Y. Ikeda
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WEAK SOLUTIONS OF A QUASI-LINEAR DEGENERATE ELLIPTIC SYSTEM WITH DISCONTINUOUS COEFFICIENTS

YOSHIAKI IKEDA

§1. Introduction

We shall discuss regularities and related topics on weak solutions of the system of the following quasi-linear elliptic differential equations (a combination of almost single equations)

$$\begin{aligned} &-\operatorname{div} A_j(x, u, \nabla u_j) + B_j(x, u, \nabla u_j) = 0 \\ &\qquad (j = 1, 2, \cdots, m) \quad u = (u_1, \cdots, u_m), \end{aligned} \tag{1.1}$$

in a bounded domain Ω in R^n $(n \geqq 2)$, where $A_j = (A_{1j}, \cdots, A_{nj})$ are given vector functions of $(x, u, \nabla u_j)$, B_j are scalar functions of the same variables, and $\nabla u_j = (\partial u_j/\partial x_1, \cdots, \partial u_j/\partial x_n)$ denote the gradients of the $u_j = u_j(x)$ $(j = 1, \cdots, m)$. We assume that there exists some $\alpha \geqq 2$ such that each A_j and B_j satisfy the inequalities

$$\begin{cases} \xi \cdot A_j(x, u, \xi) \geqq a_j(x)|\xi|^\alpha - \sum_{i=1}^m c_{ij}(x)|u_i|^\alpha - f_j(x), \\ |B_j(x, u, \xi)| \leqq b_j(x)|\xi|^{\alpha-1} + \sum_{i=1}^m d_{ij}(x)|u_i|^{\alpha-1} + g_j(x), \\ |A_j(x, u, \xi)| \leqq \tilde{a}_j(x)|\xi|^{\alpha-1} + \sum_{i=1}^m e_{ij}(x)|u_i|^{\alpha-1} + h_j(x), \end{cases} \tag{1.2}$$

for any $\xi \in R^n$. The functions $a_j, b_j, c_{ij}, \cdots, h_j$ and $\tilde{a}_j$, call them the coefficients of the structure (1.2), are all assumed to be non-negative and measurable.

Moreover we assume that

$$a_j^{-1} \in L^t(\Omega) \qquad \text{for any } t > 1,$$
$$a_j \leqq \tilde{a}_j, \quad j = 1, 2, \cdots, m,$$

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$$\tilde{a}_j,\ \tilde{a}_j^\alpha a_j^{1-\alpha},\ b_j^\alpha a_j^{1-\alpha},\ c_{ij},\ d_{ij}, f_j, g_j \in L^{p/\alpha}(\Omega),$$
$$e_{ij}^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)},\ h_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)} \in L^{p/\alpha}(\Omega)$$

where

$$\frac{\alpha}{p}+\frac{1}{t}<\frac{\alpha}{n} \quad \text{and} \quad (2\leqq)\,\alpha<p\,. \tag{1.3}$$

The class of Partial Differential Equations (1.1) that we are going to discuss involves many interesting equations whose solutions are known ([7, 10]). The purpose of this paper is to establish a systematic approach to the investigation of the solution, which may be weak solutions, of general equations in the class in question. Namely we shall discuss the topics 1) maximum principle 2) local boundedness 3) Hölder continuity 4) Harnack type inequality for the solutions.

We shall prove, under very general assumptions described above, the following theorems.

THEOREM A. *Let* $u=(u_1,\cdots,u_m)$ *be a weak solution of* (1.1) *such that* $u=M$ *on the boundary* $\partial\Omega$ *of* Ω *with* $M=(M_1,\cdots,M_m)\in R^m$, *then it holds that*

$$\sup_\Omega |u_j| \leqq C\sum_{i=1}^m \left\{\left(\int_\Omega |u_i|^\alpha dx\right)^{1/\alpha} + \left(\int_\Omega f_i^{p/\alpha} dx\right)^{1/p} + \left(\int_\Omega g_i^{p/\alpha} dx\right)^{\alpha/(\alpha-1)p} + |M_i|\right\},$$

where C *is a positive constant depending only on* n, p, t, α *and the coefficients of the structure* (1.2).

THEOREM B. *Let* $u=(u_1,\cdots,u_m)$ *be a weak solution of* (1.1) *and* $I(x_0,\rho)$ *an open ball with radius* ρ *and center at* x. *If* $I(x_0.\ 2\rho_0)\subset\Omega$, *then*

$$\sup_{I(x_0,\rho)} |u_j| \leqq C\sum_{i=i}^m \left\{\left(\rho_0^{-n}\int_{I(x_0,2\rho_0)} |u_i|^{\alpha p/(p-\alpha)} dx\right)^{(p-\alpha)/\alpha p} + \rho_0^{\ n(p-\alpha)/\alpha p}\kappa_i\right\},$$

where

$$\kappa_i = \left(\int_{I(2\rho_0)} f_i^{p/\alpha} dx\right)^{1/p} + \left(\int_{I(2\rho_0)} g_i^{p/\alpha} dx\right)^{\alpha/(\alpha-1)p} + \left(\int_{I(2\rho_0)} (h_i^{\alpha/(\alpha-1)} a_i^{1/(1-\alpha)})^{p/\alpha} dx\right)^{\alpha/(\alpha-1)p}.$$

We then proceed to prove the Hölder continuity for the weak solutions of (1.1) under the additional assumption that $\tilde{a}_j a_j^{-1}$ are bounded.

THEOREM C. *The weak solutions of* (1.1) *are locally Hölder contiunous in* Ω.

If a solution is constant on $\partial\Omega$, then it is globally Hölder continuous in Ω.

The Harnack type inequality can be proved under additional assumptions on the coefficients that are prescribed in Section 6.

We now briefly summarise the contents of each sections. Section 2 is devoted to state some lemmas which will often be used later. We shall prove the maximum principle (Theorem A) and the theorems on local boundedness (Theorem B) in Section 3 and Section 4, respectively. By using these results we shall prove in Section 5 the Hölder continuity (Theorem C). The Harnack type inequality for positive weak solutions will be obtained in Section 6. In the proofs of these results the techniques in Moser [4] and Stampacchia [8, 9] are often used.

We now pause to give some historical notes on the development of the works in this line.

J. Moser [4] and G. Stampacchia [8] first proved all these properties for linear elliptic equations of the form

$$-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j}=\sum_{i=1}^{n}(f_i)_{x_i}\,.$$

Then, G. Stampacchia [9] extended these results for the strictly elliptic equations of the form

$$(1.4)\qquad -\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i}+d_iu)_{x_j}+\sum_{i=1}^{n}(b_i(x)u_{x_i}+c(x)u)=\sum_{i=1}^{n}(f_i)_{x_i}\,,$$

which are still linear.

While, J. Serrin [7] and N.S. Trudinger [10] proved the same results for weak solutions of a quasi-linear elliptic equations. These results are particular cases of our theorems, where $m=1$ and $a=a_j=$ constant in (1.1).

Another developments were made by M.K.V. Murthy and G. Stampacchia [5] and N.S. Trudinger [11] for the linear elliptic equations (1.4) in the case where the coefficients may be degenerated. F. Mandras [1, 2, 3] considered the same problem in the case of a linear degenerate elliptic system.

Our results are actually viewed as a generalization of the above works, and our assumptions seems to be very general in order to prove the maximum principle, local boundedness and Hölder continuity.

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§ 2. Preliminaries

In this section we shall state and prove several lemmas related to the imbedding theorems.

First of all, we shall define some function spaces; let $m(x)$ be a non-negative measurable function in Ω and $m^{-1} \in L^t(\Omega)$ with $\alpha > 1 + 1/t$, $t > 1$. The space $H^{1,\alpha}(m, \Omega)$ and $H_0^{1,\alpha}(m, \Omega)$ are the completions of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ with the norm

$$\|v\|_{H^{1,\alpha}(m,\Omega)}^\alpha = \int_\Omega m|\nabla v|^\alpha dx + \int_\Omega |v|^\alpha dx\,,$$

respectively. The Sobolev spaces appear as particular cases: $H^{1,\alpha}(1, \Omega) = H^{1,\alpha}(\Omega)$ and $H_0^{1,\alpha}(1, \Omega) = H_0^{1,\alpha}(\Omega)$.

Throughout this paper, we denote by $\|f\|_p$ the $L^p(\Omega)$-norm.

LEMMA 2.1 ([6]). *Let $n \geqq 2$. If $v \in H_0^{1,p}(\Omega)$ $(1 < p)$, then*

$$\|v\|_{p^*} \leqq c_0 \|\nabla v\|_p\,.$$

Here $1/p^* = 1/p - 1/n$ if $p < n$, and p^* may be taken to be any positive number if $p \geqq n$. The constant c_0 depends on n and p^*. (If $p < n$, c_0 depends only on n.)

LEMMA 2.2. *Let $n \geqq 2$. If $v \in H_0^{1,\alpha}(m, \Omega)$, then $v \in L^{\alpha^\#}(\Omega)$ and the following inequality holds*

$$\|v\|_{\alpha^\#}^\alpha \leqq c_0 \|m^{-1}\|_t \int_\Omega m|\nabla v|^\alpha dx.$$

Here $1/\alpha^\# = (1/\alpha)(1 + 1/t) - 1/n$ if $(1/\alpha)(1 + 1/t) > 1/n$, and $\alpha^\#$ may be taken to be any positive number >1 if $(1/\alpha)(1 + 1/t) \leqq 1/n$. The constant c_0 is the same as in Lemma 2.1.

Proof. Let $(1/\alpha)(1 + 1/t) > 1/n$. Then putting $1/p = (1/\alpha)(1 + 1/t)$, we have $1/p^* = 1/\alpha^\#$. Since $v \in H_0^{1,p}(\Omega)$, by Lemma 2.1 and Hölder's inequality, we have

$$\|v\|_{\alpha^\#}^\alpha \leqq c_0 \|\nabla v\|_p^\alpha \leqq c_0 \|m^{-1}\|_t \int_\Omega m|\nabla v|^\alpha dx\,.$$

Next let $1/\alpha(1 + 1/t) \leqq 1/n$. Then for any positive number $\alpha^\# > 1$, we take β satisfying the equality $1/\alpha^{\#} = (1/\beta)(1+1/t) - 1/n$. Since $1 < \beta < \alpha$, using Lemma 2.1 and Hölder's inequality, we see

$$\|v\|_{\alpha^{\#}}^{\alpha} \leqq c_0^{1/\beta}\left(\int_\Omega |\nabla v|^{\beta t/(t+1)}dx\right)^{\alpha(t+1)/\beta t}$$
$$\leqq C\left(\int_\Omega |\nabla v|^{\alpha t/(t+1)}dx\right)^{(t+1)/t} \leqq C\|m^{-1}\|_t \int_\Omega m|\nabla v|^\alpha dx. \qquad \text{Q.E.D.}$$

The following lemma is easily obtained by using Hölder's inequality.

LEMMA 2.3. *Let r and s be such that $1 \leqq r \leqq s$. If $v \in L^s(\Omega)$, then for any p with*

$$\frac{\lambda}{s} + \frac{\mu}{r} = \frac{1}{p} \quad (\lambda, \mu > 0,\ \lambda + \mu = 1),$$

we have

$$\|v\|_p \leqq \|v\|_s^\lambda \|v\|_r^\mu .$$

LEMMA 2.4. *If $v \in H_0^{1,\alpha}(m, \Omega)$, then $v \in L^{\alpha p/(p-\alpha)}(\Omega)$ for any $p > 0$ with $\alpha/n > 1/t + \alpha/p$. Moreover, for any positive number ε, there exists a constant K depending only on $n, p, t, \alpha, \varepsilon$ and $\|m^{-1}\|_t$ satisfying the following inequality*

$$\|v\|_{\alpha p/(p-\alpha)}^\alpha \leqq \varepsilon \int_\Omega m|\nabla v|^\alpha dx + K\int_\Omega |v|^\alpha dx .$$

Here we may take

$$K = (c_0\|m^{-1}\|_t \varepsilon)^{-\gamma},\ \gamma = \frac{1-\lambda}{\lambda} \ \textit{with}\ \lambda = \frac{1}{p}\Big/\left(\frac{1}{\alpha} - \frac{1}{\alpha^{\#}}\right) < 1 .$$

Proof. Let r and s be real numbers $\geqq 1$. For $\lambda = \lambda(r, s)$ such that $0 < \lambda < 1$ and $\lambda/s + (1-\lambda)/r = (p-\alpha)/(\alpha p)$. We have by Lemma 2.3 and Young's inequality

$$\|v\|_{\alpha p/(p-\alpha)}^\alpha \leqq \|v\|_s^{\alpha\lambda}\|v\|_r^{\alpha(1-\lambda)} \leqq \lambda\varepsilon'\|v\|_s^\alpha + (1-\lambda)K'\|v\|_r^\alpha, \quad K' = \varepsilon'^{-\lambda/(1-\lambda)}$$

for any $\varepsilon' > 0$.

If $(1/\alpha)(1+1/t) > 1/n$, we set $s = \alpha^{\#}$ and $r = \alpha$ to obtain $0 < \lambda = (1/p)/(1/\alpha - 1/\alpha^{\#}) < 1$. while, if $(1/\alpha)(1+1/t) \leqq 1/n$, we take such s that $1/\alpha - 1/p > 1/s$ to obtain $0 < \lambda = (1/p)/(1/\alpha - 1/s) < 1$. With these choices of s and r Lemma 2.2 implies

$$\|v\|_s^\alpha \leqq c_0\|m^{-1}\|_t \int_\Omega m|\nabla v|^\alpha dx.$$

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Putting $\varepsilon' = \varepsilon\|m^{-1}\|_t$ we obtain the desired conclusion. Q.E.D.

