Canad. Math. Bull. Vol. **44** (1), 2001 pp. 80–86

Constructing Compacta of Different Extensional Dimensions

Michael Levin

Abstract. Applying the Sullivan conjecture we construct compacta of certain cohomological and extensional dimensions.

1 Introduction

We say that $\text{e-dim}\, X \leq K$ (the extensional dimension of X does not exceed K) if any map of any closed subset of X into K can be extended over X. Thus for the covering dimension $\dim X \leq n$ if and only if $\text{e-dim}\, X \leq \mathbb{S}^n$ and for the cohomological dimension with a coefficient group G, $\dim_G X \leq n$ if and only if $\text{e-dim}\, X \leq K(G, n)$, where $K(G, n)$ is the Eilenberg-MacLane complex.

We adopt notations of a truncated cohomology. The (reduced) truncated cohomology $T^k(X)$, $k \leq 0$ of X (with coefficients in a spectrum) generated by a space L is the set of pointed homotopy classes $[X, \Omega^{-k}L]$ $(= [\Sigma^{-k}X, L])$ of maps from X to $\Omega^{-k}L$. We write $T^k(X) = 0$ if $T^k(X)$ contains only the null-homotopic map for any base point in L.

T^k is said to be continuous if for every countable CW-complex K and every essential map $f\colon K \to \Omega^{-k}L$ there exists a finite subcomplex B of K such that $f|_B$ is essential. Let us say that T^k is strongly continuous if for every countable CW-complex K and every map $f\colon A \to \Omega^{-k}L$ of a subcomplex A of K which cannot be extended over K there exists a finite subcomplex B of K such that $f|_{A\cap B}$ cannot be extended over B.

The strong continuity implies the continuity. Indeed, let T^k be strongly continuous and let $f\colon K \to \Omega^{-k}L$ be essential. Then f cannot be extended over cone K. Hence there exists a finite subcomplex B of K such that $f|_B$ cannot be extended over cone B and therefore $f|_B$ is essential.

The finiteness of the homotopy groups of L implies the finiteness of the homotopy groups of $\Omega^{-k}L$ for every $k \leq 0$ and hence by Proposition 2.1 it also implies the strong continuity (and hence the continuity) of T^k.

Dranishnikov [1] proved

Theorem 1.1 *Let P and K be countable simplicial complexes. If T^{-2} is continuous, $T^{-2}(P) \neq 0$ and $T^k(K) = 0$ for all $k < -2$ then there exists a compactum X with* $\text{e-dim}\, X \leq K$ *such that X admits an essential map to P.*

Received by the editors February 18, 1999; revised June 17, 1999.
The author was supported by a JSPS postdoctoral fellowship for foreign researchers.
AMS subject classification: 55M10, 54F45, 55U20.
Keywords: cohomological dimension, Eilenberg-MacLane complexes, Sullivan conjecture.

In this note we generalize Dranishnikov's result in two directions:

Theorem 1.2 *Let K and P be countable simplicial complexes and let T^* be a truncated cohomology generated by L.*

(a) If T^0 is continuous, $T^0(P) \neq 0$ and $T^k(K) = 0$ for all $k < 0$ then there exists a compactum X with e-dim $X \leq K$ *such that X admits an essential map to P.*
(b) If T^0 is strongly continuous, $T^0(P) \neq 0$ and $T^k(K) = 0$ for all $k \leq 0$ then there exists a compactum X such that $P <$ e-dim $X \leq K$.

(a) is a direct generalization of Theorem 1.1, (b) covers additional areas of applications some of which are considered below. The important point of (b) is that the condition e-dim $X > P$ is stronger than the existence of an essential map to P. Note that the starting point at T^{-2} in Theorem 1.1 is imposed by a use of the Mayer-Vietoris sequence. The approach of this note does not rely on the Mayer-Vietoris sequence.

