless of which net interval containing

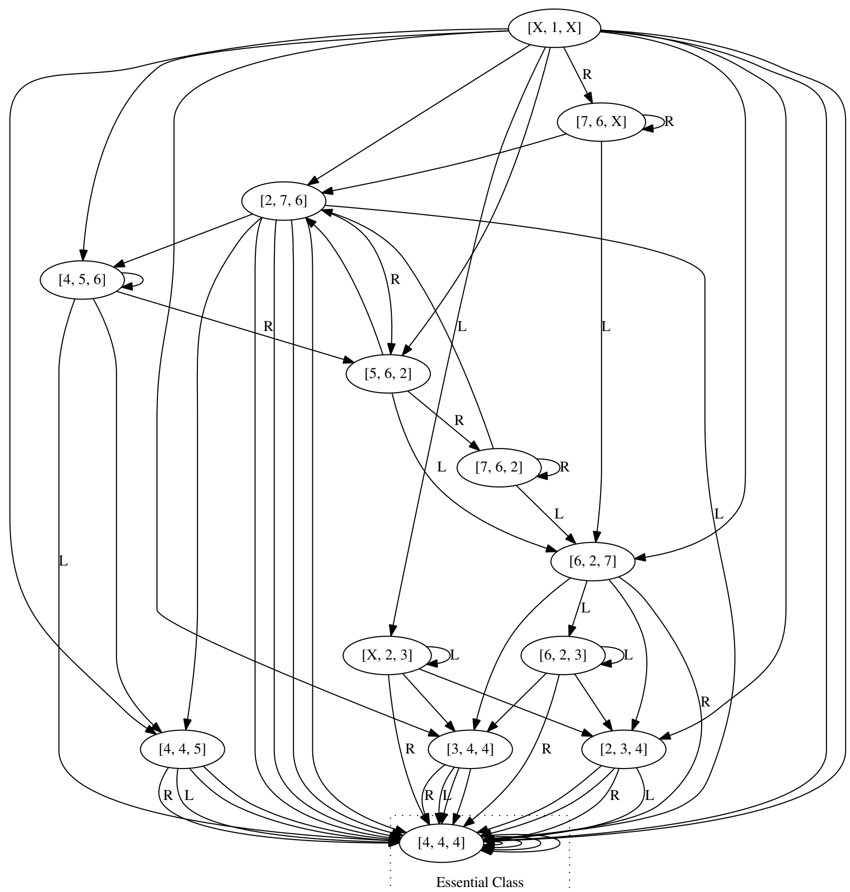

Figure 4: Triple transition diagram for example in Section 6

x we use, we see that $\Delta_{n+1}(x)$ and its two adjacent net intervals will be the reduced characteristic vector of type 4. Hence, $[\Delta^-_{n+1}(x), \Delta_{n+1}(x), \Delta^+_{n+1}(x)] = [4,4,4]$. The other inclusion is clear.

From the triple transition diagram, we can see that there are four triple maximal loop classes, in addition to the triple essential class. Three of these are singletons, $[7,6,X]$, $[X,2,3]$, and $[6,2,3]$. It is very easy to compute the local dimensions of these points. The final maximal loop class is formed by the four triples $[2,7,6]$, $[4,5,6]$, $[5,6,2]$, $[7,6,2]$ and is of positive type. See Figure 5 for the triple transition diagram of this triple maximal loop class. We have indicated on this diagram which of these transitions are right or left-most descendents.

We will determine the local dimension of points in this (non-singleton) triple maximal loop. It is important to note that this triple loop class admits no left-most descendents.

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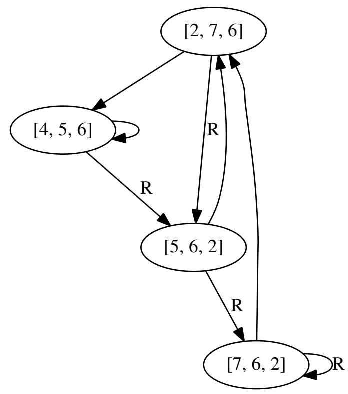

Figure 5: Triple transition diagram for maximal loop class for example in Section 6

First, assume that the symbolic representation of a point x in the loop class does not contain arbitrarily long, right-most paths, say these lengths are bounded by K. This implies that $\Delta_n(x)$ is in the interior of $\Delta_{n-K}(x)$, hence $\Delta_n(x), \Delta_n^+(x)$, and $\Delta_n^-(x)$ are all comparable to $\Delta_{n-K}(x)$ for all n. Thus, we may ignore the $\Delta_n^+(x)$ and $\Delta_n^-(x)$, and this allows us to use the techniques from [10] without modification. (We will not be able to ignore $\Delta_n^+(x)$ and $\Delta_n^-(x)$ later when we allow arbitrarily long right-most paths.)