LEMMA 2.5 ([10]). *Let A be a bounded open convex set in R^n and let $v \in H^{1,p}(A)$ $(1 < p < n)$. Then*

$$\left\{\int_A |v - v_A|^{p^*} dx\right\}^{1/p^*} \leqq K \frac{(\operatorname{diam} A)^n}{\operatorname{meas} A} \left\{\int_A |\nabla v|^p dx\right\}^{1/p},$$

where $v_A = \dfrac{1}{\operatorname{meas} A}\displaystyle\int_A v dx$ and where K is a constant depending only on n and p.

LEMMA 2.6. *Let $v \in H^{1,\alpha}(m, \Omega)$, $\alpha/n > 1/t + \alpha/p$ and assume that*

$$\frac{1}{\operatorname{meas} I(\rho)} \int_{I(\rho)} v dx = 0$$

for an open ball $I(\rho) \subset \Omega$. Then

$$\left(\rho^{-n} \int_{I(\rho)} |v|^{\alpha p/(p-\alpha)} dx\right)^{(p-\alpha)/p} \leqq C(\rho^{-n/t}\|m^{-1}\|_t)\rho^{\alpha-n} \int_{I(\rho)} m|\nabla v|^\alpha dx,$$

where C is a constant depending on n, p, t, α and $\alpha^\#$.

Proof. If $(1/\alpha)(1 + 1/t) > 1/n$, Lemma 2.5 and Hölder's inequality imply

$$\begin{aligned}\left(\rho^{-n} \int_{I(\rho)} |v|^{\alpha p/(p-\alpha)} dx\right)^{(p-\alpha)/p} &\leqq \rho^{-n(p-\alpha)/p}\left(\int_{I(\rho)} |v|^{\alpha^\#} dx\right)^{\alpha/\alpha^\#} [\operatorname{meas} I(\rho)]^{(p-\alpha)/p-\alpha/\alpha^\#} \\ &\leqq C\rho^{-\alpha n/\alpha^\#}\rho^{n/t}(\rho^{-n/t}\|m^{-1}\|_t) \int_{I(\rho)} m|\nabla v|^\alpha dx,\end{aligned}$$

where $-(\alpha n/\alpha^\#) + n/t = \alpha - n$.

If $(1/\alpha)(1 + 1/t) \leqq n/1$, we choose such β that $(p - \alpha)/(\alpha p) > (1/\beta)(1 + 1/t) - 1/n > 0$. Then, $\alpha > \beta$ and $\alpha^\# > \alpha p/(p - \alpha)$. Thus, we see

$$\begin{aligned}\left(\int_{I(\rho)} |v|^{\beta\#} dx\right)^{\alpha/\beta\#} &\leqq C\left(\int_{I(\rho)} |\nabla v|^{\beta t/(t+1)} dx\right)^{\alpha(t+1)/\beta t} \\ &\leqq C\left(\int_{I(\rho)} |\nabla v|^{\alpha t/(t+1)} dx\right)^{(t+1)/t} [\operatorname{meas} I(\rho)]^{(1-\beta/\alpha)\alpha(t+1)/\beta t} \\ &\leqq C\rho^{n(1-\beta/\alpha)\alpha(t+1)/\beta t}\|m^{-1}\|_t \int_{I(\rho)} m|\nabla v|^\alpha dx.\end{aligned}$$

Therefore we have by Hölder's inequality

$$\begin{aligned}\left(\rho^{-n} \int_{I(\rho)} |v|^{\alpha p/(p-\alpha)} dx\right)^{p/(p-\alpha)} &\leqq C\rho^{-\alpha/\beta^\#}\left(\int_{I(\rho)} |v|^{\beta^\#} dx\right)^{\alpha/\beta^\#} \\ &\leqq C\rho^{\alpha-n}(\rho^{-n/t}\|m^{-1}\|_t) \int_{I(\rho)} m|\nabla v|^\alpha dx.\end{aligned}$$

Q.E.D.

LEMMA 2.7. ([5]). *Let $G(x)$ be a uniformly Lipshitz function on R^1 such that $G(0)=0$. If $v(x)\in H_0^{1,\alpha}(m,\Omega)$, then $G(v(x))$ again belongs to $H_0^{1,\alpha}(m,\Omega)$. Further, if the derivative G' of G is continuous except a finite number of points in R^1, then we have $G(v)_{x_i}=G'(v)v_{x_i}$ in the sense of distribution.*

§ 3. Global estimates

In this section we shall prove the maximum principle for weak solutions of the system (1.1).

DEFINITION. Let

$$H^{1,\alpha}(\tilde a,\Omega)=\prod_{j=1}^{m}H^{1,\alpha}(\tilde a_j,\Omega)\quad\text{and}\quad H_0^{1,\alpha}(\tilde a,\Omega)=\prod_{j=1}^{m}H_0^{1,\alpha}(\tilde a_j,\Omega)\,.$$

We say that $u=(u_1,\cdots,u_m)$ is a weak solution of the system (1.1), if $u\in H^{1,\alpha}(\tilde a,\Omega)$ and if the equation

$$\sum_{j=1}^{m}\int_\Omega\{\nabla\Phi_j\cdot A_j(x,u,\nabla u_j)+\Phi_jB_j(x,u,\nabla u_j)\}dx=0 \tag{3.1}$$

holds for any $\Phi=(\Phi_1,\cdots,\Phi_m)\in C_0^\infty(\Omega)\times\cdots\times C_0^\infty(\Omega)$.

By Lemma 2.4, if $u\in H^{1,\alpha}(\tilde a,\Omega)$, then u is locally in $L^{\alpha p/(p-1)}(\Omega)\times\cdots\times L^{p/(p-\alpha)}(\Omega)$ for any $p>0$ with $\alpha/n>\alpha/p+1/t$ and $2\leqq\alpha<p$. Thus, from the assumption on the coefficients of the structure (1.2), we see that for any $\Phi\in H^{1,\alpha}(\tilde a,\Omega)$ with compact support in Ω

$$\sum_{j=1}^{m}\int_\Omega\{|\Phi_jA_j(x,u,\nabla u_j)|+|\Phi_jB_j(x,u,\nabla u_j)|\}dx<\infty\,.$$

Therefore it follows that if u is a weak solution of (1.1), then (3.1) holds not only for $\Phi\in C_0^\infty(\Omega)\times\cdots\times C_0^\infty(\Omega)$, but in fact for any $\Phi\in H^{1,\alpha}(\tilde a,\Omega)$ with compact support in Ω.

For a function $u=(u_1,\cdots,u_m)$ belonging to $H^{1,\alpha}(\tilde a,\Omega)$, we shall simply say $u=0$ on the boundary $\partial\Omega$ of Ω if u_j, $j=1,2,\cdots,m$, belong to the space $H_0^{1,\alpha}(\tilde a_j,\Omega)$. Similarly, $u=M$ for $M=(M_1.\cdots,M_m)\in R^m$ on a boundary $\partial\Omega$ of Ω, if $u_j-M_j\in H_0^{1,\alpha}(\tilde a_j,\Omega)$, $j=1,2,\cdots,m$.

THEOREM 3.1. *Let $u=(u_1,\cdots,u_m)$ be a weak solution of (1.1) such that $u=0$ on $\partial\Omega$. Then there exists a constant C depending only on n,p,t,α and the coefficients of the structure (1.2) such that*

$$\sup_\Omega|u_j|\leqq\sum_{i=1}^{m}\sup_\Omega|u_i|\leqq C\sum_{i=1}^{m}(\|u_i\|_\alpha+\|f_i\|_{p/\alpha}^{1/\alpha}+\|g_i\|_{p/\alpha}^{1/(\alpha-1)})\,. \tag{3.2}$$

Proof. Without loss of generality, we may assume that meas $\Omega = 1$. For, if meas $\Omega \neq 1$, then we can introduce new variable $x' = x(\text{meas}\,\Omega)^{1/n}$, so that u satisfies the system of the form (1.1) in a domain Ω' with meas $\Omega' = 1$.

Now we put $\kappa_j = \|f_j\|_{p/\alpha}^{1/\alpha} + \|g_j\|_{p/\alpha}^{1/(\alpha-1)} + \varepsilon$ for any positive number ε and $\bar{u}_j = |u_j| + \kappa_j$, and define the functions

$$G_j(u_j) = \begin{cases} \bar{u}_j^{\alpha q-\alpha+1} - \kappa_j^{\alpha q-\alpha+1} & \text{for } |u_j| \leqq \ell_j - \kappa_j\,, \\ \ell_j^{\alpha q-\alpha}\bar{u}_j - \kappa_j^{\alpha q-\alpha+1} & \text{for } |u_j| \geqq \ell_j - \kappa_j\,, \end{cases}$$

$j = 1, 2, \cdots, m$, where $q \geqq 1$ is any fixed number and ℓ_j are constants greater than κ_j.

Next, we define $\Phi_j = G_j \operatorname{sign}(u_j)$, $j = 1, \cdots, m$. It is clear that $\Phi = (\Phi_1, \cdots, \Phi_m) \in H_0^{1,\alpha}(\tilde{a}, \Omega)$. Thus, we have

$$\sum_{j=1}^{m}\int_{\Omega}\{\nabla\Phi_j\cdot A_j + \Phi_j B_j\}dx = 0\,.$$

If we put

$$H_j(u_j) = \begin{cases} (\alpha q - \alpha + 1)\bar{u}_j^{\alpha q-\alpha} & \text{for } |u_j| \leqq \ell_j - \kappa_j\,, \\ \ell_j^{\alpha q-\alpha} & \text{for } |u_j| \geqq \ell_j - \kappa_j\,, \end{cases}$$

then we see

$$\nabla\Phi_j = H_j\nabla u_j\,, \qquad |\nabla u_j| = |\nabla\bar{u}_j| \quad \text{and} \quad G_j \leqq \bar{u}_j^q H_j^{(\alpha-1)/\alpha}\,.$$

Therefore, by (1.2), we have

$$\begin{aligned} \sum_{j=1}^{m}\int_{\Omega} H_j a_j |\nabla u_j|^\alpha dx \leqq \sum_{j=1}^{m}\int_{\Omega}\Big\{ & G_j b_j |\nabla\bar{u}_j|^{\alpha-1} + H_j\sum_{i=1}^{m} c_{ij}\bar{u}_i^\alpha \\ & + G_j\sum_{i=1}^{m} d_{ij}\bar{u}_i^{\alpha-1} + H_j f_j + G_j g_j\Big\}dx\,, \end{aligned} \tag{3.3}$$

and

$$G_j b_j |\nabla\bar{u}_j|^{\alpha-1} \leqq \frac{1}{2}H_j a_j|\nabla\bar{u}_j|^\alpha + 2^{\alpha-1}b_j^\alpha a_j^{1-\alpha}\bar{u}_j^{\alpha q}\,.$$

Moreover, since $G_j \leqq \bar{u}_j^{\alpha q-\alpha+1}$, $H_j \leqq (\alpha q - \alpha + 1)\bar{u}_j^{\alpha q-\alpha}$, $\bar{u}_j^{\alpha q-\alpha+1}\bar{u}_i^{\alpha-1} \leqq \bar{u}_j^{\alpha q} + \bar{u}_i^{\alpha q}$ and $\bar{u}_j^{\alpha q-\alpha}\bar{u}_i^\alpha \leqq \bar{u}_j^{\alpha q} + \bar{u}_i^{\alpha q}$, we have

$$H_j\sum_{i=1}^{m} c_{ij}\bar{u}_i^\alpha + G_j\sum_{i=1}^{m} d_{ij}\bar{u}_i^{\alpha-1} \leqq (\alpha q - \alpha + 1)\sum_{i=1}^{m}(c_{ij} + d_{ij})(\bar{u}_i^{\alpha q} + \bar{u}_j^{\alpha q})\,.$$

We also have

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$$H_j f_j + G_j g_j \leqq (\alpha q - \alpha + 1)\left(\frac{f_j}{\kappa_j^\alpha} + \frac{g_j}{\kappa_j^{\alpha-1}}\right)\bar{u}_j^{\alpha q}$$

since $\kappa_j \leqq \bar{u}_j$.

Therefore, it follows from (3.3) that

$$\sum_{j=1}^m \int_\Omega H_j a_j |\nabla \bar{u}_j|^\alpha dx \leqq C(\alpha q - \alpha + 1) \sum_{j=1}^m \int_\Omega B_j(x) \bar{u}_j^{\alpha q} dx\,, \tag{3.4}$$

where

$$B_j(x) = b_j^\alpha a_j^{1-\alpha} + \sum_{i=1}^m (c_{ij} + c_{ji} + d_{ij} + d_{ji}) + \frac{f_j}{\kappa_j^\alpha} + \frac{g_j}{\kappa_j^{\alpha-1}} \in L^{\alpha/p}(\Omega)$$

and C is a constant depending only on α.

The right-hand side of (3.4) is independent of $\ell_j \geqq 0$ $(j = 1, \cdots, m)$. Since H_j are non-decreasing functions and since $\lim_{\ell_j \to \infty} H_j = (\alpha q - \alpha + 1)\bar{u}_j^{\alpha q}$, the monotone convergence theorem proves that

$$\sum_{j=1}^m \int_\Omega a_j(\bar{u}_j^{q-1}|\nabla \bar{u}_j|)^\alpha dx \leqq C \sum_{j=1}^m \int_\Omega B_j(x) \bar{u}_j^{\alpha q} dx\,.$$

Since $|\nabla(\bar{u}_j^q)| = q\bar{u}_j^{q-1}|\nabla \bar{u}_j|$, putting $v_j = \bar{u}_j^q$, we have

$$\sum_{j=1}^m \int_\Omega a_j |\nabla v_j|^\alpha dx \leqq Cq^\alpha \sum_{j=1}^m \int_\Omega B_j(x) v_j^\alpha dx\,.$$

By Hölder's inequality we have

$$\int_\Omega B_j(x) v_j^\alpha dx \leqq \|B_j\|_{p/\alpha} \|v_j\|_{\alpha p/(p-\alpha)}^\alpha \leqq \|B_j\|_{p/\alpha}(\|v_j - \kappa_j^q\|_{\alpha p/(p-\alpha)} + \kappa_j^{\alpha q})\,.$$

Since $v_j - \kappa_j^q \in H_0^{1,\alpha}(a_j, \Omega)$ and $v_i - \kappa_j^\alpha \leqq v_j$, by Lemma 2.4, we have

$$\|v_j - \kappa_j^q\|_{\alpha p/(p-\alpha)}^\alpha \leqq 2^{-1}C^{-1}q^{-\alpha}\|B_i\|_{p/\alpha}^{-1} \int_\Omega a_j |\nabla v_j|^\alpha dx + Cq^{\alpha\tau}\|v_j\|_\alpha^\alpha\,,$$

and since $\kappa_j^q \leqq v_j$ and meas $\Omega = 1$, it follows that $\kappa_j^{\alpha q} \leqq \|v_j\|_\alpha^\alpha$. Thus we have

$$\sum_{j=1}^m \int_\Omega a_j |\nabla v_j|^\alpha dx \leqq Cq^{\alpha\tau} \sum_{j=1}^m \|v_j\|_\alpha^\alpha\,, \tag{3.5}$$

where C is a constant depending only on n, p, t, α and $\|B_j\|_{p/\alpha}$, $(j = 1, \cdots, m)$.