Let us show how (b) leads to constructing infinite dimensional compacta of $\dim_{\mathbb{Z}} = 2$. Let $K = K(\mathbb{Q}, 1) \vee \big(\bigvee\{K(\mathbb{Z}_p, 1) : p \text{ prime}\}\big)$, $P = \mathbb{S}^2$ and $L = M(\mathbb{Z}_2, 2)$ (= Moore space of type $(\mathbb{Z}_2, 2)$). Then $T^0(P) = \pi_2\big(M(\mathbb{Z}_2, 2)\big) = \mathbb{Z}_2 \neq 0$. Note that by the generalized Hurewicz isomorphism theorem the homotopy groups of $M(\mathbb{Z}_2, 2)$ are finite and hence T^k is strongly continuous for every $k \leq 0$. In order to apply Theorem 1.2 (b) we have to check that $T^k\big(K(\mathbb{Q}, 1)\big) = T^k\big(K(\mathbb{Z}_p, 1)\big) = 0$. For $K(\mathbb{Q}, 1)$ it can be done directly.

Recall that a model for $K(\mathbb{Q}, 1)$ is the infinite telescope of $\mathbb{S}^1 \to \mathbb{S}^1 \to \cdots$ where the i-th bonding map is of degree $i!$. Let K_i be the mapping telescope of the first i maps. Then K_i is homotopy equivalent to $\mathbb{S}^1$ and $K(\mathbb{Q}, 1) = \bigcup K_i$. Note that $\mathbb{Q}$ admits only the trivial homomorphism to a finite group. Let $f\colon K(\mathbb{Q}, 1) \to \Omega^{-k}M(\mathbb{Z}_2, 2)$. Then $f|_{K_i}$ is null-homotopic for every i. Since T^k is continuous, f is null-homotopic. Thus $T^k\big(K(\mathbb{Q}, 1)\big) = 0$.

In the case of $K(\mathbb{Z}_p, 1)$ we fortunately have the following powerful theorem of Miller [4]. The importance of Miller's theorem for cohomological dimension was realized by Dydak and Walsh [3], see also [1].

Theorem 1.3 **(Miller's theorem (The Sullivan conjecture) [4])** *Let K be a connected CW-complex such that $\pi_i(K)$ is locally finite and such that $\pi_i(K)$ is non-zero for only finitely many i. Let L be a connected finite dimensional CW-complex. Then the space of pointed maps from K to L has the weak homotopy type of a point or, equivalently, $[\Sigma^n K, L] = [K, \Omega^n L] = 0$ for all $n \geq 0$.*

By Miller's theorem $T^k\big(K(\mathbb{Z}_p, 1)\big) = [K(\mathbb{Z}_p, 1), \Omega^{-k}M(\mathbb{Z}_2, 2)] = 0$ and we have verified that $T^k(K) = 0$ for every $k \leq 0$. Then by Theorem 1.2 (b) there exists a compactum X of dim ≥ 3, $\dim_{\mathbb{Q}} X \leq 1$ and $\dim_{\mathbb{Z}_p} X \leq 1$ for p prime. By the Bockstein theorems $\dim_{\mathbb{Z}} X \leq 2$. Therefore $\dim_{\mathbb{Z}} X = 2$ and $\dim X = \infty$. Since $\mathbb{Q}$ and $\mathbb{Z}_p$ are fields, $\dim_{\mathbb{Q}}(X \times X) \leq 2$ and $\dim_{\mathbb{Z}_p}(X \times X) \leq 2$ for p prime. Hence $\dim_{\mathbb{Z}}(X \times X) \leq 3$. Thus we have obtained the Dydak-Walsh example [3], see also [1].

Another application concerns the following question related to the mapping intersection problem [2] which is still open in the case of codim $= 2$. Does $\dim_{\mathbb{Z}_2} X \leq 1$ imply e-dim $X \leq \mathbb{R}\mathrm{P}^2$ for a finite dimensional compactum X?