In this case, the relevant transition matrices are:

$$T(5,5) = T(7,5) = \begin{bmatrix} 1/14 & 1/14 \\ 1/14 & 1/14 \end{bmatrix} \qquad T(5,6) = T(7,6) = \begin{bmatrix} 1/14 \\ 1/14 \end{bmatrix}$$
$$T(6,7) = \begin{bmatrix} 1/14 & 1/14 \end{bmatrix} \qquad T(6,6) = \begin{bmatrix} 1/14 \end{bmatrix}.$$

For these matrices, the minimal column sum is 1/14, and the maximal sum is 1/7. These numbers are also the eigenvalues of $T(6,6)$ and $T(5,5)$, respectively. As we are only concerned with $\Delta_n(x)$ and do not need to worry about $\Delta_n^+(x)$ or $\Delta_n^-(x)$, we see that the standard convexity argument can be used to show that the set of local dimensions is an interval. Consequently, such points produce the interval

$$\left[\frac{\log 7}{\log 4}, \frac{\log 14}{\log 4}\right] \approx [1.403677461, 1.903677461]$$

as the set of local dimensions.

To consider the the case when x contains arbitrarily long right-most paths, we now need to consider $\Delta_n^+(x)$ and $\Delta_n^-(x)$. We will need to know about the additional transition matrices

$$T(2,2) = [1/2] \quad \text{and} \quad T(6,2) = [1/14]$$

First, consider an x whose tail consists of the right-most branch of the triples $[7,6,2], [7,6,2], [7,6,2], \ldots$. We observe in this case that $\Delta_n^-(x)$ is comparable to $\Delta_n(x)$ as they share the common ancestor $\Delta_{n-1}(x)$, so that $\mu(\Delta_n(x))$ and $\mu(\Delta_n^-(x))$ are comparable to $\|T(6,6)^n\| = 1/14^n$. We further see that $\Delta_n^+(x)$ is not comparable to $\Delta_n(x)$, as it does not share a common ancestor a bounded number of generations back. In fact, the symbolic representation of $\Delta_n^+(x)$ has tail $(2,2,2,\ldots,2)$ and hence $\mu(\Delta_n^+(x))$ is comparable to $\|T(2,2)^n\| = 1/2^n$. This gives us that the local dimension at x is

$$\begin{aligned}\dim_{\mathrm{loc}}\mu(x) &= \lim_n \frac{\log M_n(x)}{n\log 1/4} = \lim_n \frac{\log(\mu(\Delta_n^-(x)) + \mu(\Delta_n(x)) + \mu(\Delta_n^+(x))}{n\log 1/4} \\ &= \lim_n \frac{\log((1/14)^n + (1/14)^n + (1/2)^n)}{n\log 1/4} = 1/2.\end{aligned}$$

Next, consider the case where x has arbitrarily long, but not infinitely-long, right-most paths from $[7,6,2] \to [7,6,2]$. We claim that in this case the upper local dimension must be greater than $\log 7/\log 4 \sim 1.403677461$. To see this, we note that for all n where $\Delta_n(x)$ is not a right-most child of $\Delta_{n-1}(x)$ (which happens infinitely often) the value of $\mu(M_n(x)) \sim \mu(\Delta_n(x))$, as $\mu(\Delta_n(x)) \sim \mu(\Delta_n^+(x)) \sim \mu(\Delta_n^-(x))$. As on this subsequence we have that the lim sup must be greater than $\log 7/\log 4$, it follows that the set of upper local dimensions is not an interval.

This is in contrast to the lower local dimension, where we can achieve any value z in the interval $\left[\frac{1}{2}, \frac{\log 14}{\log 4}\right]$. We will prove this by constructing an x in this maximal loop class such that $\underline{\dim}_{\mathrm{loc}}\mu(x) = z$.