By Lemma 2.2 and (3.5) we see the inequalities

$$\left\|\sum_{j=1}^m v_j\right\|_{\alpha^\#}^\alpha \leqq C\sum_{j=1}^m (\|v_j-\kappa_j^q\|_{\alpha^\#}^\alpha + \|v_j\|_\alpha^\alpha)$$
$$\leqq C\sum_{j=1}^m \left(\int_\Omega a_j|\nabla v_j|^\alpha dx + \|v_j\|_\alpha^\alpha\right)$$
$$\leqq Cq^{\alpha\gamma}\left\|\sum_{j=1}^m v_j\right\|_\alpha^\alpha,$$

which prove

$$\left(\int_\Omega\left(\sum_{j=1}^m \bar{u}_j\right)^{\alpha q\alpha^\#/\alpha}dx\right)^{1/\alpha^\# q} \leqq C^{1/\alpha q}q^{\gamma(1/q)}\left(\int_\Omega\left(\sum_{j=1}^m \bar{u}_j\right)^{\alpha q}dx\right)^{1/\alpha q}.$$

Thus, putting $r=\alpha^\#/\alpha$ and $q=r^s$, $s=0,1,2,\cdots$, we have

$$\left\|\sum_{j=1}^m \bar{u}_j\right\|_{\alpha r^{s+1}} \leqq C^{(1/\alpha)r^s}r^{\gamma s/r^s}\left\|\sum_{j=1}^m \bar{u}_j\right\|_{\alpha r^s}$$

(3.6)
$$\leqq C^{(1/\alpha)(\Sigma_{s=0}^\infty 1/r^s)}r^{\gamma(\Sigma_{s=0}^\infty s/r^s)}\left\|\sum_{j=1}^m \bar{u}_j\right\|_\alpha$$
$$\leqq C\sum_{j=1}^m\|\bar{u}_j\|_\alpha.$$

Now, let s tend to infinity to have

$$\sup_\Omega \sum_{j=1}^m \bar{u}_j \leqq C\sum_{j=1}^m \|\bar{u}_j\|_\alpha,$$

Note that $\|f_j/\kappa_j^\alpha + g_j/\kappa_j^{\alpha-1}\|_{p/\alpha} \leqq 1$ for any $\varepsilon>0$. Therefore letting ε tend to zero, we have (3.2). Q.E.D.

THEOREM 3.2. *Let $u=(u_1,\cdots,u_m)$ be a weak solution of* (1.1) *such that $u=M$ on $\partial\Omega$ for $M=(M_1,\cdots,M_m)\in R^m$. Then it holds*

(3.7) $$\sup_\Omega|u_j| \leqq C\sum_{i=1}^m (\|u_i\|_\alpha + \|f_i\|_{p/\alpha}^{1/\alpha} + \|g_i\|_{p/\alpha}^{1/(\alpha-1)} + |M_i|),$$

where C is a constant depending only on n, p t, α and the coefficients of the structure (1.2).

Proof. Consider $U=u-M=(u_1-M_1,\cdots,u_m-M_m)$. Then $U=0$ on $\partial\Omega$. Since u is a weak solution of (1.1), we must have

$$-\operatorname{div}\tilde{A}_j(x,U,\nabla U_j)+\tilde{B}_j(x,U,\nabla U_j)=0 \quad j=1,2,\cdots,m,$$

where

$$\tilde{A}_j(x,U,\nabla U_j)=A_j(x,U+M,\nabla U_j) \text{ and } \tilde{B}_j(x,U,\nabla U_j)=B_j(x,U+M,\nabla U_j).$$

In view of (1.2)

$$\begin{cases} \xi \cdot \tilde{A}_j(x, U, \xi) \geqq a_j|\xi|^\alpha - \sum_{i=1}^m c_{ij}|U_j + M_j|^\alpha - f_j, \\ |B_j(x, U, \xi)| \leqq b_j|\xi|^{\alpha-1} + \sum_{i=1}^m d_{ij}|U_i + M_i|^{\alpha-1} + g_j, \end{cases}$$

$j = 1, 2, \cdots, m$, for any $\xi \in R^n$. From these, we immediately have the following;

There exist positive constants λ and μ depending only on α such that for any $\xi \in R^n$

$$\begin{cases} \xi \cdot \tilde{A}_j(x, U, \xi) \geqq a_j|\xi|^\alpha - \sum_{i=1}^m \lambda c_{ij}|U_i|^\alpha - F_j, \\ |\tilde{B}_j(x, U, \xi)| \leqq b_j|\xi|^{\alpha-1} + \sum_{i=1}^m \mu d_{ij}|U_i|^{\alpha-1} + G_j, \end{cases}$$

where

$$F_j = f_j + \sum_{i=1}^m \lambda M_i^\alpha c_{ij}, \qquad G_j = g_j + \sum_{i=1}^m \mu M_i^{\alpha-1} d_{ij} \in L^{p/\alpha}(\Omega).$$

Therefore we can apply (3.6) to obtain

$$\sup_\Omega \bar{U}_j \leqq \sup_\Omega \left(\sum_{i=1}^m \bar{U}_j\right) \leqq C \sum_{i=1}^m \|\bar{U}_j\|_\alpha$$

for $\bar{U}_j = |U_j| + \|F_j\|_{p/\alpha}^{1/\alpha} + \|G_j\|_{p/\alpha}^{1/(\alpha-1)} + \varepsilon$. Since constant C does not depend on $\|F_j\|_{p/\alpha}^{1/\alpha} + \|G_j\|_{p/\alpha}^{1/(\alpha-1)} + \varepsilon$, we obtain (3.7), by letting ε tend to zero.

Q.E.D.

In the particular case where $m = 1$, more sharp results can be obtained.

THEOREM 3.3. *Let u be a weak solution of* (1.1) *where $m = 1$ and let $u \leqq M$ on $\partial\Omega$. Then it holds that*

$$\sup_\Omega u \leqq (1 + C)M^+ + C\{\|u\|_\alpha + \|f\|_{p/\alpha}^{1/\alpha} + \|g\|_{p/\alpha}^{1/(\alpha-1)}\},$$

where $M^+ = \max(0, M)$ and C is a constant depending only on n, p, t, α and the coefficients of the structure (1.2).

§ 4. Local boundedness

In this section, we shall derive the local estimates both in the interior and near the boundary of Ω for the weak solutions of (1.1).

Let $I(x_0, \rho) = I(\rho)$ be an open ball with center at x_0, radius ρ and let $\Omega(x_0, \rho) = \Omega \cap I(x_0, \rho)$. We set

$$B_j = B_j(x) = \tilde{a}_j + \tilde{a}_j^\alpha a_j^{1-\alpha} + b_j^\alpha a_j^{1-\alpha} + \sum_{i=1}^m \{c_{ij} + c_{ji} + d_{ij} + d_{ji} + (e_{ij}^{\alpha/(\alpha-1)} + e_{ji}^{\alpha/(\alpha-1)})\tilde{a}_j^{1/(1-\alpha)}\} + \kappa_j^{-\alpha}(f_j + h_j^{\alpha/(1-\alpha)}\tilde{a}_j^{1/(1-\alpha)}) + \kappa_j^{1-\alpha} g_j$$

for positive constants κ_j. Moreover we put $\tilde{p} = (p/\alpha)' = p/(p-\alpha)$,

$$a_j(\rho) = \left(\rho^{-n}\int_{I(p)} a_j(x)^{-t}dx\right)^{1/t}, \qquad B_j(\rho) = \left(\rho^{-n}\int_{I(\rho)} B_j(x)^{p/\alpha}dx\right)^{\alpha/p},$$

and

$$\zeta = \zeta(x, \rho', \rho) = \begin{cases} 1 & \text{in } I(\rho') \\ \dfrac{\rho - |x|}{\rho - \rho'} & \text{for } \rho' \leqq |x| \leqq \rho \\ 0 & \text{outside of } I(\rho) \end{cases}$$

for $0 < \rho' < \rho$.

We obtain the following lemma by the same argument as in the proof of Lemma 2.2, using Hölder's inequality

LEMMA 4.1. *Let $v \in H^{1,\alpha}(a_j, \Omega)$. Then for any α with $1 + 1/t < \alpha < p$, and for any ρ_0 with $0 < \rho' < \rho \leqq \rho_0$*

$$\left(\rho_0^{-n}\int_{I(\rho')} |v|^{\alpha^\#}dx\right)^{\alpha/\alpha^\#} \leqq c_0\rho_0^{\alpha-n}a_j(\rho_0)\int_{I(p)} a_j|\nabla v|^\alpha\zeta^\alpha dx + \rho_0^\alpha(\rho-\rho')^{-\alpha}\left(\rho_0^{-n}\int_{I(\rho)}|v|^{\alpha\tilde{p}}dx\right)^{1/\tilde{p}},$$

where $\zeta = \zeta(x, \rho', \rho)$.

THEOREM 4.1. *Let $u = (u_1, \cdots, u_m)$ be a weak solution of (1.1), and $I(x_0, 2\rho_0) \subset \Omega$. Then it holds that*

$$\sup_{I(x_0,\rho_0)} |u_j| \leqq C\sum_{i=1}^m \left\{\left(\rho_0^{-n}\int_{I(x_0,2\rho_0)}|u_i|^{\alpha\tilde{p}}dx\right)^{1/\alpha\tilde{p}} + \rho_0^{-n/\alpha\tilde{p}}K_i\right\}, \tag{4.1}$$

where

$$K_i = \left(\int_{I(2\rho_0)} f_i^{p/\alpha}dx\right)^{1/p} + \left(\int_{I(2\rho_0)} g_i^{p/\alpha}dx\right)^{\alpha/(\alpha-1)p} + \left(\int_{I(2\rho_0)} (h_i^{\alpha/(\alpha-1)}\tilde{a}_i^{1/(1-\alpha)})^{p/\alpha}dx\right)^{\alpha/(\alpha-1)p},$$

and C is a constant depending only on $n, p, t, \alpha, a_j(2\rho_0)$ and $B_j(2\rho_0)$ $(j = 1, \cdots, m)$.

Proof. Put $\kappa_j = K_j + \varepsilon$ for a positive number ε and put $\bar{u}_j = |u_j| + \kappa_j$

$(j=1,\cdots,m)$. Define Φ_j by $\Phi_j=\operatorname{sign}(u_j)G_j\zeta^\alpha(x,\rho',\rho)$ for $0<\rho'<\rho\leqq 2\rho_0$, where

$$G_j=G_j(u_j)=\begin{cases}\bar u_j^{\alpha q-\alpha+1}-\kappa_j^{\alpha q-\alpha+1} & \text{for } |u_j|\leqq \ell_j-\kappa_j\,,\\ \ell_j^{\alpha q-\alpha}u_j-\kappa_j^{\alpha q-\alpha+1} & \text{for } |u_j|\geqq \ell_j-\kappa_j\,,\end{cases}$$

$j=1,2,\cdots,m,\ q\geqq 1,\ \ell_j>\kappa_j$.