Let $\mathbb{S}^1 \subset \mathbb{R}\mathrm{P}^2$ generate $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$. Since $\pi_1\big(\Omega M(\mathbb{Z}_2, 2)\big) = \pi_2\big(M(\mathbb{Z}_2, 2)\big) = \mathbb{Z}_2$ there is an essential map from $\mathbb{S}^1$ to $\Omega M(\mathbb{Z}_2, 2)$ which extends over $\mathbb{R}\mathrm{P}^2$ and hence $\mathbb{R}\mathrm{P}^2$ admits an essential map to $\Omega M(\mathbb{Z}_2, 2)$. Theorem 1.2 (b) applied to $K = K(\mathbb{Z}_2, 1)$, $P = \mathbb{R}\mathrm{P}^2$ and $L = \Omega M(\mathbb{Z}_2, 2)$ produces a compactum X with $\dim_{\mathbb{Z}_2} \leq 1$ and e-dim $>$ $\mathbb{R}\mathrm{P}^2$. Thus without the finite dimensional restriction on X the answer to the question is "No".

2 Proofs

Proposition 2.1 *Let K be a countable CW-complex, let A be a subcomplex of K and let L be such that $\pi_n(L, l_0)$ is finite for all $n \geq 0$ and $l_0 \in L$. If a map $f\colon A \to L$ cannot be extended over K then there exists a finite subcomplex B of K such that $f|_{A\cap B}\colon A\cap B \to K$ cannot be extended over B.*

Proof Represent K as the union $K = \bigcup K_i$ of an increasing sequence of finite subcomplexes K_i such that $K_0 = \varnothing$ and K_{i+1} is obtained from K_i by adjoining only one cell, and assume that f extends over $A_i = A \cup K_i$ for every i.

Denote by $[A_i, L]_f$ the homotopy classes of all possible extensions of f over A_i with respect to a homotopy relative to A. Let $p_{i+1}\colon [A_{i+1}, L]_f \to [A_i, L]_f$ be the correspondence defined by the restriction $g \to g|_{A_i}$.

We will show by induction that $[A_i, L]_f$ is finite for every i. Since $A_0 = A$, $[A_0, L]_f$ contains only one element. Assume that $[A_i, L]_f$ is finite and $A_{i+1} \neq A_i$. Then $A_{i+1} = A_i \cup C$ is obtained from A_i by adjoining an n-dimensional cell C defined by a characteristic map $h\colon (B, \partial B) \to (C, \partial C)$ from an n-dimensional ball B.

Let $g\colon A_i \to L$ be such that $g|_A = f$ and assume that $g_1, g_2, \ldots\colon C \to L$ is an infinite sequence of maps such that $g_j|_{\partial C} = g|_{\partial C}$. Let $\mathbb{S}^n = B_1 \cup B_2$, $B_1 \cap B_2 = \partial B_1 = \partial B_2$ $(= \mathbb{S}^{n-1})$ be a decomposition of an n-dimensional sphere $\mathbb{S}^n$ into two n-dimensional balls B_1 and B_2, and let $h_1\colon B_1 \to B$, $h_2\colon B_2 \to B$ be homeomorphisms such that $h_1|_{\partial B_1} = h_2|_{\partial B_2}$. For each pair of maps g_{j_1} and g_{j_2} define the map $\alpha(j_1, j_2)\colon \mathbb{S}^n \to L$ by $\alpha(j_1, j_2)(x) = g_{j_1}\Big(h\big(h_1(x)\big)\Big)$ for $x \in B_1$ and $\alpha(j_1, j_2)(x) = g_{j_2}\Big(h\big(h_2(x)\big)\Big)$ for $x \in B_2$. Since the homotopy groups of L are finite there exist $1 < j_1 < j_2$ such that the maps $\alpha(1, j_1)$ and $\alpha(1, j_2)$ are homotopic. Then $\alpha(j_1, j_2)$ is null-homotopic and hence $g_{j_1} \cong g_{j_2}$ rel ∂C. Thus g admits only finitely many extensions over A_{i+1} representing different elements of $[A_{i+1}, L]_f$ and therefore $[A_{i+1}, L]_f$ is finite.