Let $A = T((7,6,2),(2,7,6)) \cdot T((2,7,6),(5,6,2)) \cdot T((5,6,2),(7,6,2))$ be a triple of the transition matrices for the path through

$$(7,6,2) \to (2,7,6) \to_R (5,6,2) \to_R (7,6,2).$$

We note here that these transition matrices may work on the middle or the right-most matrix of the previous transition, depending upon the nature of the transition. Consider the path with transition matrices

$$\begin{aligned}T_k := {}& T\big((7,6,2),(7,6,2)\big)^{n_1} \cdot A \cdot T\big((7,6,2),(7,6,2)\big)^{n_2} \\ & \cdot A \cdots A \cdot T\big((7,6,2),(7,6,2)\big)^{n_k}.\end{aligned}$$

We let x be the point in K with symbolic path $\lim_k T_k$. Let L_k be the length of T_k, that is, $L_k = n_1 + 2 + n_2 + 2 + \cdots + 2 + n_k$. We see that the three matrices associated with T_1 are

$$\big(\big[14^{-n_1} \quad 14^{-n_1}\big], \big[14^{-n_1}\big], \big[2^{-n_1}\big]\big) = \big(\big[14^{-L_1} \quad 14^{-L_1}\big], \big[14^{-L_1}\big], \big[2^{-L_1}\big]\big).$$

The three matrices associated with T_2 are

$$\begin{aligned}&\big(2\big[14^{-(n_1+2+n_2)} \quad 14^{-(n_1+2+n_2)}\big], 2\big[14^{-(n_1+2+n_2)}\big], \big[2^{-(n_2+1)}14^{-(n_1+1)}\big]\big) = \\ &\qquad\qquad \big(2\big[14^{-L_2} \quad 14^{-L_2}\big], 2\big[14^{-L_2}\big], \big[2^{-(n_2+1)}14^{-(L_1+1)}\big]\big).\end{aligned}$$

In general, for $k \geq 2$, we have that the three matrices associated to T_k are

$$\big(2^{k-1}\big[14^{-L_k} \quad 14^{-L_k}\big], 2^{k-1}\big[14^{-L_k}\big], 2^{k-2}\big[2^{-(n_k+1)}14^{-(L_{k-1}+1)}\big]\big)$$

So, on the subsequence associated with L_k we see that $M_{L_k}(x)$ is approximately $2^{k-2}2^{-(n_k+1)}14^{-(L_{k-1}+1)}$. Choosing the n_k such that

$$z = \lim_{k\to\infty} \frac{\log(2^{k-2}2^{-(n_k+1)}14^{-(L_{k-1}+1)})}{\log(4^{-L_k})}$$

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gives that the local dimension, computing along this subsequence, is equal to z. For example taking

$$n_k \approx \frac{\log 14 - z \log 4}{(2z-1)\log 2} L_{k-1}$$

will suffice. Note: so long as $z \in (\frac{1}{2}, \frac{\log 14}{\log 4})$, we see that this is always a positive constant times L_{k-1}.

It is straightforward to see that this subsequence of lower local dimension estimates gives a lower bound for the sequence, which proves the desired result. To see this just note that if we consider a path for x of length $N \in (L_{k-1}, L_k)$, then $M_{L_k}(x))^{1/L_k} < M_N(x))^{1/N}$.

Thus, the set of lower local dimensions at points in the loop class is the interval $\left[\frac{1}{2}, \frac{\log 14}{\log 4}\right]$. This is in contrast to the set of upper local dimensions at points in the loop class, which is the union of the interval $\left[\frac{\log 7}{\log 4}, \frac{\log 14}{\log 4}\right]$ together with the singleton $1/2$.

7 When Finite Type IFS have Pisot Contractions

In this section we explore the connection between finite type and Pisot contraction factors. This was motivated by Feng's observation in [6] showing that the IFS $\{S_j(x) = \rho x + j(1-\rho)/m : j = 0, \dots, m\}$ satisfies the finite type condition if and only if ρ^{-1} is Pisot.

In Example 7.1, the IFS is of finite type, does not satisfy the open set condition, but the contraction factor is not necessarily the inverse of a Pisot number. This example also illustrates that we can have a measure of finite type whose support is not the full interval $[0,1]$, yet every row of each primitive transition matrix admits a non-zero entry. In addition, it has the interesting property that every element of the self-similar set is truly essential.