Then we have

$$\sum_{j=1}^m\int_{I(\rho)}\{\nabla\Phi_j\cdot A_j(x,u,\nabla u_j)+\Phi_jB_j(x,u,\nabla u_j)\}dx=0\,.$$

Put

$$H_j=\begin{cases}(\alpha q-\alpha+1)\bar u_j^{\alpha q-\alpha} & \text{for } |u_j|\leqq \ell_j-\kappa_j\,,\\ \ell_j^{\alpha q-\alpha} & \text{for } |u_j|\geqq \ell_j-\kappa_j\,.\end{cases}$$

Since $\nabla\Phi_j=H_j\zeta^\alpha\nabla u_j+\alpha G_j\zeta^{\alpha-1}\nabla\zeta$, we have, by (1.2),

$$\begin{aligned}&\sum_{j=1}^m\int_{I(\rho)}H_ja_j|\nabla\bar u_j|^\alpha\zeta^\alpha dx\\ &\quad\leqq\sum_{j=1}^m\int_{I(\rho)}\Big\{\alpha G_j\zeta^{\alpha-1}|\nabla\zeta|(\tilde a_j|\nabla\bar u_j|^{\alpha-1}+\sum_{i=1}^m e_{ij}\bar u_i^{\alpha-1}+h_j)\\ &\qquad+G_j\zeta^\alpha\Big(b_j|\nabla\bar u_j|^{\alpha-1}+\sum_{i=1}^m d_{ij}\bar u_i^{\alpha-1}+g_j\Big)+H_j\zeta^\alpha\sum_{i=1}^m c_{ij}\bar u_i^\alpha+H_j\zeta^\alpha f_j\Big\}dx\,.\end{aligned}$$

It follows by Young's inequality that

$$G_j\zeta^{\alpha-1}|\nabla\zeta|\tilde a_j|\nabla\bar u_j|^{\alpha-1}\leqq\frac{H_j}{4}a_j|\nabla\bar u_j|^\alpha\zeta^\alpha+4^{\alpha-1}\tilde a_j^\alpha a_j^{1-\alpha}\bar u_j^{\alpha q}|\nabla\zeta|^\alpha\,,$$

$$G_j\zeta^{\alpha-1}|\nabla\zeta|e_{ij}\bar u_i^{\alpha-1}\leqq e_{ij}^{\alpha/(\alpha-1)}\tilde a_j^{1/(1-\alpha)}\bar u_i^{\alpha q}\zeta^\alpha+\tilde a_j\bar u_j^{\alpha q}|\nabla\zeta|^\alpha$$

and

$$\begin{aligned}G_j\zeta^{\alpha-1}|\nabla\zeta|h_j&\leqq G_j\zeta^{\alpha-1}|\nabla\zeta|h_j\kappa_j^{1-\alpha}\bar u_j^{\alpha-1}\\ &\leqq\kappa_j^{-\alpha}h^{\alpha/(\alpha-1)}\tilde a_j^{1/(1-\alpha)}\bar u_j^{\alpha q}\zeta^\alpha+\tilde a_j\bar u_j^{\alpha q}|\nabla\zeta|^\alpha\,.\end{aligned}$$

Put $v_j=\bar u_j^q$. By the same argument as in Theorem 3.1, we see that

$$\begin{aligned}\sum_{j=1}^m\int_{I(\rho)}a_j|\nabla v_j|^\alpha\zeta^\alpha dx&\leqq Cq^\alpha(\rho-\rho')^{-\alpha}\sum_{j=1}^m\int_{I(\rho)}B_j(x)v_j^\alpha dx\\ &\leqq Cq^\alpha(\rho-\rho')^{-\alpha}\rho_0^\alpha\sum_{j=1}^m B_j(2\rho_0)\Big(\rho_0^{-n}\int_{I(\rho)}v_j^{\alpha\bar p}dx\Big)^{1/\bar p}\,.\end{aligned}\tag{4.2}$$

This result, Lemma 4.1 and (4.2) imply

$$\Big(\rho_0^{-n}\int_{I(\rho')}v_j^{\alpha^\#}dx\Big)^{\alpha/\alpha^\#}\leqq Cq^\alpha(\rho-\rho')^{-\alpha}\rho_0^\alpha\Big(\rho_0^{-n}\int_{I(\rho)}v_j^{\alpha\bar p}dx\Big)^{1/\bar p},$$

where C is a constant depending only on $n, p, t, \alpha, \alpha^{\#}, a_j(2\rho_0)$ and $\sum_{j=1}^{m} B_j(2\rho_0)$.

Now put $r = \alpha/\alpha^{\#}$, $q_s = (r/p)^s$ and $\rho_s = \rho_0 + 2^{-s}\rho_0$ $(s = 0, 1, 2, \cdots)$.

Then

$$\left(\rho_0^{-n}\int_{I(\rho s+1)}\left(\sum_{j=1}^{m}\bar{u}_j\right)^{\alpha q_{s+1}\tilde{p}}dx\right)^{1/\alpha q_{s+1}\tilde{p}}$$
$$= \left(\rho_0^{-n}\int_{I(\rho s+1)}\left(\sum_{j=1}^{m}\bar{u}_j\right)^{\alpha/q_s r}dx\right)^{1/\alpha q_{s+1}\tilde{p}}$$
$$\leqq C^{1/q_{s+1}\tilde{p}}2^{(s+1)/r^{s+1}\tilde{p}}\left(\rho_0^{-n}\int_{I(\rho_s)}\left(\sum_{j=1}^{m}\bar{u}_j\right)^{\alpha q_s r}dx\right)^{1/\alpha q_s\tilde{p}}$$
$$\leqq C^{1/\alpha\sum_{k=0}^{\infty}(\tilde{p}/r)^k)}2^{(1/\alpha\sum_{k=0}^{\infty}k(\tilde{p}/r)^k)}\left(\rho_0^{-n}\int_{I(2\rho_0)}\left(\sum_{j=1}^{m}\bar{u}_j\right)^{\alpha\tilde{p}}dx\right)^{1/\alpha\tilde{p}}.$$

from which, letting s tend to infinity and ε tend to zero, we have the desired inequality (4.1). Q.E.D.

THEOREM 4.2. *Let $u = (u_1, \cdots, u_m)$ be a weak solution of* (1.1) *and $x_0 \in \partial\Omega$. Suppose that for some positive number ρ_0, $u = M$ on $\partial\Omega \cap I(x_0, 2\rho_0)$, $M = (M_1, \cdots, M_m) \in R^m$. Then*

$$\sup_{\Omega(x_0,\rho_0)}|u_j| \leqq \rho_0^{-\theta}\, C\sum_{i=1}^{m}\left\{\left(\int_{\Omega(x_0,2\rho_0)}|u_i|^{\alpha\tilde{p}}dx\right)^{1/\alpha\tilde{p}} + |M_i| + K_i\right\}, \quad j = 1, \cdots, m,$$

where

$$K_i = \left(\int_{\Omega(x_0,2\rho_0)} f_i^{p/\alpha}dx\right)^{1/p} + \left(\int_{\Omega(x_0,2\rho_0)} g_i^{p/\alpha}dx\right)^{\alpha/(\alpha-1)p}$$
$$+ \left(\int_{\Omega(x_0,2\rho_0)} (h_i^{\alpha/(\alpha-1)}\tilde{a}_i^{1/(1-\alpha)})^{p/\alpha}dx\right)^{\alpha/(\alpha-1)p}.$$

The constant C depends only on $n, p, t, \alpha, \|a_j^{-1}\|_t$ and $\|B_j\|_{p/\alpha}$ $(j = 1, \cdots, m)$, while θ depends on n, p, t and α.

Proof. Let $U = u - M = (u_1 - M_1, \cdots, u_m - M_m)$. Then U is a weak solution of a system of the form (1.1) such that $U = 0$ on $\partial\Omega \cap I(x_0, 2\rho_0)$. Therefore the same method of the proof of Theorem 4.1 can be applied to prove the assertion. Q.E.D.

§ 5. Hölder continuity

In this section we shall prove that weak solutions of (1.1) are Hölder continuous in Ω under the assumption

$$(5.1) \qquad \tilde{a}_j a_j^{-1} \in L^{\infty}(\Omega), \qquad j = 1, 2, \cdots, m.$$

First we shall state some lemmas.

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LEMMA 5.1 ([9]). *Let $I(x_0, \rho_0) \subset \Omega$ and v be an $H^{1,1}(I(x_0, \rho_0))$-function. Put*

$$A(k_0, \rho_0) = \{x \in I(x_0, \rho_0) \mid v \geqq k_0\}.$$

Suppose that there are two constants k_0 and θ $(0 \leqq \theta < 1)$ such that

$$\text{meas } A(k_0, \rho_0) < \theta \text{ meas } I(x_0, \rho_0)$$

holds. Then for any h and k with $h > k > k_0$, there exists a positive constant C depending only on θ and n such that the following inequality hold:

$$(5.2)\qquad (h - k)\,[\text{meas } A(h, \rho_0)]^{(n-1)/n} \leqq C \int_{[A(k,\rho_0) - A(h,\rho_0)]} |\nabla v(t)|\, dt .$$

This lemma immediately implies the following

LEMMA 5.2. *Under the same hypothesis of Lemma 5.1, we have*

$$(5.3)\qquad \begin{aligned} &(h - k)^\alpha\,[\text{meas } A(h, \rho_0)]^{\alpha(n-1)/n} \leqq c_0 \rho_0^{n/t} \Big(\rho_0^{-n} \int_{I(\rho_0)} m^{-t} dx\Big)^{1/t} \\ &\quad \times \int_{A(k,\rho_0)} m |\nabla v|^\alpha dx\,[\text{meas } A(k, \rho_0) - \text{meas } A(h, \rho_0)]^{\alpha - 1 - (1/t)} \end{aligned}$$

for $v \in H^{1,\alpha}(m, \rho)$.

LEMMA 5.3 ([8]). *Let $\phi(h, \rho)$ be a non-negative function on the strip $(h \geqq k_0 \geqq 0) \times (0 \leqq \rho < R_0)$ such that*

(i) *for every fixed ρ, $\phi(h, \rho)$ is non-increasing on h*

(ii) *for every fixed h, (h, ρ) is non-decreasing in ρ and that there exist positive constant C, α, β, γ $(\beta > 1)$ with*

$$\phi(h, \rho) \leqq C[\phi(k, R)]^\beta (h - k)^{-\alpha} (R - \rho)^{-\gamma}$$

for $h > k \geqq k_0$, $\rho < R \leqq R_0$. Then we have

$$\phi(k_0 + d, R_0 - \sigma R_0) = 0$$

for any σ with $0 < \sigma < 1$ and for

$$d = C^{1/\alpha} (\sigma^{-1} R_0^{-1})^{\gamma/\alpha} [\phi(k_0, R_0)]^{(\beta-1)/\alpha} 2^{\beta(\alpha+\beta)/(\beta-1)\alpha} .$$

Remark. Lemma 5.3 is valid not only for $k_0 \geqq 0$, but also $k_0 < 0$.

From now on, we fix j and use simplified notations:

$$M(\rho) = M_j(\rho) = \sup_{I(\rho)} u_j , \quad m(\rho) = m_j(\rho) = \inf_{I(\rho)} u_j ,$$
$$\omega(\rho) = \omega_j(\rho) = M_j(\rho) - m_j(\rho) .$$

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THEOREM 5.1. *Let* $u = (u_1, \cdots, u_m)$ *be a weak solution of* (1.1) *and* $I(R) = I(x_0, R) \subset \Omega$ *for* $0 < 4\rho_0 < R$. *Then under the assumption* (5.1) *there exist positive constants* K *and* λ *such that for any* ρ *with* $0 < \rho < \rho_0$

$$\omega_j(\rho) \leqq K\rho^\lambda \quad (j = 1, 2, \cdots, m), \tag{5.4}$$

which means the local Hölder continuity of u_j *in* Ω.

Proof. We put $F_j(x) = \sum_{i=1}^m c_{ij}|u_i|^\alpha + f_j$, $G_j(x) = \sum_{i=1}^m d_{ij}|u_i|^{\alpha-1} + g_j$ and $H_j = \sum_{i=1}^m e_{ij}|u_i|^{\alpha-1} + h_j$. Then from (1.2), we have

$$\begin{cases} \xi\cdot A_j(x, u, \xi) \geqq a_j|\xi|^\alpha - F_j, \\ |B_j(x, u, \xi)| \leqq b_j|\xi|^{\alpha-1} + G_j, \\ |A_j(x, u, \xi)| \leqq \tilde{a}_j|\xi|^{\alpha-1} + H_j, \end{cases} \tag{5.5}$$

$j = 1, 2, \cdots, m$, for any $\xi \in R^n$. Moreover, since $u = (u_1, \cdots, u_m)$ is bounded in $I(R)$, $F_j, G_j, H_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)} \in L^{p/\alpha}(I(R))$.

We define the functions

$$\begin{cases} \Phi_j = (u_j - k)^+\zeta^\alpha, & \text{for any choice of } k, \\ \Phi_i = 0 \quad (i \neq j), \end{cases}$$

where $\zeta = \zeta(x, \rho', \rho)$ with $\rho_0 < \rho' < \rho \leqq 2\rho_0$ and $(u_j - k)^+ = \max(u_j - k, 0)$. Then $\Phi = (\Phi_1, \cdots, \Phi_m) \in H_0^{1,\alpha}(\tilde{a}, \Omega)$ and hence the following equality holds:

$$\int_\Omega \{\nabla\Phi_j\cdot A_j + \Phi_j B_j\}dx = 0.$$

Now we put $v = (u_j - k)^+$. Then $\nabla u_i = \nabla v$ on the set $A(k, \rho) = \{x \in I(\rho)\,|\,u_j \geqq k\}$ and $\Phi_j = 0$ outside of $A(k, \rho)$. Thus, in the set $A(k, \rho)$, we have, by (5.5),

$$\int_{A(k,\rho)} a_j|\nabla v|^\alpha\zeta^\alpha dx \leqq C\int_{A(k,\rho)} \{\tilde{a}_j v^\alpha|\nabla\zeta|^\alpha + B_j(x)v^\alpha\zeta^\alpha + T_j(x)\}dx,$$

where $B_j(x) = b_j^\alpha a_j^{1-\alpha} + G_j$, $T_j(x) = (F_j + G_j + H_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)})\zeta^\alpha \in L^{p/\alpha}(I(R))$ and where C is a constant depending only on α and $\sup \tilde{a}_j a_j^{-1}$.

We now put $\tilde{a}_j(\rho) = \left(\rho^{-n}\int_{I(\rho)} \tilde{a}_j^{p/\alpha}dx\right)^{\alpha/p}$, $a_j(\rho) = \left(\rho^{-n}\int_{I(\rho)} a_j^{-t}dx\right)^{1/t}$ and $\Lambda(\rho) = \tilde{a}_j(\rho)a_j(\rho)$. Then by (5.1), $\Lambda(\rho)$ is bounded.