Since $[A_i, L]_f$ is finite for every i, $\varprojlim([A_i, L]_f, p_i) \neq \varnothing$. Then any element of $\varprojlim([A_i, L]_f, p_i)$ defines an extension of f over K. This contradiction shows that there exists A_i such that f does not extend over A_i. Set $B = K_i$ and we are done. ∎

Proposition 2.2 *Assume that for a countable simplicial complex K_1 and a space K_2, $[\Sigma^n K_1, K_2] = 0$ for all $n \geq 0$. Let A be a finite simplicial complex and let $f_1\colon A_1 \to K_1$ be $f_2\colon A_2 \to K_2$ be maps of a closed subset A_1 of A and a subcomplex A_2 of A such that f_2 cannot be extended over A. Then there exist a countable simplicial complex B and a*

https://doi.org/10.4153/CMB-2001-009-2 Published online by Cambridge University Press

map $p\colon B \to A$ such that for $B_1 = p^{-1}(A_1)$ and $B_2 = p^{-1}(A_2)$, B_2 is a subcomplex of B, the map $g_1 = f_1 \circ (p|_{B_1})\colon B_1 \to K_1$ can be extended over B while the map $g_2 = f_2 \circ (p|_{B_2})\colon B_2 \to K_2$ cannot be extended over B.

Proof Taking a small triangulation of A we may assume that A_1 is a subcomplex of A and f_1 is a simplicial map. We may also assume that A_1 contains the 0-skeleton of A. For each simplex $\sigma \subset A$ we are going to construct a CW-complex B_σ such that $B_{\sigma'}$ is a subcomplex of B_σ if $\sigma' \subset \sigma$. We will also construct maps $f_\sigma\colon B_\sigma \to K_1$ and $p_\sigma\colon B_\sigma \to \sigma \subset A$ such $f_\sigma|_{B_{\sigma'}} = f_{\sigma'}$ and $p_\sigma|_{B_{\sigma'}} = p_{\sigma'}$ for $\sigma' \subset \sigma$. Define $B^i = \bigcup\{B_\sigma : \dim\sigma \le i\}$ with the topology induced by B_σ's and let $f^i\colon B^i \to K_1$ and $p^i\colon B^i \to A$ be the maps induced by f_σ's and p_σ's respectively. By ∂B_σ we mean $\partial B_\sigma = \bigcup\{B_{\sigma'} : \dim\sigma' = \dim\sigma - 1 \text{ and } \sigma' \subset \sigma\} \subset B^{\dim\sigma-1}$.

For a 0-simplex σ define $B_\sigma = \sigma$, $f_\sigma(B_\sigma) = f_1(\sigma)$ and $p_\sigma(B_\sigma) = \sigma$. Assume that for each $\sigma \subset A$ of $\dim \le i$, B_σ, f_σ and p_σ have been constructed and take $\sigma \subset A$ of $\dim = i+1$. Define B_σ as the mapping cylinder of $f^i|_{\partial B_\sigma}$. We will refer to ∂B_σ and K_1 as the 0 and 1 levels of B_σ respectively. Define p_σ as the map sending the 1-level of B_σ to the barycenter b_σ of σ and linearly extending $p^i|_{\partial B_\sigma}$ along the intervals connecting the 0-level points with the corresponding points of the 1-level. If $\sigma \subset A_1$ set $f_\sigma = f_1 \circ p_\sigma$. If σ is not contained in A_1 define f_σ as the natural extension of $f^i|_{\partial B_\sigma}$ which is constant on each interval connecting a 0-level point with the corresponding point of the 1-level and the identity on the 1-level.

Set $B = B^k$, $p = p^k$ where $k = \dim A$ and let us show that the required properties are satisfied. From the construction it follows that f^k is an extension of g_1 and there exists a countable triangulation of B for which B_2 is a subcomplex of B. Aiming at a contradiction assume that there exists an extension $g\colon B \to K_2$ of g_2. Denote by A^0 the quotient space of B obtained by identifying $p^{-1}(x)$ with a singleton for every $x \in A_2$, *i.e.*, we identify the points of $p^{-1}(A_2)$ with A_2 according to the map p. Note that since there is a finite subcomplex of B mapped by p onto A we indeed obtain from $p^{-1}(A_2)$ a space homeomorphic to A_2. Then A^0 is a CW-complex, A_2 can be considered as a subspace of A^0, and g and p factor through the maps $\psi^0\colon A^0 \to K_2$ and $h^0\colon A^0 \to A$. Note that $\psi^0|_{A_2} = f_2$.