Example 7.1 Pick any positive number $\varepsilon < 1/8$. Let $0 < \rho < 1$ be a root of $\varepsilon - 2x^2 + 4x - 1$ and consider the self-similar set K generated by the contractions $S_i(x) = \rho x + d_i$, with $d_0 = 0$, $d_1 = -\rho^2 + \rho$, $d_2 = \varepsilon - \rho^2 + 2\rho$, and $d_3 = \varepsilon - 2\rho^2 + 3\rho$. Consider the associated probability measure with uniform probabilities, $p_i = 1/4$ for $i = 0, \dots, 3$. There are 5 reduced characteristic vectors: $(1, (0))$, $(1-\rho, (0))$, $(\rho, (0, 1-\rho))$, $(1-\rho, (\rho))$, and $(1-2\rho, (\rho))$. Figure 6 shows the transition diagram. The essential class consists of all the characteristic vectors except 1, and there are no loop classes outside of the essential class. Hence K is the truly essential set. As this satisfies the positive row property, the set of local dimensions is a closed interval.

We list below the transition matrices that are not equal to $[1/4]$:

$$T(1,3) = T(2,3) = T(4,3) = \begin{bmatrix} 1/4 & 1/4 \end{bmatrix}, \quad T(3,5) = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix},$$
$$T(3,3) = \begin{bmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{bmatrix}, \quad T(3,3) = \begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{bmatrix}.$$

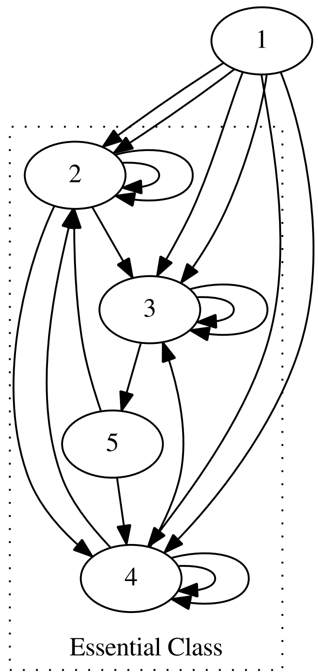

Figure 6: Transition diagram for Example 7.1

Using techniques similar to [10] one can show that the minimal local dimension is

$$\frac{\log\left(\operatorname{sp}\begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 1/4 & 1/4 \end{bmatrix}\right)}{2\log\rho} = \frac{\log\frac{3+\sqrt{5}}{32}}{2\log\rho},$$

and the maximal local dimension is

$$\frac{\log\left(\operatorname{sp}\begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{bmatrix}\right)}{\log\rho} = \frac{\log 1/4}{\log\rho}.$$

The details are left to the reader.

The incidence matrix of the essential class is

$$I = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Its spectral radius is $2+\sqrt{2}$, thus the formula from Proposition 3.8 gives that $\dim_H K = \log(2+\sqrt{2})/|\log\rho|$.

In this example the overlap was "perfect"; that is, all overlaps were of the form $\rho^n K$ for some integer n, (ρ the contraction factor, K the self-similar set). But $K \neq [0,1]$. In our final proposition we show that if the self-similar set is a full interval and the overlaps are perfect, in this sense, then ρ is Pisot.

Proposition 7.2 *Suppose $[0,1]$ is the self-similar set associated with contractions S_j, each with contraction factor ρ. Assume that, for each j, the length of the interval $S_j([0,1]) \cap S_{j+1}([0,1]))$ is either equal to ρ^{k_j} for some integer k_j or has length equal to 0. Then ρ is Pisot.*

Proof Assume that we have n contractions. As the self-similar set is $[0,1]$, we have that

$$n\rho - \sum_{i=1}^{n-1} \rho^{k_i} = 1.$$

Let $N = \max(k_j)$ and $q = \rho^{-1}$. Let $f(z) = z^N - nz^{N-1}$ and $g(z) = \sum_j z^{N-k_j}$. Then $(f+g)(q) = 0$. Clearly, $f(n) = 0$ and all other zeros of f are inside the unit disc (namely, at 0). Further, on the unit disc, $|f(z)| \geq n-1 \geq |g(z)|$. By Rouche's theorem, $f+g$ has $n-1$ zeros in the closure of the unit disk and therefore its other root, q, is a Pisot number. ∎

Remark 7.3 It would be interesting to fully understand the connection between finite type and a Pisot contraction factor. Note that if $\dim_H K = 1$, then as $\dim_H K = \log(\mathrm{sp}(I))/|\log\rho|$, and the incidence matrix I is integer valued, it follows that ρ^{-1} is an algebraic integer.

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