By the same argument as in Lemma 2.4, we obtain

$$\int_{A(k,\rho)} B_j(x)v^\alpha\zeta^\alpha dx \leqq \|B_j\|_{p/\alpha}\Big\{\varepsilon\Big(\int_{A(k,\rho)} a_j|\nabla v|^\alpha\zeta^\alpha dx$$
$$+ [\text{meas } A(k, \rho)]^{1/t}\int_{A(k,\rho)} v^\alpha|\nabla\zeta|^\alpha dx + K\int_{A(k,\rho)} v^\alpha\zeta^\alpha dx\Big)\Big\}.$$

Note that $v \leqq M(4\rho_0) - k_0$ in $I(2\rho_0)$ for $k_0 \leqq M(4\rho_0)$.
Take $\varepsilon = 2^{-1}\|B_j\|_{p/\alpha}^{-1}$ to have

$$\int a_j|\nabla v|^\alpha\zeta^\alpha dx \leqq C(\rho-\rho')^{-\alpha}\{[(2\rho_0)^{\alpha n/p}\tilde{a}_j(\rho) + (2\rho_0)^{n(1/t+\alpha/p)} + (2\rho_0)^{\alpha+\alpha n/p}](M(4\rho_0)-k_0)^\alpha + \rho_0^\alpha T^\alpha\}[\text{meas } A(k,\rho)]^{1-\alpha/p} \tag{5.6}$$

for any $k_0 \leqq M(4\rho_0)$, where $T^\alpha = \|T_j(x)\|_{p/\alpha}$ and C is a constant depending only on α and $\|B_j\|_{p/\alpha}$.

Let $k_0 < k < h$. Then by Lemma 4.1 with $p = 1$, (5.6) and Hölder's inequality, we have

$$\begin{aligned}(h-k)^\alpha \text{ meas } A(k,\rho') &\leqq \int_{A(k,\rho)}(u_j-k)^\alpha\zeta^\alpha dx\\ &\leqq \left(\int_{A(k,\rho)}(v\zeta)^{\alpha^\#}dx\right)^{\alpha/\alpha^\#}[\text{meas } A(k,\rho)]^{\alpha/n-1/t}\\ &\leqq \left((2\rho_0)^{n/t}a_j(\rho)\int_{A(k,\rho)}a_j|\nabla v|^\alpha\zeta^\alpha dx + (2\rho_0)^{n/t}\int_{A(k,\rho)}v^\alpha|\nabla\zeta|^\alpha dx\right)\\ &\qquad\times[\text{meas } A(k,\rho)]^{\alpha/n-1/t}\\ &\leqq C(\rho-\rho')^{-\alpha}\rho_0^{n(1/t+\alpha/p)}\{(\Lambda_j(\rho)+\|a_j^{-1}\|_t)(M(4\rho_0)-k_0)^\alpha + \rho_0^{\alpha-(n/t)}T^\alpha\}\\ &\qquad\times[\text{meas } A(k,\rho)]^\beta\end{aligned}$$

for any α with $2 \leqq \alpha < p$, where $\beta = 1 + \alpha/n - 1/t - \alpha/p > 1$. Thus, by Lemma 5.3, we see that meas $A(k_0+d, \rho_0) = 0$, where

$$d = C_0\rho_0^{-1}\rho_0^{n/\alpha(1/t+\alpha/p)}\{M(4\rho_0) - k_0 + \rho_0^\theta T\}[\text{meas } A(k_0, 2\rho_0)]^{(\beta-1)/\alpha},$$

where $\theta = 1 - (n/\alpha t)$ and C_0 is a constant depending only on $n, p, t, \alpha, \|a_j^{-1}\|_t$ and $\sup_{0<R}\Lambda_j(R)$. It is noted that C_0 is independent of ρ_0.

Hence we have

$$M(\rho_0) \leqq k_0 + C_0[\rho_0^{-n} \text{ meas } A(k_0, 2\rho_0)]^{(\beta-1)/\alpha}\{M(4\rho_0) - k_0 + \rho_0^\theta T\} \tag{5.7}$$

for any $k_0 \leqq M(4\rho_0)$.

It is easily verified that (5.7) remains valid even when $-u_j$ is taken instead of u_j. In this case,

$$M(\rho_0) = \sup_{I(\rho_0)}(-u_j) \quad \text{and} \quad A(k,\rho) = \{x \in I(\rho) \mid u_j \leqq k\}.$$

Now we can assume that the inequality

$$\text{meas } A(k, 2\rho_0) < \theta_0 \text{ meas } I(2\rho_0) \quad (0 \leqq \theta_0 < 1) \tag{5.8}$$

holds for some θ_0 and for $k = \frac{1}{2}(M(4\rho_0) + \rho_0^\theta T)$. In fact, in case (5.8) is

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not valid, we may take $u'_j = -u_j$ instead of u_j. We can therefore use Lemma 5.2 and (5.6) with the choices $\rho' = 2\rho_0$ and $\rho = 3\rho_0$. We have, then

$$(5.9)\qquad \begin{aligned}&(h-k)^\alpha\{\rho_0^{-n} \operatorname{meas} A(h, 2\rho_0)\}^{\alpha(n-1)/n}\\ &\quad \leqq C\{M(4\rho_0) - k + \rho_0^\theta T\}^\alpha\{\rho_0^{-n} [\operatorname{meas} A(k, 3\rho_0) - \operatorname{meas} A(h, 3\rho_0)]^{\alpha-1-1/t}\end{aligned}$$

for $k < h \leqq M(4\rho_0)$.

Put $h_s(\rho_0) = M(4\rho_0) + \rho_0^\theta T - 2^{-(s+1)}\{\omega(4\rho_0) + \rho_0^\theta T\}$ for $s = 0, 1, 2, \cdots$. We fix a natural number N such that

$$(C_0C_1/N)^{(\beta-1)/\alpha\delta} \leqq \frac{1}{2} \quad \left(\delta = \alpha(n-1)\Big/n\left(\alpha - 1 - \frac{1}{t}\right)\right)$$

for a constant C_1 with $C2^\gamma 3^n\omega_n \leqq C_1$, where $\gamma = 1/(\alpha - 1 - 1/t)$ and ω_n is a volume of unit ball in R^n. First we consider it in the case when ρ_0 is a number such that $h_s(\rho_0) < M(4\rho_0)$ when $s \leqq N$. Then the inequality (5.9) is valid for $h = h_s(\rho_0)$ and $k = h_{s-1}(\rho_0)$. Since

$$\begin{aligned}h_s(\rho_0) - h_{s-1}(\rho_0) &= 2^{-(s+1)}\{\omega(4\rho_0) + \rho_0^\theta T\}\,,\\ M(4\rho_0) + \rho_0^\theta T - h_{s-1}(\rho_0) &= 2^{-s}\{\omega(4\rho_0) + \rho_0^\theta T\}\,,\end{aligned}$$

(5.9) induces

$$(5.10)\qquad \begin{aligned}&[\rho_0^{-n} \operatorname{meas} A(h_s(\rho_0), 2\rho_0)]^\delta\\ &\quad \leqq C2^\gamma\rho_0^{-n} [\operatorname{meas} A(h_{s-1}(\rho_0), 3\rho_0) - \operatorname{meas} A(h_s(\rho_0), 3\rho_0)]\,.\end{aligned}$$

Summing up each side of (5.10) for $s = 1, 2, \cdots, N$ we have

$$N[\rho_0^{-n} \operatorname{meas} A(h_N(\rho_0), 2\rho_0)]^\delta \leqq C2^\gamma\rho_0^{-n} \operatorname{meas} A(h_0(\rho_0), 3\rho_0) \leqq C_1\,.$$

Thus we obtain

$$C_0[\rho_0^{-n} \operatorname{meas} A(h_N(\rho_0), 2\rho_0)]^{(\beta-1)/\alpha} \leqq (C_0C_1/N)^{(\beta-1)/\alpha\delta} \leqq \frac{1}{2}\,,$$

from which we get

$$M(\rho_0) \leqq M(4\rho_0) + \rho_0^\theta T - 2^{-(N+2)}\{\omega(4\rho_0) + \rho_0^\theta T\}$$

if we take $k_0 = h_N(\rho_0)$ in (5.7). Namely we obtain

$$(5.11)\qquad \omega(\rho_0) \leqq (1 - 2^{-(N+1)})\{\omega(4\rho_0) + \rho_0^\theta T\}\,,$$

since $m(\rho_0) \leqq M(\rho_0)$. Finally we consider it in the case when ρ_0 is a number such that $h_{N_0}(\rho_0) \geqq M(4\rho_0)$ for some N_0 with $N_0 < N$. It then follows that

$$M(\rho_0) \leqq M(4\rho_0) \leqq M(4\rho_0) + \rho_0^\theta T - 2^{-(N_0+1)}\{\omega(4\rho_0) + \rho_0^\theta T\}\,,$$

and we have the inequality (5.11). Thus (5.11) is valid for any ρ_0 with $0 < \rho_0 < 1$, and hence we have the inequality (5.4) using Lemma 5.4 below. Q.E.D.

LEMMA 5.4 ([9]). *If there exist an η with $0 < \eta < 1$ and $H > 0, \theta > 0$ such that*

$$\omega(\rho) \leqq \eta\omega(4\rho) + \rho^{\theta}H \qquad \text{for any } 0 < \rho < 1$$

then

$$\omega(\rho) \leqq K\rho^{\lambda}$$

holds for some positive number K and λ.

Our next attention is focused on the behavior of u near the boundary $\partial\Omega$ of Ω. Namely, we shall now investigate the Hölder continuity of the weak solutions of (1.1) near $\partial\Omega$.

DEFINITION ([9]). A bounded open set Ω of R^n is said to be $H_0^1(\Omega)$-admissible if for any $\rho < \rho_0$ and $x_0 \in \partial\Omega$, and for any $v(x) \in C^1(\overline{\Omega(x_0, \rho)})$ such that $v(x) = 0$ on $\partial\Omega \cap I(x_0, \rho)$, there exists a positive constant satisfying β

$$|v(x)| \leqq \beta \int_{\Omega(x_0,\rho)} \frac{|\nabla V(t)|}{|x - t|^{n-1}} dt\,.$$

LEMMA 5.5 ([9]). *Let $v \in H^{1,1}(\Omega(x_0, R))$ with $x_0 \in \partial\Omega$ and let Ω be $H_0^1(\Omega)$-admissible. If $v = 0$ on $\partial\Omega \cap I(x_0, R)$, then the formula* (5.2) *is valid for $h > k > 0$.*

THEOREM 5.2. *Let $u = (u_1, \cdots, u_m)$ be a weak solution of* (1.1). *For some $R > 0$, if $\Omega(x_0, R)$, $x_0 \in \partial\Omega$, is $H_0^1(\Omega)$-admissible and if $u = M$ on $\partial\Omega \cap I(x_0, R)$, $(M = (M_1, \cdots, M_m) \in R)$, then, under the assumption* (5.1) *u_j $(j = j, 2, \cdots, m)$ are Hölder continuous in $\overline{\Omega} \cap I(x_0, R)$.*

Proof. We may assume that $u_j = 0$ on $\partial\Omega \cap I(x_0, R)$. (If $u_j \neq 0$, we subtract the constant from u_j.) The proof is obtained in parallel with that of Theorem 5.1, where we note that

$$\frac{1}{2}[M(4\rho_0) + m(4\rho_0) + \rho_0^{\theta}T] \geqq 0$$

may be assumed. Q.E.D.

In what follows, we shall assume that $\partial\Omega$ is Lipshitz continuous. Obviously Ω is $H_0^1(\Omega)$-admissible.

If $u=(u_1,\cdots,u_m)$ is a weak solution of (1.1) and if $u_j-w_j\in H_0^{1,\alpha}(\tilde{a}_j,\Omega)$ for a Lipshitz continuous function w_j, $j=1,2,\cdots,m$, then $U=(u_1-w_1,\cdots,u_m-w_m)\in H_0^{1,\alpha}(\tilde{a},\Omega)$ and U is a weak solution of the system

$$-\operatorname{div}\tilde{A}_j(x,U,\nabla U_j)+\tilde{B}_j(x,U,\nabla U_j)=0 \qquad j=1,\cdots,m,$$

where

$$\tilde{A}_j(x,U,\nabla U_j)=A_j(x,U+w,\nabla(U_j+w_j))$$

and

$$\tilde{B}_j(x,U,\nabla U_j)=B_j(x,U+w,\nabla(U_j+w_j)),\quad (w=(w_1,\cdots,w_m)).$$

Moreover, we have for any $\xi\in R^n$

$$\begin{cases}\xi\tilde{A}_j(x,U,\xi)\geqq c_0a_j|\xi|^\alpha-\sum_{i=1}^m C_{ij}|U_i|^\alpha-F_j\,,\\ |\tilde{B}_j(x,U,\xi)|\leqq c_1b_j|\xi|^{\alpha-1}+\sum_{i=1}^m D_{ij}|U_i|^{\alpha-1}+G_j\,,\\ |\tilde{A}_j(x,U,\xi)|\leqq c_2\tilde{a}_j|\xi|^{\alpha-1}+\sum_{i=1}^m E_{ij}|U_i|^{\alpha-1}+H_j\,,\end{cases}$$

where c_0, c_1 and c_2 are some constants with $0<c_0<1, 1<c_1$ and $1<c_2$, and

$$C_{ij}=c_3(c_{ij}+e_{ij}),\quad D_{ij}=c_4d_{ij},\quad E_{ij}=c_5e_{ij}\,,$$

$$F_{ij}=c_6\Big((a_j+\tilde{a}_j)|\nabla w_j|^\alpha+\sum_{i=1}^m(c_{ij}+e_{ij})|w_i|^\alpha\Big)+f_j+|w_j|h_j\,,$$

$$G_{ij}=c_7\Big(b_j|\nabla w_j|^{\alpha-1}+\sum_{i=1}^m d_{ij}|w_i|^{\alpha-1}\Big)g_j\,,$$

$$H_{ij}=c_8\Big(\tilde{a}_j|\nabla w_j|^{\alpha-1}+\sum_{i=j}^m e_{ij}|w_i|^{\alpha-1}\Big)+h_j\,.$$

(Here $c_3,\cdots,c_8$ are positive constants depending only on α.)