We are going to construct CW-complexes $A^1, \dots, A^k$ and maps $h^i\colon A^i \to A$, $h^i_{i+1}\colon A^i \to A^{i+1}$ and $\psi^i\colon A^i \to K_2$ such that each A^i contains A_2 as a subspace, $h^i_{i+1}(A_2) = A_2$, $h^i(A_2) = A_2$ and $h^i_{i+1}|_{A_2}$, $h^i|_{A_2}$ are the identity maps, $h^i = h^{i+1} \circ h^i_{i+1}$ and, finally, $\psi^{i+1} \circ h^i_{i+1}$ is homotopic to ψ^i. The construction below will imply that $h^k\colon A^k \to A$ is a homeomorphism and this proves the proposition because for $\psi^k\colon A^k \to K_2$ we have that $\psi^k|_{A_2}$ is homotopic to f_2.

By a barycentric simplex $\beta = \beta(\sigma_0, \sigma_1, \dots, \sigma_n)$ we mean the simplex spanned by the barycenters $b_{\sigma_0}, \dots, b_{\sigma_n}$ of an increasing sequence $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$ of distinct simplexes in A with $\dim\sigma_0 > 0$. Note that $p^{-1}(x)$ is a singleton if x belongs to no barycentric simplex. We will define A^{i+1} as the quotient space of A^0 obtained by identifying the points of $(h^0)^{-1}(\beta)$ with β according to the map h^0 for each barycentric simplex β of $\dim \le i$. Then h^0 naturally factors through $h^{i+1}\colon A^{i+1} \to A$ and define h^i_{i+1} such that $h^i = h^{i+1} \circ h^i_{i+1}$.

Note that for a 0-dimensional barycentric simplex β, $X_\beta = p^{-1}(\beta)$ is homeomorphic to K_1. For an n-dimensional barycentric simplex $\beta = \beta(\sigma_0, \dots, \sigma_n)$,

$n > 0$, denote by X_β the space obtained from $p^{-1}(\beta)$ after identifying the points of $p^{-1}(\partial\beta)$ with $\partial\beta$ by the map p (where $\partial\beta = \bigcup\{\beta(\sigma_0,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n) : i = 0,\ldots,n\}$). Let us show by induction that X_β is homotopy equivalent to $\Sigma^n K_1$. Indeed, from the construction of B it follows that X_β is a CW-complex which can be obtained from $X_{\beta(\sigma_0,\ldots,\sigma_{n-1})} \times [0,1]$ by identifying the points of $X_{\beta(\sigma_0,\ldots,\sigma_{n-1})} \times \{0\}$ with $\beta(\sigma_0,\ldots,\sigma_{n-1})$ and the points of $X_{\beta(\sigma_0,\ldots,\sigma_{n-1})} \times \{1\}$ with a singleton. By the induction assumption $X_{\beta(\sigma_0,\ldots,\sigma_{n-1})}$ is homotopy equivalent to $\Sigma^{n-1}K_1$ and hence X_β is homotopy equivalent to $\Sigma^n K_1$.

Let β be an i-dimensional barycentric simplex which is not contained in A_2. It is easy to see that $(h^i)^{-1}(\beta)$ is homeomorphic to X_β and for $x \in A^{i+1}$ such that $h^{i+1}(x)$ belongs to no barycentric simplex of dim $= i$, $(h^i_{i+1})^{-1}(x)$ is a singleton. Since $[(h^i)^{-1}(\beta), K_2] = [X_\beta, K_2] = [\Sigma^i K_1, K_2] = 0$ and $(h^i)^{-1}(\beta)$ is a subcomplex of A^i, ψ^i is homotopic to a map which is constant on $(h^i)^{-1}(\beta)$ and hence which admits a factorization through $\psi^i_\beta\colon A^i_\beta \to K_1$ where the CW-complex A^i_β is obtained from A^i by identifying $(h^i)^{-1}(\beta)$ with β by h^i. Clearly h^i factors through $h^i_\beta\colon A^i_\beta \to A$ and h^i_β factors through $(h^i_{i+1})_\beta\colon A^i_\beta \to A^{i+1}$. The same procedure can be carried out for another i-dimensional barycentric complex not lying in A_2 but this time with respect to A^i_β, ψ^i_β, h^i_β and $(h^i_{i+1})_\beta$ instead of A^i, ψ^i, h^i and h^i_{i+1} respectively. Thus passing through all the i-dimensional barycentric simplexes not lying in A_2 we will end up with $\psi^{i+1}\colon A^{i+1} \to K_1$, $h^{i+1}\colon A^{i+1} \to A$ and $h^i_{i+1}\colon A^i \to A^{i+1}$ having the required properties. Clearly $A^k = A$ and the proposition follows. ∎