It is clear that $C_{ij}, D_{ij}, E_{ij}^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)}, F_j, G_j, H_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)}\in L^{p/\alpha}(\Omega)$. By applying Theorem 5.2, we therefore have the following

THEOREM 5.3. *Let $u=(u_1,\cdots,u_m)$ be a weak solution of* (1.1) *and let w_j $(j=1,\cdots,m)$ be Lipshitz continuous functions on Ω. If $\partial\Omega$ is Liphistz continuous, and if $u_j-w_j\in H_0^{1,\alpha}(\tilde{a}_j,\Omega)$ $(j=i,\cdots,m)$, then under the condition* (5.1) *u_j are uniformly Hölder continuous on $\bar{\Omega}$.*

§ 6. Harnack type inequality

In this section we shall prove the Harnack type inequality for positive weak solutions of the following system:

(6.1) $$-\operatorname{div} A_j(x, u, \nabla u_j) + B_j(x, u, \nabla u_j) = 0 \quad (j = 1, 2, \cdots, m),$$

under some restrictions on the coefficients. Here each A_j and B_j satisfy the following three conditions (I)–(III):

(I) For any $\xi \in R^n$ it holds that

(6.2) $$\begin{cases} \xi \cdot A_j(x, u, \xi) \geqq \lambda_j a_j(x)|\xi|^\alpha - c_j(x)|u_j|^\alpha - \sum_{\substack{i=1 \\ i \neq j}}^m c_{ij}(x)|u_i|^{\alpha-1} - f_j(x), \\ |B_j(x, u, \xi)| \leqq b_j(x)|\xi|^{\alpha-1} + \sum_{i=1}^m d_{ij}(x)|u_i|^{\alpha-1} + g_j(x), \\ |A_j(x, u, \xi)| \leqq \tilde{a}_j(x)|\xi|^{\alpha-1} + \sum_{i=1}^m e_{ij}(x)|u_i|^{\alpha-1} + h_j(x), \end{cases}$$

with some $0 < \lambda_j < 1$. The functions $a_j, b_j, c_{ij}, \cdots, h_j$ and $\tilde{a}_j$ are non-negative and measurable. We assume that

$$a_j^{-1} \in L^t(\Omega) \qquad \text{for any } t > 1.$$
$$a_j \leqq a_j,$$
$$\tilde{a}_j,\ \tilde{a}_j a_j^{1-\alpha},\ b_j a_j^{1-\alpha},\ c_j,\ c_{ij},\ d_{ij},\ e_{ij}^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)},\ f_j,\ g_j,\ h_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)} \in L^{p/\alpha}(\Omega),$$

where

(6.3) $$\frac{\alpha}{p} + \frac{1}{t} < \frac{\alpha}{n} \quad \text{and} \quad (2 \leqq)\alpha < p.$$

(II) There exists another system of functions $\bar{A}_j(x, u_j, \nabla u_j)$ $(j = 1, \cdots, m)$ such that

(6.4) $$\begin{cases} \xi \cdot \bar{A}_j(x, u_j, \xi) \geqq a_j(x)|\xi|^\alpha - c_j(x)|u_j|^\alpha - \bar{f}_j(x), \\ |\bar{A}_j(x, u_j, \xi)| \leqq \tilde{a}_j(x)|\xi|^{\alpha-1} + e_j(x)|u_j|^{\alpha-1} + \bar{h}_j(x), \ \xi \in R^n \end{cases}$$

and $e_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)}, \bar{f}_j, \bar{h}_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)} \in L^{p/\alpha}(\Omega)$ with the inequalities (6.3).

(III) We assume that for any non-negative function $\Phi_j \in C_0^\infty(\Omega)$ $(1 \leqq j \leqq m)$ and for $u > 0$ it holds that

(6.5) $$\begin{aligned} &\int_\Omega \{\nabla\Phi_j \cdot A_j(x, u, \nabla u_j) + \Phi_j B_j(x, u, \nabla u_j)\} dx \\ &\qquad \leqq \int_\Omega \{\nabla\Phi_j \cdot \bar{A}_j(x, u_j, \nabla u_j) + \Phi_j(b_j|\nabla u_j|^{\alpha-1} + d_j u_j^{\alpha-1} + \bar{g}_j)\} dx \end{aligned}$$

with a suitable choice of d_j and $\bar{g}_j$ in $L^{p/\alpha}(\Omega)$.

It is noted that the system (6.1) under the condition (I)–(III) involves general degenerate quasi-linear equations when $m = 1$.

Throughout this section we consider a positive weak solution

$u=(u_1,\cdots,u_m)$ of (6.1) under the condition (I)–(III), and we assume that $I(4\rho_0)\subset\Omega$ for some positive number $\rho_0<1$.

Put

$$\begin{aligned}&\kappa_j=\|f_j+\bar f_j\|_{p/\alpha}^{1/\alpha}+\|g_j+\bar g_j\|_{p/\alpha}^{1/(\alpha-1)}+\|(h_j+\bar h_j)^{\alpha/(\alpha-1)}\tilde a_j^{1/(1-\alpha)}\|_{p/\alpha}^{1/(\alpha-1)},\\&\bar u_j=u_j+\kappa_j+\varepsilon\ (\varepsilon>0),\ \bar c_j=c_j+(f_j+\bar f_j)/\kappa_j^\alpha,\ \bar d_{jj}=d_{jj}+g_j/\kappa_j^{\alpha-1},\\&\bar d_j=d_j+\bar g_j/\kappa_j^{\alpha-1},\ \bar e_{jj}=e_{jj}+h_j/\kappa_j^{\alpha-1},\ \bar e_j=e_j+\bar h_j/\kappa_j^{\alpha-1},\ j=1,\cdots,m.\end{aligned}$$

Then from (6.2) and (6.4) it follows that

$$\text{(6.6)}\qquad\begin{cases}\xi\cdot A_j(x,u,\xi)\geqq\lambda_j a_j|\xi|^\alpha-\bar c_j\bar u_j^\alpha-\sum\limits_{\substack{i=1\\i\neq j}}^m c_{ij}\bar u_i^{\alpha-1},\\|B_j(x,u,\xi)|\leqq b_j|\xi|^{\alpha+1}+\sum\limits_{i=1}^m\bar d_{ij}\bar u_i^{\alpha-1},\\|A_j(x,u,\xi)|\leqq\tilde a_j|\xi|^{\alpha-1}+\sum\limits_{i=1}^m\bar e_{ij}\bar u_i^{\alpha-1},\qquad\xi\in R^n,\end{cases}$$

where $\bar d_{ij}=d_{ij}$, $\bar e_{ij}=e_{ij}$ $(i\neq j)$, and that

$$\text{(6.7)}\qquad\begin{cases}\xi\cdot\bar A_j(x,u_j,\xi)\geqq a_j|\xi|^\alpha-\bar c_j\bar u_j^\alpha,\\|\bar A_j(x,u_j,\xi)|\leqq\tilde a_j|\xi|^{\alpha-1}+\bar e_j\bar u_j^{\alpha-1},\qquad\xi\in R^n.\end{cases}$$

We then put $\tilde p=(p/\alpha)'=p/(p-\alpha)$,

$$\begin{cases}B_j(x)=\tilde a_j+\tilde a_j a_i^{1-\alpha}+b_j a_j^{1-\alpha}+\bar c_j+\bar d_j+\bar e_j^{\alpha/(\alpha-1)}\tilde a_j^{1/(1-\alpha)},\\\tilde B_j(x)=B_j(x)+\sum\limits_{i=1}^m(c_{ij}+c_{ji}+\bar d_{ij}+\bar d_{ji}+(\bar e_{ij}^{\alpha/(\alpha-1)}+\bar e_{ji}^{\alpha/(\alpha-1)})\tilde a_j^{1/(1-\alpha)}),\end{cases}$$

and

$$\begin{cases}a_j(\rho)=\left(\rho^{-n}\displaystyle\int_{I(\rho)}a_j^{-t}dx\right)^{1/t},\quad B_j(\rho)=\left(\rho^{-n}\displaystyle\int_{I(\rho)}B_j^{p/\alpha}dx\right)^{\alpha/p},\\\tilde B_j(\rho)=\left(\rho^{-n}\displaystyle\int_{I(\rho)}\tilde B_j^{p/\alpha}dx\right)^{\alpha/p}.\end{cases}$$

THEOREM 6.1. *Let $u=(u_1,\cdots,u_m)$ be a positive weak solution of* (6.1). *Then*

$$\text{(6.8)}\qquad\sup_{I(\rho_0)}\bar u_j\leqq\sum_{i=1}^m\sup_{I(\rho_0)}\bar u_i\leqq C\sum_{i=1}^m\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar u_i^{\alpha\tilde p}dx\right)^{1/\alpha\tilde p},$$

where C is a constant depending only on n, p, t, α, $a_j(\rho)$ and $\tilde B_j(\rho)$.

Proof. Young's inequality with the help of (6.2) proves that the condition (1.2) is naturally satisfied by the system (6.1). Therefore, from Theorem 4.1, we have (6.8). Q.E.D.

LEMMA 6.1. *For any positive number* $q_0 > 0$,

$$\inf_{I(\rho_0)} \bar{u}_j \geqq C\Big(\rho_0^{-n}\int_{I(2\rho_0)} \bar{u}_j^{-q_0}dx\Big)^{-1/q_0}, \tag{6.9}$$

where C *is a constant depending only on* $n, p, t, \alpha, a_j(2\rho_0)$ *and* $B_j(2\rho_0)$.

Proof. Take $q<0$ and ρ', ρ such that $\rho_0 \leqq \rho' \leqq 2\rho_0$. Put

$$\Phi_j = \bar{u}_j^{\alpha q-\alpha+1}\zeta^\alpha(x,\rho',\rho) \quad \text{and} \quad \Phi_i = 0 \quad \text{for } i \neq j.$$

Then, by (6.5), we have

$$0 \leqq \int_{I(\rho)} \{\nabla\Phi_j\cdot\bar{A}_j(x,u_j,\nabla u_j) + \Phi_j(b_j|\nabla\bar{u}_j|^{\alpha-1} + \bar{d}_j\bar{u}_j^{\alpha-1})\}dx.$$

Thus we see

$$\begin{aligned}&-(\alpha q-\alpha+1)\int_{I(\rho)} \bar{u}_j^{\alpha q-\alpha}\zeta^\alpha\nabla\bar{u}_j\cdot\bar{A}_j dx\\ &\leqq \int_{I(\rho)}\{\alpha\bar{u}_j^{\alpha q-\alpha+1}\zeta^{\alpha-1}|\nabla\zeta||\bar{A}_j| + \bar{u}_j^{\alpha q-\alpha+1}\zeta^\alpha(b_j|\nabla\bar{u}_j|^{\alpha-1} + \bar{d}_j\bar{u}_j^{\alpha-1})\}dx,\end{aligned}$$

from which, we have, by using (6.7),

$$\int_{I(\rho)} a_j|\nabla v_j|^\alpha\zeta^\alpha dx \leqq C(\rho-\rho')^{-\alpha}\rho_0^n|q|^\alpha\Big(\rho_0^{-n}\int_{I(\rho)} v_j^{\alpha\tilde{p}}dx\Big)^{1/\tilde{p}}, \tag{6.10}$$

where $v_j = \bar{u}_j^q$ and C is a constant depending only on n, p, t, α, and $B_j(2\rho_0)$.

Another inequalities established in Lemma 4.1 are

$$\begin{aligned}\Big((2\rho_0)^{-n}\int_{I(\rho')}|v_j|^{\alpha^\#}dx\Big)^{\alpha/\alpha^\#} &\leqq \Big(\rho_0^{-n}\int_{I(p)}|v_j\zeta|^{\alpha^\#}dx\Big)^{\alpha/\alpha^\#}\\ &\leqq c_0(2\rho_0)^{\alpha-n}a_j(2\rho_0)\int_{I(\rho)} a_j|\nabla v_j|^\alpha\zeta^\alpha dx\\ &\quad + (2\rho_0)^\alpha(\rho-\rho')^{-\alpha}\Big((2\rho_0)^{-n}\int_{I(\rho)}|v_j|^{\alpha\tilde{p}}dx\Big)^{1/\tilde{p}}.\end{aligned} \tag{6.11}$$

By combining (6.10) and (6.11), we have

$$\begin{aligned}&\Big(\rho_0^{-n}\int_{I(\rho')}|v_j|^{\alpha^\#}dx\Big)^{\alpha/\alpha^\#}\\ &\leqq Ca_j(2\rho_0)(\rho-\rho')^{-\alpha}\rho_0^\alpha|q|^\alpha\Big(\rho_0^{-n}\int_{I(\rho)}|v_j|^{\alpha\tilde{p}}dx\Big)^{1/\tilde{p}}.\end{aligned} \tag{6.12}$$

Put $r=\alpha^\#/\alpha$, $q_s = (r/\tilde{p})^s(-q_0)$ and $\rho_s = \rho_0 + \rho_0/2^s$, $s = 0, 1, 2, \cdots$. With these notations the inequality (6.12) means that

$$\left(\rho_0^{-n}\int_{I(\rho_{s+1})}\bar{u}_j^{\alpha q_{s+1}\tilde{p}}dx\right)^{1/r}=\left(\rho_0^{-n}\int_{I(\rho_{s+1})}\bar{u}_j^{\alpha q_s r}dx\right)^{1/r}$$
$$\leqq C2^{(s+1)\alpha}|q_s|^{\alpha}\left(\rho_0^{-n}\int_{I(\rho_s)}\bar{u}_j^{\alpha q_s\tilde{p}}dx\right)^{1/\tilde{p}},$$

that is,

$$\left(\rho_0^{-n}\int_{I(\rho_{s+1})}\bar{u}_j^{\alpha q_{s+1}p}dx\right)^{1/\alpha q_{s+1}\tilde{p}}$$
$$\geqq C^{1/\alpha q_s}|q_s|^{1/\alpha q_s}\left(\rho_0^{-n}\int_{I(\rho_s)}\bar{u}_j^{\alpha q_s\tilde{p}}dx\right)^{1/\alpha q_s\tilde{p}}$$
$$\geqq C^{-(1/\alpha q_0)\Sigma_{i=0}^{\infty}(\tilde{p}/r)^i}\left(|q_0|\frac{r}{p}\right)^{-(1/q_0)\Sigma_{i=0}^{\infty}i(\tilde{p}/r)^i}\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar{u}_j^{-\alpha q_0\tilde{p}}dx\right)^{-1/\alpha q_0\tilde{p}}$$
$$\leqq C\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar{u}_j^{-\alpha q_0\tilde{p}}dx\right)^{-1/\alpha q_0\tilde{p}}.$$

Letting s tend to infinity we have (6.9) for $q_0=q_0/\alpha\tilde{p}$. Q.E.D.

LEMMA 6.2. *Set* $k=(\text{meas}\, I(3\rho_0))^{-1}\int_{I(3\rho_0)}\bar{u}_j dx$, *and let* $\tilde{u}_j=\bar{u}_j k^{-1}$ *and* $v_j=\log\tilde{u}_j$. *Then, it follows that*

$$\left(\rho_0^{-n}\int_{I(3\rho_0)}|v_j|^{\alpha\tilde{p}}dx\right)^{1/\alpha\tilde{p}}\leqq C_0\quad (j=1,2,\cdots,m),\tag{6.13}$$

where C_0 *is a constant depending only on* $n, p, t, \alpha, a_j(4\rho_0)$ *and* $B_j(4\rho_0)$.

Proof. We put $\Phi_j=\tilde{u}_j^{1-\alpha}\zeta^{\alpha}(x,3\rho_0,4\rho_0)$ and $\Phi_i=0$ $(i\neq j)$. Then by (6.5), we see

$$0=\int_{I(4\rho_0)}\{\nabla\Phi_j\cdot A_j+\Phi_jB_j\}dx$$
$$\leqq\int_{I(4\rho_0)}\{\nabla\Phi_j\cdot\bar{A}_j+\Phi_j(b_j|\nabla\bar{u}_j|^{\alpha-1}+\bar{d}_j\bar{u}_j^{\alpha-1})\}dx.$$

Since $\nabla\Phi_j=(1-\alpha)\tilde{u}_j^{-\alpha}\zeta^{\alpha}\nabla\tilde{u}_j+\alpha\tilde{u}_j^{1-\alpha}\zeta^{\alpha-1}\nabla\zeta$, it follows that

$$\begin{aligned}&\int_{I(4\rho_0)}\tilde{u}_j^{-\alpha}\zeta^{\alpha}\nabla\tilde{u}_j\cdot\bar{A}_jdx\\&\leqq\int_{I(4p_0)}\{\alpha\tilde{u}_j^{1-\alpha}\zeta^{\alpha-1}|\nabla\zeta||\bar{A}_j|+\tilde{u}_j^{1-\alpha}\zeta^{\alpha}(b_j|\nabla\bar{u}_j|^{\alpha-1}+\bar{d}_j\bar{u}_j^{\alpha-1})\}dx.\end{aligned}\tag{6.14}$$

Remind the condition (6.7) to have

$$(\alpha-1)\tilde{u}_j^{-\alpha}\zeta^{\alpha}\nabla\tilde{u}_j\,\bar{A}_j\geqq(\alpha-1)k^{\alpha-1}(a_j|\nabla v_j|^{\alpha}-\bar{c}_j)\zeta^{\alpha},$$
$$\alpha\tilde{u}_j^{1-\alpha}\zeta^{\alpha-1}|\nabla\zeta||\bar{A}_j|\leqq\left\{\left(\frac{\alpha-1}{4}\right)a_j|\nabla v_j|^{\alpha}\zeta^{\alpha}+\left(\frac{4}{\alpha-1}\right)^{\alpha-1}\tilde{a}_j^{\alpha}a_j^{1-\alpha}|\nabla\zeta|^{\alpha}\right.$$
$$\left.+\alpha e_j^{\alpha/(\alpha-1)}\tilde{a}_j^{1/(1-\alpha)}\zeta^{\alpha}+\alpha|\nabla\zeta|^{\alpha}\tilde{a}_j\right\}k^{\alpha-1}$$

and

$$\tilde{u}_j^{1-\alpha}\zeta^\alpha(b_j|\nabla \bar{u}_j|^{\alpha-1} + \bar{d}_j\bar{u}_j^{\alpha-1})$$
$$\leqq \left\{\left(\frac{\alpha-1}{4}\right)a_j|\nabla v_j|^\alpha + \left(\frac{4}{\alpha-1}\right)^{\alpha-1}(b_j a_j^{1-\alpha}\zeta^\alpha + \bar{d}_j\zeta^\alpha)\right\}k^{\alpha-1}.$$

With these estimate (6.14) becomes

$$\int_{I(3\rho_0)} a_j|\nabla v_j|^\alpha dx \leqq C\rho_0^{-\alpha+n}B_j(4\rho_0).$$

Since $(\text{meas } I(3\rho_0))^{-1}\int_{I(3\rho_0)} v_j dx = 0$, by Lemma 2.6, we finally have

$$\left(\rho_0^{-n}\int_{I(3\rho_0)} |v_j|^{\alpha\bar{p}}dx\right)^{1/\bar{p}} \leqq Ca_j(3\rho_0)\rho_0^{n-\alpha}\int_{I(3\rho_0)} a_j|\nabla v_j|^\alpha dx$$
$$\leqq Ca_j(4\rho_0)B_j(4\rho_0).$$

Q.E.D.

LEMMA 6.3. *Put $v_j = \log \tilde{u}_j$. Then for a sufficiently small $p_0 > 0$,*

(6.15) $$\rho_0^{-n}\int_{I(2\rho_0)} e^{p_0|v_j|}dx \leqq C,$$

where C is a constant depending only on $n, p, t, \alpha, a_j(3\rho_0)$ and $B_j(3\rho_0)$.

Proof. Take $q \geqq 1$ arbitrarily. Put

$$\Phi_j = \bar{u}_j^{1-\alpha}\left(|v_j|^{\alpha q-\alpha+1} + \left[\left(\frac{\alpha}{\alpha-1}\right)(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right)\zeta^\alpha \quad \text{and} \quad \Phi_i = 0 \quad (i \neq j),$$

where $\zeta = \zeta(x, \rho', \rho)$ with $2\rho_0 \leqq \rho' < \rho \leqq 3\rho_0$. Then, we see

$$0 \leqq \int_{I(\rho)} \{\nabla\Phi_j\cdot\bar{A}_j + \Phi_j(b_j|\nabla\bar{u}_j|^{\alpha-1} + d_j\bar{u}_j^{\alpha-1})\}dx.$$

The exact form of $\nabla\Phi_j$ is given by

$$\nabla\Phi_j = \left\{(1-\alpha)\left(|v_j|^{\alpha q-\alpha+1} + \left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right.\right.$$
$$\left.\left. + (\alpha q-\alpha+1)|v_j|^{\alpha q-\alpha}\operatorname{sign} v_j\right)\right\}\bar{u}_j^{-\alpha}\zeta^\alpha\nabla\bar{u}_j$$
$$+ \alpha\left(|v_j|^{\alpha q-\alpha+1} + \left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right)\bar{u}_j^{1-\alpha}\zeta^{\alpha-1}\nabla\zeta.$$

While, the next inequality comes from Young's inequality

(6.16) $$(\alpha q-\alpha+1)|v_j|^{\alpha q-\alpha}$$
$$\leqq \left(\frac{\alpha-1}{\alpha}\right)\left(|v_j|^{\alpha q-\alpha+1} + \left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right).$$

Thus we see that

$$(1-\alpha)\left(|v_j|^{\alpha q-\alpha+1}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right) + (\alpha q-\alpha+1)|v_j|^{\alpha q-\alpha}\operatorname{sign} v_j \leqq -\frac{(\alpha-1)^2}{\alpha}\left(|v_j|^{\alpha q-\alpha+1}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right)<0.$$

Therefore, by the structure (6.7), we have

$$\frac{(\alpha-1)^2}{\alpha}\int_{I(\rho)}\left(|v_j|^{\alpha q-\alpha+1}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right)a_j|v_j|\zeta^\alpha dx \tag{6.17}$$
$$\leqq C\int_{I(\rho)}\left(|v_j|^{\alpha q-\alpha+1}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right) \times\{\bar u_j^{-\alpha}\zeta^\alpha\bar c_j\bar u_j^\alpha+\alpha\bar u_j^{1-\alpha}\zeta^{\alpha-1}|\nabla\zeta|(\tilde a_j|\nabla\bar u_j|^{\alpha-1}+\bar e_j\bar u_j^{\alpha-1}) + \bar u_j^{1-\alpha}\zeta^\alpha(b_j|\nabla\bar u_j|^{\alpha-1}+\bar d_j\bar u_j^{\alpha+1})\}dx$$

By an obvious equality $\bar u_j^{-\alpha}|\nabla\bar u_j|=|\nabla v_j|$ we can see the right-hand side of (6.17)

$$=C\int_{I(\rho)}\left(|v_j|^{\alpha q-\alpha+1}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right)\{\alpha\tilde a_j|\nabla v_j|^{\alpha-1}\zeta^{\alpha-1}|\nabla\zeta| + b_j|\nabla v_j|^{\alpha-1}\zeta^\alpha+\bar c_j\zeta^\alpha+\bar d_j\zeta^\alpha+\bar e_j\zeta^{\alpha-1}|\nabla\zeta|\}dx$$
$$\leqq C\int_{I(\rho)}\left(|v_j|^{\alpha q-\alpha+1}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right) \times\left(\frac{1}{2C}\frac{(\alpha-1)^2}{\alpha}a_j|\nabla v_j|^\alpha\zeta^\alpha+|\nabla\zeta|^\alpha B_j(x)\right)dx.$$

Then, we have by (6.17), using (6.16) again,

$$\int_{I(\rho)}a_j|v_j|^{\alpha q-\alpha}|\nabla v_j|^\alpha\zeta^\alpha dx \leqq(\alpha q-\alpha+1)^{-1}C\int_{I(\rho)}\left(|v_j|^{\alpha q}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}\right)|\nabla\zeta|^\alpha B_j(x)dx.$$

Here we have used the fact that

$$|v_j|^{\alpha q-\alpha+1}\leqq|v_j|^{\alpha q}+\left[\frac{\alpha}{\alpha-1}(\alpha q-\alpha+1)\right]^{\alpha q-\alpha+1}.$$

Put $V_j=v_j^q$. Then, by an obvious equality $|\nabla V_j|^\alpha=q^\alpha|v_j|^{\alpha q-\alpha}$, we have

$$\int_{I(\rho)}a_j|\nabla V_j|^\alpha\zeta^\alpha dx\leqq C\int_{I(\rho)}(|V_j|^\alpha+(\gamma q)^{\alpha q})|\nabla\zeta|B_j(x)dx \leqq C(\rho-\rho')^{-\alpha}\rho_0^{-n}\left\{\left(\rho_0^{-n}\int_{I(\rho)}|V_j|^{\alpha p}dx\right)^{1/p}+(\gamma q)^{\alpha q}\right\},$$

where C is a constant depending only on n, p, t, α and $B_j(3\rho_0)$, and γ depends on α.

This results and Lemma 4.1 guarantee the following

$$\left(\rho_0^{-n}\int_{I(\rho')}|V_j|^{\alpha^\#}dx\right)^{\alpha/\alpha^\#} \leqq C(\rho-\rho')^{-\alpha}\rho_0^{\alpha}\left\{\left(\rho_0^{-n}\int_{I(\rho)}|V_j|^{\alpha\tilde{p}}dx\right)^{1/\tilde{p}} + (\gamma q)^{\alpha q}\right\},$$

where C depends only on $n, p, t, \alpha, a_j(3\rho_0)$ and $B_j(3\rho_0)$.

Putting $q_s = (r/p)^s$ $(r = \alpha^\#/\alpha)$ and $\rho_s = 2\rho_0 + 2^{-s}\rho_0$, $s = 0, 1, \cdots$, the above inequality proves

$$\begin{aligned}\left(\rho_0^{-n}\int_{I(\rho_{s+1})}|v_j|^{\alpha q_{s+1}\tilde{p}}dx\right)^{1/\alpha q_{s+1}\tilde{p}} &\leqq C^{1/\alpha q_s}2^{s/\alpha q_s}\left\{\left(\rho_0^{-n}\int_{I(\rho_s)}|v_j|^{\alpha q_s\tilde{p}}dx\right)^{1/\alpha q_s\tilde{p}} + \gamma q_s\right\} \\ &\leqq C\left\{\left(\rho_0^{-n}\int_{I(\rho_s)}|v_j|^{\alpha q_s\tilde{p}}dx\right)^{1/\alpha q_s p} + \sum_{\ell=0}^{s}\gamma q_\ell\right\},\end{aligned}$$

that is, for any positive number $q \geqq 1$

$$\left(\rho_0^{-n}\int_{I(2\rho_0)}|v_j|^{q}dx\right)^{1/q} \leqq C\left\{\left(\rho_0^{-n}\int_{I(3\rho_0)}|v_j|^{\alpha\tilde{p}}dx\right)^{1/\alpha\tilde{p}} + q\right\} \leqq C\{C_0 + q\},$$

from which we have (6.15). Q.E.D.

LEMMA 6.4. *Let $u = (u_1, \cdots, u_m)$ be a positive solution of* (6.1). *Then*

$$\inf_{I(2\rho_0)} \bar{u}_j \geqq C_1, \tag{6.18}$$

where C_1 is a positive constant depending only on $n, p, t, \alpha, a_j(3\rho_0)$ and $B_j(3\rho_0)$.