Proof of Theorem 1.2 (b) Let $f\colon P \to L$ be an essential map. Since T^0 is strongly continuous there exists a finite simplicial complex $N_0 \subset P$ such that the map $f_0 = f|_{N_0}\colon N_0 \to L$ is essential and hence f_0 cannot be extended over $M_0 = \text{cone}\, N_0$.

Assume that we have constructed for $i = 0, 1, \ldots, n$ finite simplicial pairs (M_i, N_i) and maps $p^{i+1}_i\colon (M_{i+1}, N_{i+1}) \to (M_i, N_i)$ such that $f_i = f_0 \circ (p^i_0|_{N_i})\colon N_i \to L$ does not extend over M_i where $p^i_j = p^{j+1}_j \circ p^{j+2}_{j+1} \circ \cdots \circ p^i_{i-1}\colon (M_i, N_i) \to (M_j, N_j)$ for $i > j$ and p^i_i is the identity.

Construct (M_{n+1}, N_{n+1}) and p^{n+1}_n as follows. Take a map $\psi\colon C \to K$ of a closed subset C of M_i for some $0 \le i \le n$. Define $\psi_n = \psi \circ (p^n_i|_{C_n})\colon C_n = (p^n_i)^{-1}(C) \to K$. By Proposition 2.2 there exist a countable simplicial complex M_{n+1}, a subcomplex N_{n+1} of M_{n+1} and a map $p^{n+1}_n\colon (M_{n+1}, N_{n+1}) \to (M_n, N_n)$ such that f_{n+1} cannot be extended over M_{n+1} and $\psi_{n+1} = \psi_n \circ (p^{n+1}_n|_{C_{n+1}})\colon C_{n+1} = (p^{n+1}_n)^{-1}(C_n) \to K$ admits an extension over M_{n+1}. By the strong continuity of T^0 we may assume that (M_{n+1}, N_{n+1}) is a pair of finite simplicial complexes.

Since for each M_i we need to solve only countably many extension problems the map ψ on each step of the construction can be chosen such that $X = \varprojlim(M_i, p^i_{i-1})$ will be of e-dim $\le K$. Let $p\colon X \to M_0$ be the projection and let $X' = p^{-1}(N_0)$. Then $p|_{X'}\colon X' \to N_0 \subset P$ cannot be extended over X as a map to P since otherwise for a sufficiently large i, $p^i_0|_{N_i}$ would extend over M_i as a map to P and this would imply that f_i also extends over M_i. Hence e-dim $X > P$ and the theorem follows. ∎

Proposition 2.3 *Assume that for a countable simplicial complex K_1 and a space K_2, $[\Sigma^n K_1, K_2] = 0$ for all $n \ge 1$. Let A be a finite simplicial complex, let $f_1\colon A_1 \to K_1$ be a*

https://doi.org/10.4153/CMB-2001-009-2 Published online by Cambridge University Press

map of a closed subset A_1 of A and let $f_2\colon A \to K_2$ be an essential map. Then there exists a countable simplicial complex B and a map $p\colon B \to A$ such that for $B_1 = p^{-1}(A_1)$ the map $g_1 = f_1 \circ (p|_{B_1})\colon B_1 \to K_1$ can be extended over B and the map $g_2 = f_2 \circ p\colon B \to K_2$ is essential.

Proof Note that if we assume that $[\Sigma^n K_1, K_2] = 0$ for all $n \geq 0$, then Proposition 2.3 would follow from Proposition 2.2 by embedding A into cone A. In order to avoid the use of $[K_1, K_2] = 0$ for proving Proposition 2.3 we need to make the following adjustments in the proof of Proposition 2.2.