Proof. Put $v_j = \log\{\min(1, \tilde{u}_j)\}$ and take $\Phi_j = \bar{u}_j^{1-\alpha}|v_j|^{\alpha q-\alpha+1}\zeta^\alpha$ with $q \leqq 1$, where $\zeta = \zeta(x, \rho', \rho)$ with $2\rho_0 \leqq \rho' < \rho \leqq 3\rho_0$. Then, we have

$$\begin{aligned}\nabla\Phi_j = ((1-\alpha)|v_j|^{\alpha q-\alpha+1} - (\alpha q-\alpha+1)|v_j|^{\alpha q-\alpha})\bar{u}_j^{-\alpha}\zeta^\alpha\nabla\bar{u}_j \\ + \alpha\bar{u}_j^{1-\alpha}|v_j|^{\alpha q-\alpha+1}\zeta^{\alpha-1}|\nabla\zeta|.\end{aligned}$$

Thus, by (6.5) and the structure (6.7),

$$\begin{aligned}&(\alpha q-\alpha+1)\int_{I(\rho)} a_j|v_j|^{\alpha q-\alpha}|\nabla v_j|^\alpha\zeta^\alpha dx \\ &\leqq \int_\Omega \{\alpha\bar{u}_j^{1-\alpha}|v_j|^{\alpha q-\alpha+1}\zeta^{\alpha-1}|\nabla\zeta|(\tilde{a}_j|\nabla\bar{u}_j|^{\alpha-1} + \bar{e}_j\bar{u}_j^{\alpha-1}) \\ &\quad + \bar{u}_j^{1-\alpha}|v_j|^{\alpha q-\alpha+1}\zeta^\alpha(b_j|\nabla\bar{u}_j|^{\alpha-1} + \bar{d}_j\bar{u}_j^{\alpha-1}) \\ &\quad + ((\alpha-1)|v_j|^{\alpha q-\alpha+1} + (\alpha q-\alpha+1)|v_j|^{\alpha q-\alpha})\bar{u}_j^{-\alpha}\zeta^\alpha\bar{c}_j\bar{u}_j^\alpha\}dx.\end{aligned}$$

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Since $|v_j|^{\alpha q-\alpha+1} \leqq |v_j|^{\alpha q} + (1/(\alpha q-\alpha+1))$ and $|v_j|^{\alpha q-\alpha} \leqq |v_j|^{\alpha q} + 1/q$ hold, we have

$$\int_{I(\rho)} a_j|\nabla(v_j^q)|^\alpha \zeta^\alpha dx \leqq Cq^\alpha(\rho-\rho')^{-\alpha}\rho_0^n\left\{\left(\rho_0^{-n}\int_{I(\rho)}|v_j|^{\alpha q\bar{p}}dx\right)^{1/\bar{p}} + \frac{1}{q}\right\},$$

where C is a constant depending only on n, p, t, α, $a_j(3\rho_0)$ and $B_j(3\rho_0)$. From which we have

$$\sup_{I(2\rho_0)}|v_j| \leqq C\{C_0 + \gamma\}$$

in a similar manner to the case of Theorem 4.1, that is, (6.18) holds.

Q.E.D.

We are now ready to state the Harnack theorem.

THEOREM 6.2. *Let* $u = (u_1, \cdots, u_m)$ *be a positive weak solution of* (6.1) *under the condition* (I)–(III). *Then*

$$\sum_{j=1}^m \sup_{I(\rho_0)} u_j \leqq C\sum_{j=1}^m (\inf_{I(\rho_0)} u_j + \kappa_j), \tag{6.9}$$

where C *is a positive constant depending only on* n, p, t, α, $a_j(4\rho_0)$ *and* $B_j(4\rho_0)$ $(j = 1, 2, \cdots, m)$.

Proof. From (6.15), we have

$$\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar{u}_j^{p_0}dx\right)^{1/p_0} \leqq C\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar{u}_j^{-p_0}dx\right)^{-1/p_0}.$$

Combining this inequality with (6.9), we obtain

$$\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar{u}_j^{p_0}dx\right)^{1/p_0} \leqq C\inf_{I(\rho_0)}\bar{u}_j, \qquad j = 1, 2, \cdots, m.$$

Hence Theorem 6.2 can be proved if we show

$$\sum_{j=1}^m \sup_{I(\rho_0)}\bar{u}_j \leqq C\sum_{j=1}^m\left(\rho_0^{-n}\int_{I(2\rho_0)}\bar{u}_j^{p_0}dx\right)^{1/p_0}. \tag{6.20}$$

In fact this is true as is shown below. Put $\Phi_j = \bar{u}_j^{\alpha q-\alpha+1}\zeta^\alpha$ with $q > 0$, $q \neq (\alpha-1)/\alpha$, where $\zeta = \zeta(x, \rho', \rho)$ with $\rho_0 \leqq \rho' < \rho \leqq 2\rho_0$.

In case $0 < q < (\alpha-1)/\alpha$, Φ_i $(i \neq j)$ is defined to be 0, and

$$0 \leqq \int_{I(\rho)}\{\nabla\Phi_j\cdot\bar{A}_j + \Phi_j(b_j|\nabla\bar{u}_j|^{\alpha-1} + \bar{d}_j\bar{u}_j^{\alpha-1})\}dx$$

is shown by (6.5). Since $\nabla\Phi_j = (\alpha q-\alpha+1)\bar{u}_j^{\alpha q-\alpha}\zeta^\alpha\nabla\bar{u}_j + \alpha\bar{u}_j^{\alpha q-\alpha+1}\zeta^{\alpha-1}\nabla\zeta$, by

(6.7), we have

$$-(\alpha q-\alpha+1)\int_{I(\rho)} a_j\bar u_j^{\alpha q-\alpha}|\nabla \bar u_j|^\alpha\zeta^\alpha dx$$
$$\leqq \int_{I(\rho)}\{\alpha\bar u_j^{\alpha q-\alpha+1}\zeta^{\alpha-1}|\nabla\zeta|(\tilde a_j|\nabla\bar u_j|^{\alpha-1}+\bar e_j\bar u_j^{\alpha-1})$$
$$+\bar u_j^{\alpha q-\alpha+1}\zeta^\alpha(b_j|\nabla\bar u_j|^{\alpha-1}+\bar d_j\bar u_j^{\alpha-1})+|\alpha q-\alpha+1|\bar c_j\bar u_j^{\alpha q}\}dx\,.$$

Put $v_j=\bar u_j^q$. Then, by the same argument as before, we have

$$\int_{I(\rho)} a_j|\nabla v_j|^\alpha\zeta^\alpha dx\leqq C_q\int_{I(\rho)}|\nabla\zeta|^\alpha B_j(x)v_j^\alpha dx\,.$$

Therefore,

$$\sum_{j=1}^m\int_{I(\rho)} a_j|\nabla v_j|^\alpha\zeta^\alpha dx\leqq C_q(\rho-\rho')^{-\alpha}\rho_0^n\Big(\rho_0^{-n}\int_{I(\rho)}\Big(\sum_{j=1}^m v_j\Big)^{\alpha\tilde p}dx\Big)^{1/\tilde p},$$

where C_q is a constant depending only on $n, p, q, t, \alpha, a_j(2\rho^0)$ and $B_j(2\rho_0)$ $(j=1,2,\cdots,m)$.

In case $(\alpha-1)/\alpha<q$, we define $\Phi_j=\bar u_j^{\alpha q-\alpha+1}\zeta^\alpha$ $(j=1,2,\cdots,m)$ to have

$$\sum_{j=1}^m\int_{I(\rho)}\{\nabla\Phi_j\cdot A_j+\Phi_jB_j\}dx=0\,.$$

Since $\nabla\Phi_j=(\alpha q-\alpha+1)\bar u_j^{\alpha q-\alpha}\zeta^\alpha\nabla u_j+\alpha\bar u_j^{\alpha q-\alpha+1}\zeta^{\alpha-1}\nabla\zeta$,

$$(\alpha q-\alpha+1)\sum_{j=1}^m\lambda_j\int_{I(\rho)} a_ju_j^{\alpha q-\alpha}|\nabla\bar u_j|^\alpha\zeta^\alpha dx$$
$$\leqq\sum_{j=1}^m\int_{I(\rho)}\Big\{\alpha\bar u_j^{\alpha q-\alpha+1}\zeta^{\alpha-1}|\nabla\zeta|\Big(\tilde a_j|\nabla\bar u_j|^{\alpha-1}+\sum_{i=1}^m\bar e_{ij}\bar u_i^{\alpha-1}\Big)$$
$$+\bar u_j^{\alpha q-\alpha+1}\zeta^\alpha\Big(b_j|\nabla\bar u_j|^{\alpha-1}+\sum_{i=1}^m\bar d_{ij}\bar u_i^{\alpha-1}\Big)$$
$$+(\alpha q-\alpha+1)\zeta^\alpha\Big(\bar c_j\bar u_j^{\alpha q}+\sum_{\substack{i=1\\ i\neq j}}^m c_{ij}\bar u_i^{\alpha-1}\bar u_j^{\alpha q-\alpha}\Big)\Big\}dx$$

is proved by using the condition (6.6). By Lemma 6.4, $\bar u_j^{\alpha q-\alpha}\leqq C_1\bar u_j^{\alpha q-\alpha+1}$ is true. Noting $\alpha q/(\alpha-1)\geqq 1$, we see, by using Young's inequality,

$$\bar u_i^{\alpha-1}\bar u_j^{\alpha q-\alpha+1}\leqq\bar u_i^{\alpha q}+\bar u_j^{\alpha q}\quad\text{and}\quad\bar u_i^{\alpha-1}\bar u_j^{\alpha q-\alpha}\leqq C_1^{-1}(\bar u_i^{\alpha q}+\bar u_j^{\alpha q})\,.$$

Thus, putting $v_j=\bar u_j^q$ $(j=1,2,\cdots,m)$, we have

$$\sum_{j=1}^m\int_{I(\rho)} a_j|\nabla v_j|^\alpha\zeta^\alpha dx\leqq C_q(\rho-\rho')^{-\alpha}\rho^n\sum_{j=1}^m\Big(\rho_0^{-n}\int_{I(\rho)}\Big(\sum_{j=1}^m v_j\Big)^{\alpha\tilde p}dx\Big)^{1/\tilde p},\tag{6.21}$$

where C_q is a constant depending only on $n, p, q, t, \alpha, a_j(2\rho_0)$ and $B_j(2\rho_0)$ $(j = 1, 2, \cdots, m)$.

We have therefore obtained (6.21) for any $q > 0$ with $q \neq (\alpha - 1)/\alpha$.

Let q_0 be a sufficiently small number such that $0 < \alpha q_0 \tilde{p} \leqq p_0$ and $q_0(r/p)^s \neq (\alpha - 1)/\alpha$, $s = 0, 1, 2, \cdots$. Then, (6.20) is obtained in a similar manner to the case of Theorem 4.1, which was to be proved. Q.E.D.

To close this section we give an illustrative example.

Take a vector function

$$C_j(x, u) = \Bigl(\sum_{\substack{i=1\\ i\neq j}}^{m} c_{ij}^1 u_i^{(\alpha-1)2/\alpha} \operatorname{sign} \frac{\partial u_i}{\partial x_1}, \cdots, \sum_{\substack{i=1\\ i\neq j}}^{m} c_{ij}^n u_i^{(\alpha-1)2/\alpha} \operatorname{sign} \frac{\partial u_i}{\partial x_n}\Bigr).$$

Let $\bar{A}_i(x, u_j, \nabla u_j) = (A_{1j}, \cdots, A_{nj})$ and a scalar function $\bar{B}_j(x, u, \nabla u_j)$ be given and satisfy the following inequalities

$$\begin{cases} \xi \cdot \bar{A}_j(x, u_j, \xi) \geqq a_j|\xi|^\alpha - c_j|u_j|^\alpha - \bar{f}_j\,, \\ |\bar{B}_j(x, u, \xi)| \leqq b_j|\xi|^{\alpha-1} + \sum_{i=1}^{m} d_{ij}|u_i|^{\alpha-1} + g_j\,, \\ |A_j(x, u_j, \xi)| \leqq \tilde{a}_j|\xi|^{\alpha-1} + e_j|u_j|^{\alpha-1} + \bar{h}_j\,, \qquad \xi \in R^n \end{cases}$$

$j = 1, \cdots, m$. Here $a_j, b_j, \cdots, \bar{h}_j$ and $\tilde{a}_j$ are non-negative measurable functions in Ω, $a_j^{-1} \in L^t(\Omega)$ for any $t > 1$, $a_j \leqq \tilde{a}_j$, and $\tilde{a}_j$, $\tilde{a}_j^\alpha a_j^{1-\alpha}$, $b_j a_j^{1-\alpha}$, c_j, d_{ij}, $\bar{f}_j$, g_j, $(c_{ij}^k)^{\alpha/(\alpha-1)} a_j^{1/(1-\alpha)}$, $e_j^{\alpha/(\alpha-1)} \tilde{a}_j^{1/(1-\alpha)}$, $\bar{h}_j^{\alpha/(\alpha-1)} \tilde{a}_j^{1/(1-\alpha)} \in L^{p/\alpha}(\Omega)$, and $c_{ij} \in H^{1,p/\alpha}(\Omega)$.

Define $A_j(x, u, \nabla u_j) = A_j(x, u_j, \nabla u_j) + C_j(x, u)$ and

$$B_j(x, u, \nabla u_j) = \bar{B}_j(x, u, \nabla u_j) + \sum_{i=1}^{m} \tilde{d}_{ij} u_i^{\alpha-1} + \sum_{\substack{i=1\\ i\neq j}}^{m} \tilde{c}_{ij} u_i^{(\alpha-1)2/\alpha}\,,$$

for some choices of functions, $\tilde{d}_{ij}$, $\tilde{c}_{ij} \in L^{p/\alpha}(\Omega)$.

We assume that for any i $(i \neq j)$

$$\sum_{k=1}^{n} \left|\frac{c_{ij}^k}{x_k}\right| + \tilde{c}_{ij} \leqq 0 \quad \text{and} \quad d_{ij} + \tilde{d}_{ij} \leqq 0\,.$$

Then, the system

$$-\operatorname{div} A_j(x, u, \nabla u_j) + B_j(x, u, \nabla u_j) = 0 \qquad (j = 1, 2, \cdots, m)$$

satisfies the condition (I)–(III).

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Department of Mathematics
Aichi University of Education
Kariya, 448 Japan