Let B be constructed as in the proof of Proposition 2.2. Starting from the construction of A^0 replace A, B and p by $A' = \operatorname{cone} A$, $B' = \operatorname{cone} B$, $p' = \operatorname{cone}(p)\colon B' \to A'$ and consider A_2 as A embedded in $A' = \operatorname{cone} A$ and f_2 as a map of A_2.

In the part of the proof of Proposition 2.2 where $A^1, A^2, \dots$ are constructed replace σ, $\partial\sigma$, β by $\sigma' = \operatorname{cone}(\sigma)$, $\partial'\sigma = \operatorname{cone}(\partial\sigma)$, $\beta'(\sigma_0, \dots, \sigma_n) = \operatorname{cone}\bigl(\beta(\sigma_0, \dots, \sigma_n)\bigr)$ respectively. Then for $\beta' = \beta'(\sigma_0, \dots, \sigma_n)$, $X_{\beta'}$ will be homotopy equivalent to $\Sigma^{n+1} K_1$. Thus we show that if $[\Sigma^{n+1} K_1, K_2] = 0$ for $n \geq 0$ then g_2 cannot be extended over $A' = \operatorname{cone} A$ and hence g_2 is essential. ■

Proof of Theorem 1.2 (a) Let $f\colon P \to L$ be an essential map. By the continuity of T^0 there exists a finite simplicial complex $M_0 \subset P$ such that the map $f_0 = f|_{M_0}\colon M_0 \to L$ is essential.

Assume that we have constructed for $i = 0, 1, \dots, n$ finite simplicial complexes M_i and maps $p_i^{i+1}\colon M_{i+1} \to M_i$, such that $f_i = f_0 \circ p_0^i\colon M_i \to L$ is essential where $p_j^i = p_j^{j+1} \circ p_{j+1}^{j+2} \circ \dots \circ p_{i-1}^i\colon M_i \to M_j$ for $i > j$ and p_i^i is the identity.

Construct M_{n+1} and p_n^{n+1} as follows. Take a map $\psi\colon C \to K$ of a closed subset C of M_i for some $0 \leq i \leq n$. Define $\psi_n = \psi \circ (p_i^n|_{C_n})\colon C_n = (p_i^n)^{-1}(C) \to K$. By Proposition 2.3 there exist a countable simplicial complex M_{n+1} and a map $p_n^{n+1}\colon M_{n+1} \to M_n$ such that f_{n+1} is essential and $\psi_{n+1} = \psi_n \circ (p_n^{n+1}|_{C_{n+1}})\colon C_{n+1} = (p_n^{n+1})^{-1}(C_n) \to K$ admits an extension over M_{n+1}. By the continuity of T^0 we may assume that M_{n+1} is a finite simplicial complex.

Since for each M_i we need to solve only countably many extension problems the map ψ on each step of the construction can be chosen such that $X = \varprojlim(M_i, p_{i-1}^i)$ will be of e-dim $\leq K$. Let $p\colon X \to M_0 \subset P$ be the projection. p is essential as a map to P since otherwise for a sufficiently large i, p_0^i would be null-homotopic as a map to P and this would imply that f_i is also null-homotopic. The theorem is proved. ■

Remark Note that in the proof of Theorem 1.3 (a) the map $f_X = f_0 \circ p\colon X \to L$ is essential if L is a CW-complex. Also note that with no restriction on L, f_X may fail to be essential.

References

[1] A. Dranishnikov, *Basic elements of the cohomological dimension theory of compact metric spaces.* Topology Atlas.

[2] ———, *On the mapping intersection problem.* Pacific J. Math. (2) **173**(1996), 403–412.

[3] J. Dydak and J. John Walsh, *Infinite-dimensional compacta having cohomological dimension two: an application of the Sullivan conjecture.* Topology (1) **32**(1993), 93–104.
[4] H. Miller, *The Sullivan conjecture on maps from classifying spaces.* Ann. of Math. (2) **120**(1984), 39–87.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305
Japan
email: mlevin@math.tsukuba.ac.jp